LE MATEMATICHE
Vol. LXXVII (2022) – Issue I, pp. 33–45
doi: 10.4418/2022.77.1.2

IDEMPOTENT FACTORIZATION OF MATRICES OVER A PRÜFER DOMAIN OF RATIONAL FUNCTIONS

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We consider the smallest subring D of $\mathbb{R}(X)$ containing every element of the form $1/(1+x^2)$, with $x \in \mathbb{R}(X)$. D is a Prüfer domain called the *minimal Dress ring of* $\mathbb{R}(X)$. In this paper, addressing a general open problem for Prüfer non Bézout domains, we investigate whether 2×2 singular matrices over D can be decomposed as products of idempotent matrices. We show some conditions that guarantee the idempotent factorization in $M_2(D)$.

1. Introduction

In 1965 Andreas Dress [7] introduced a family of Prüfer domains constructed as subrings D_K of a field K containing every element of the form $1/(1+x^2)$, for $x \in K$. Given a field K not containing square roots of -1, the subring of K generated by $\{(1+x^2)^{-1} : x \in K\}$ is said to be the *minimal Prüfer-Dress ring* (or simply the *minimal Dress ring*) of K. We refer to [7] and [4] for more details on these domains. In the paper [4], the authors investigated minimal Dress rings of

Received on February 2, 2021

AMS 2010 Subject Classification: 15A23, 13F05, 13A15

Keywords: Factorization of matrices, idempotent matrices, minimal Dress rings, Prüfer domains.

This research was partially supported by the "Ernst Mach Grant - worldwide" of the Austria's Agency for Education and Internationalisation (OeAD), under the project ICM-2020-00054 "Factorization theory in matrix rings".

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special classes of fields: Henselian fields, ordered fields and formally real fields (e.g., $\mathbb{R}(\mathcal{A})$, with $\mathcal{A}$ a set of indeterminates). They focused in particular on the minimal Dress ring D of the field of real rational functions $\mathbb{R}(X)$, characterizing its elements [4, Prop. 2.1] and ideals [4, Prop. 2.4] and proving that D is a Dedekind domain (i.e., a Noetherian Prüfer domain) that is not a principal ideal domain [4, Th. 2.3]. They also identified a family of 2×2 singular matrices over D that can be written as a product of idempotent factors [4, Th. 3.3]. The study of the factorization of singular square matrices over rings as product of idempotent matrices has raised a remarkable interest both in the commutative and non-commutative setting since the middle of the 1960's (see [8, 11]). We say that an integral domain R satisfies the property (ID_2) if every 2×2 singular matrix over R is a product of idempotent factors. A natural and well motivated conjecture, proposed by Salce and Zanardo in [11] and then investigated in [3] and [5], asserts that every domain R satisfying (ID_2) must be a Bézout domain, namely, every finitely generated ideal of R must be principal. Note that the reverse implication is false: not every Bézout domain verifies (ID_2) (see [2, 6]). In [3] it is proved that if R satisfies (ID_2), then every finitely generated ideal of R is invertible and so R is a Prüfer domain. Therefore, it is not restrictive to study (ID_2) within this class of domains and, in view of the above conjecture, we expect that for every Prüfer non-Bézout domain R there exists at least one singular matrix in $M_2(R)$ that cannot be written as a product of idempotent factors.

In this paper we develop the investigation started in [4] on idempotent factorizations of 2×2 matrices over the minimal Dress ring D of $\mathbb{R}(X)$. In Section 2 we fix the notation and recall some preliminary results and definitions. In Section 3 we focus on the factorizations in $M_2(D)$ and, in Theorems 3.3, 3.8 and 3.10, we identify several conditions on a pair of elements $p, q \in D$ under which the matrix $\begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix}$ factors into idempotents. In this way we supplement the results in [4] by providing further families of 2×2 matrices over D that admit idempotent factorizations. Moreover, in Example 3.14 we exhibit a singular matrix in $M_2(D)$ for which the failure of the above conditions prevents an "easy" decomposition into idempotent factors. However the general problem whether D satisfies (ID_2) remains open.

2. Preliminaries and notation

Let R be a (commutative) integral domain. We will use the standard notations $R^\times$ to denote its multiplicative group of units and $M_n(R)$ to denote the R-algebra of $n \times n$ matrices over R. A square matrix $\mathbf{T}$ over R is said to be idempotent if $\mathbf{T}^2 = \mathbf{T}$. A direct computation shows that a singular nonzero matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

over an arbitrary integral domain is idempotent if and only if $d = 1-a$. For a singular matrix $\mathbf{S} \in M_n(R)$, the property of being a product of idempotent factors is preserved by similarity. This immediately leads to the following lemma.

Lemma 2.1 (Lemma 3.1 of [4])**.** *Let R be an integral domain, $p,q \in R$. The matrix $\begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix}$ is a product of idempotent matrices if and only if such is $\begin{pmatrix} q & p \\ 0 & 0 \end{pmatrix}$.*

The next result will also be useful in the following.

Lemma 2.2. *Let p and q be nonzero elements of an integral domain R, and $\boldsymbol{M} = \begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix} \in M_2(R)$. If $\boldsymbol{M} = \boldsymbol{S} \cdot \boldsymbol{T}$, with $\boldsymbol{S} = \begin{pmatrix} p' & q' \\ z & t \end{pmatrix}$ a singular matrix and $\boldsymbol{T} = \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}$ an idempotent matrix over R, then $\boldsymbol{S}$ has the form $\boldsymbol{S} = \begin{pmatrix} p' & q' \\ 0 & 0 \end{pmatrix}$.*

We omit the proof, since it is essentially contained in that of Lemma 3.1 in [6].

Finally, we recall below two immediate factorizations in $M_2(R)$:

$$\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1-p & 0 \end{pmatrix} \; ; \; \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & q \\ 0 & 1 \end{pmatrix}. \qquad (1)$$

From now on D will denote the minimal Dress rings of the field of rational functions $\mathbb{R}(X)$. In accordance with [4], we define the degree of a rational function f/g, with $f,g \in \mathbb{R}[X]$, as $\deg(f/g) := \deg(f) - \deg(g)$.

Following the notation in [4], let Γ be the set of the polynomials in $\mathbb{R}[X]$ that have no roots in $\mathbb{R}$. Then $\Gamma = \{\alpha \prod_i \gamma_i\}$, where the γ_i are monic degree-two polynomials irreducible over $\mathbb{R}[X]$ and $0 \neq \alpha$ is a real number. Set $\Gamma^+ = \{f \in \mathbb{R}[X] : f(r) > 0, \forall r \in \mathbb{R}\}$ and, correspondingly, $\Gamma^- = \{-f : f \in \Gamma^+\}$. By Proposition 2.1 in [4],

$$D = \{f/\gamma : f \in \mathbb{R}[X], \gamma \in \Gamma, \deg f \leq \deg \gamma\},$$

and

$$D^\times = \{\gamma_1/\gamma_2 : \gamma_1, \gamma_2 \in \Gamma, \deg \gamma_1 = \deg \gamma_2\}.$$

As recalled in the introduction, we know from Theorem 2.3 and Proposition 2.4 of [4] that D is a Dedekind domain which is not a principal ideal domain. As an example, the ideal generated by $1/\gamma$ and X/γ, with $\gamma \in \Gamma \setminus \mathbb{R}$, is not principal. It is worth remarking that a non-constant polynomial of $\mathbb{R}[X]$ *never* lies in D.

Given a polynomial $f \in \mathbb{R}[X]$ we will denote as l.c.(f) its leading coefficient. In the following, given an element $p = f/\gamma \in D$ we will always assume that γ is a product of monic irreducible polynomials of degree 2. We will then define the leading coefficient of a rational function $p \in D$ as the leading coefficient of its numerator.

3. Idempotent factorizations in $M_2(D)$

In this section we investigate property (ID_2) over D. We find sufficient conditions on the entries of a singular matrix over D to get a factorization into idempotents. We start recalling two results of [4] that we will need in our discussion.

Lemma 3.1 (Lemma 3.2 of [4]). *Let x, y be non-zero polynomials in $\mathbb{R}[X]$ with* $\deg x = \deg y$.

(a) If $y(u) > 0$ (or $y(u) < 0$) for every u root of x, then there exists $\beta \in \Gamma$ such that $\delta = x^2 + y\beta \in \Gamma^+$, $\deg x - 1 \leq \deg \beta \leq \deg x = \deg \delta/2$.

(b) If $x(z) > 0$ (or $x(z) < 0$) for every z root of y, then there exists $\eta \in \Gamma$ such that $\delta = x\eta + y^2 \in \Gamma^+$ and $\deg y - 1 \leq \deg \eta \leq \deg y = \deg \delta/2$.

Theorem 3.2 (Th. 3.3 of [4]). *Let p, q be elements of D. Then the matrix $\begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix}$ is a product of idempotent matrices if one of the following holds:*

(i) $\deg p \geq \deg q$ *and $q(u) > 0$ (or $q(u) < 0$) for every u root of p*

(ii) $\deg q \geq \deg p$ *and $p(z) > 0$ (or $p(z) < 0$) for every z root of q.*

Two polynomials $x, y \in \mathbb{R}[X]$ are said to be *weakly comaximal* if $\gcd(x, y) \in \Gamma$, i.e., if x and y have no common roots in $\mathbb{R}$. Let p and q be two elements of D. Then we can always write $p = x/\gamma$ and $q = y/\gamma$, with $\gamma \in \Gamma^+$ and $x, y \in \mathbb{R}[X]$. We say that p and q are *weakly comaximal* if so are x and y. Given an element $p \in D$, we will write $p \geq 0$ (resp. $p \leq 0$) if $p(r) \geq 0$ (resp. $p(r) \leq 0$) for each $r \in \mathbb{R}$.

Theorem 3.3. *Let p and q be weakly comaximal elements of D. If either $p \geq 0$ or $q \geq 0$, then the matrix $\begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix}$ is a product of idempotent matrices.*

Proof. By Lemma 2.1, $\begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix}$ is a product of idempotent matrices if and only if such is $\begin{pmatrix} q & p \\ 0 & 0 \end{pmatrix}$, therefore we can safely assume that $p \geq 0$.

Let us first consider the case $\deg p \geq \deg q$. We can assume without loss of generality that $\deg p = \deg q$. In fact, since $\begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix}$ is similar to $\begin{pmatrix} p & p+q \\ 0 & 0 \end{pmatrix}$ and p and $p+q$ are still weakly comaximal, if $\deg p > \deg q$, it not restrictive to replace q with $p+q$. Thus, let $\deg p = \deg q$ and set $p = x/\gamma$ and $q = y/\gamma$, with $x, y \in \mathbb{R}[X]$, $\gamma \in \Gamma^+$.

As a further reduction, we may assume that $\deg p = \deg q = 0$. In fact, being $p \geq 0$, every root of p has even multiplicity and $\deg x$ is even. Moreover, if $\deg x < \deg \gamma$, taking any $\tau \in \Gamma^+$ such that $\deg x = \deg \tau$,

$$\begin{pmatrix} x/\gamma & y/\gamma \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \tau/\gamma & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x/\tau & y/\tau \\ 0 & 0 \end{pmatrix}$$

is a factorization in $M_2(D)$ and, by (1), the matrix on the left of the above equality is a product of idempotents if such is the second factor of the product on the right.

Since for every z root of y, $x(z)$ is always > 0, we have got in the position to apply Lemma 3.1 (ii) to x and y. Therefore, there exists $\eta \in \Gamma$ such that $\delta = x\eta + y^2 \in \Gamma^+$ where $\deg \eta = \deg x$ and $\deg \delta = 2 \deg \eta$.

Then, since $\deg x = \deg y = \deg \gamma = \deg \eta$, $\delta/\gamma\eta \in D^\times$ and $x\eta/\delta$, $y\eta/\delta$, xy/δ, $y^2/\delta \in D$. Moreover, the relation $1 - x\eta/\delta = y^2/\delta$ implies that $\mathbf{T} = \begin{pmatrix} x\eta/\delta & y\eta/\delta \\ xy/\delta & y^2/\delta \end{pmatrix}$ is an idempotent matrix over D. Therefore

$$\begin{pmatrix} x/\gamma & y/\gamma \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \delta/\gamma\eta & 0 \\ 0 & 0 \end{pmatrix} \mathbf{T},$$

and using (1) we conclude that $\begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix}$ is a product of idempotent matrices over D. On the other hand, if $\deg q > \deg p$, it suffices to apply Theorem 3.2 (ii). $\square$

Remark 3.4. The matrix $\begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix} \in M_2(D)$ is a product of idempotent matrices even if p and q are two comaximal elements of D such that either $p \leq 0$ or $q \leq 0$. The proof is basically the same as that of Theorem 3.3.

In what follows the symbol $f^{(j)}$ denotes the j-th derivative of the polynomial $f \in \mathbb{R}[X]$.

Lemma 3.5. *Let $x, y \in \mathbb{R}[X]$ and $\varepsilon \in \mathbb{R}^+$ be such that:*

- *y has 0 as unique root with multiplicity k;*

- *$x(0) \neq 0$;*
- *$y^{(i)} > 0$ in $(0,\varepsilon]$ for $0 \leq i \leq k-1$;*
- *$y^{(k)}$ does not change sign in $(0,\varepsilon]$;*
- *$x^{(j)}$ is either zero or does not change sign in $(0,\varepsilon]$ for $0 \leq j \leq k$.*

Then, there exists a real number $r_0 > 0$ such that, for every $r \in (0,r_0]$, $rx+y$ has at most one root in $(0,\varepsilon]$, and exactly one root if $x < 0$ in $(0,\varepsilon]$.

Proof. Note that $y^{(k)}(0) \neq 0$, hence $y^{(k)}$ has nonzero max and min in $[0,\varepsilon]$. Since, by assumption, $x^{(k)}$ is either zero or does not change sign in $(0,\varepsilon]$, an easy direct check shows that, for all possible signs, there exists $r_0 > 0$ such that, for every $r \in (0,r_0]$, $rx^{(k)} + y^{(k)}$ is either strictly positive or negative in $[0,\varepsilon]$. It follows that $\forall r \in (0,r_0]$, $rx^{(k-1)} + y^{(k-1)}$ is either increasing or decreasing in the interval and hence it has at most one root.

Let us consider the $(k-1)$-th derivative of $rx+y$. We distinguish three cases.

(i) If $x^{(k-1)} \geq 0$ in $(0,\varepsilon]$, then $rx^{(k-1)} + y^{(k-1)} > 0$ in this neighborhood for every $r \in (\,0,\ r_0\,]$. Therefore, being increasing, $rx^{(k-2)} + y^{(k-2)}$ has at most a unique root in the interval.

(ii) If $x^{(k-1)} < 0$ in $(0,\varepsilon]$ and $y^{(k-1)}(0) \neq 0$, then, by possibly choosing a smaller r_0, $rx^{(k-1)} + y^{(k-1)}$ is either strictly positive or negative in $[0,\varepsilon]$ for every $r \in (0,r_0]$, and again we get that $rx^{(k-2)} + y^{(k-2)}$ has at most a unique root.

(iii) If $x^{(k-1)} < 0$ in $(0,\varepsilon]$ and $y^{(k-1)}(0) = 0$, by possibly choosing a smaller r_0, $rx^{(k-1)}(\varepsilon) + y^{(k-1)}(\varepsilon) > 0$ for every $r \in (0,r_0]$. Since $rx^{(k-1)}(0) + y^{(k-1)}(0) \leq 0$, we have two possibilities: or $rx^{(k-1)} + y^{(k-1)} > 0$ in $(0,\varepsilon]$ for every $r \in (0,r_0]$, or there exists $x^r_{k-1} \in (0,\varepsilon)$ such that $rx^{(k-1)}(x^r_{k-1}) + y^{(k-1)}(x^r_{k-1}) = 0$ and this zero is unique since $rx^{(k-1)} + y^{(k-1)}$ has at most one root. As a consequence, in the first case $rx^{(k-2)} + y^{(k-2)}$ is strictly increasing and admits at most one root, in the second case it decreases on $(0,x^r_{k-1})$ and increases on $(x^r_{k-1},\varepsilon]$.

Let us now distinguish three more cases for the $(k-2)$-th derivative of $rx+y$.

(i) If $x^{(k-2)} \geq 0$ in $(0,\varepsilon]$, then $rx^{(k-2)} + y^{(k-2)} > 0$ in this neighborhood for every $r \in (0,r_0]$, therefore $rx^{(k-3)} + y^{(k-3)}$ is increasing and it has at most a unique root in the interval.

(ii) If $x^{(k-2)} < 0$ in $(0,\varepsilon]$ and $y^{(k-2)}(0) \neq 0$, then, by possibly choosing a smaller r_0, $rx^{(k-2)} + y^{(k-2)}$ is either strictly positive or negative in $[0,\varepsilon]$ for every $r \in (0,r_0]$, and again we get that $rx^{(k-3)} + y^{(k-3)}$ has at most a unique root.

(iii) If $x^{(k-2)} < 0$ in $(0,\varepsilon]$ and $y^{(k-2)}(0) = 0$, by possibly choosing a smaller r_0 $rx^{(k-2)}(\varepsilon) + y^{(k-2)}(\varepsilon) > 0$ for every $r \in (0,r_0]$. Since $rx^{(k-2)}(0) + y^{(k-2)}(0) \leq 0$ we have two possibilities: $rx^{(k-2)} + y^{(k-2)} > 0$ in $(0,\varepsilon]$ for every $r \in (0,r_0]$ or, for every $r \in (0,r_0]$, there exists $x^r_{k-2} \in (0,\varepsilon)$ zero of $rx^{(k-2)} + y^{(k-2)}$. By the previous step $rx^{(k-2)} + y^{(k-2)}$ is either increasing or has a unique critical point on $(0,\varepsilon]$. In both this cases we cannot have other roots besides x^r_{k-2}. As a consequence $rx^{(k-3)} + y^{(k-3)}$ is either strictly increasing or it decreases on $(0,x^r_{k-2})$ and increases on $(x^r_{k-2},\varepsilon]$.

Iterating the procedure, after k steps we obtain that there exists a real number $r_0 > 0$ such that, for every $r \in (0,r_0]$, $rx + y$ has at most a unique root in $(0,\varepsilon]$ and exactly one root if $x < 0$ in $(0,\varepsilon]$. □

Remark 3.6. In the hypothesis of the above Lemma 3.5, if $k = 1$, the proof becomes much easier. If $x > 0$ in $(0,\varepsilon]$, $rx + y > 0$ for every positive $r \in \mathbb{R}$. Let us assume henceforth that $x < 0$ in $(0,\varepsilon]$. There always exists a suitable $r_0 > 0$ such that $rx(\varepsilon) + y(\varepsilon) > 0$ for every $r \in (0,r_0]$. Since $y(0) = 0$ and $y(\varepsilon) > 0$, it must be $y' > 0$ on $(0,\varepsilon]$. If in the same interval $x' \geq 0$ then $rx' + y' > 0$ and since $rx(0) + y(0) < 0$, $rx + y$ has a unique root in $(0,\varepsilon]$ for every $r \in (0,r_0]$. If $x' < 0$ in $(0,\varepsilon]$, by possibly choosing a smaller r_0, $rx' + y'$ is still strictly positive in $(0,\varepsilon]$ for every $r \in (0,r_0]$ and we conclude as before.

Lemma 3.7. *Let x,y be polynomials in $\mathbb{R}[X]$ without common roots, such that* $\deg x$ *and* $\deg y$ *are odd,* $\deg x > \deg y$*, y has a unique root and there exist $x_1,x_2 \in \mathbb{R}$ roots of x such that $y(x_1)y(x_2) < 0$. Then, there exists a suitable $r \in \mathbb{R}$, such that $rx + y$ has a unique root.*

Proof. It is not restrictive to assume $\lim_{X\to\pm\infty} xy = +\infty$. If the leading coefficients l.c.(x) and l.c.(y) are discordant the proof can be accordingly adapted by replacing r with $-r$.

Let $\lim_{X\to\pm\infty} x,y = \pm\infty$. The case $\lim_{X\to\pm\infty} x,y = \mp\infty$ is analogous. Up to a suitable translation we can assume $y(0) = 0$, $x_1 < 0$ and $x_2 > 0$. Let k be the (odd) multiplicity of 0 as a root of y. Let $I_0 = (-\varepsilon,\varepsilon)$ be a sufficiently small neighborhood of 0 such that $x < 0$ and y is strictly increasing in I_0. The case $x > 0$ in I_0 can be treated similarly in the interval $[-\varepsilon,0)$. By possibly choosing a smaller ε, we may assume that in the interval $(0,\varepsilon]$ $y^{(i)} > 0$ for $1 \leq i \leq k-1$, $y^{(k)}$ does not change sign and $x^{(j)}$ is either zero or does not change sign for

$1 \le j \le k$. By Lemma 3.5 there exists a real number $r_0 > 0$ such that, for any $r \in (0, r_0]$, $rx + y$ has a unique root on $[0, \varepsilon]$. Let us observe that, in $[-\varepsilon,\ 0\ ]$, $rx + y < 0$ for every positive r.

Now take $M \in \mathbb{R}^+$ such that $xy > 0$ for all X such that $|X| > M$. Clearly, for every $r > 0$, $rx + y$ has no roots for $|X| > M$. We consider the closed intervals $I_1 = [\ -M,\ -\varepsilon\ ]$ and $I_2 = [\ \varepsilon,\ M\ ]$. Let $M_x = \max_{I_1 \cup I_2} |x| > 0$ and $m_y = \min_{I_1 \cup I_2} |y|$. Since y has no roots other than 0, clearly $m_y > 0$. By choosing $0 < r < m_y/M_x$ the polynomial $rx + y$ has no zeroes in $I_1 \cup I_2$.

We conclude by choosing any $0 < r < \min\{r_0, m_y/M_x\}$. □

Theorem 3.8. *Let p and q be elements of D. If $\deg p, \deg q$ are odd and either p or q has a unique root, then the matrix $\begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix}$ is a product of idempotent matrices.*

Proof. By Lemma 2.1, we can safely assume that q has a unique root. We first consider the case p and q weakly comaximal.

If $\deg q \ge \deg p$, since q has a unique root and p and q have no common factors, the hypothesis of Th. 3.2 (ii) are satisfied.

If $\deg p > \deg q$ we distinguish two cases. If q does not change sign on the roots of p, we are done by Theorem 3.2 (i). Otherwise, by Lemma 3.7, it is always possible to find a suitable $r \in \mathbb{R}$ such that $rp + q$ has a unique root. Therefore, by Theorem 3.2 (ii), the matrix $\begin{pmatrix} p & rp+q \\ 0 & 0 \end{pmatrix}$, similar to $\begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix}$, is a product of idempotent matrices.

Now consider the case of p and q not weakly comaximal. If $p = x/\gamma$ and $q = y/\gamma$, with $\gamma \in \Gamma^+$ and $x, y \in \mathbb{R}[X]$, x and y have a common root. Since q has odd degree and a unique root $z \in \mathbb{R}$, we have $x = (X - z)^h \bar{x}$ and $y = (X - z)^k \bar{y}$ with k, h positive integers, k odd, $\bar{y} \in \Gamma$, $\bar{x} \in \mathbb{R}[X]$ and $\gcd(X - z, \bar{x}) = 1$. Let us choose any $\delta \in \Gamma^+$ such that either $\deg \delta = \min\{k, h\}$ or $\deg \delta = \min\{k, h\} + 1$, in accordance with the parity of $\min\{k, h\}$. Since $\max\{\deg p, \deg q\} < 0$, $\begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} (X-z)^{\min\{k,h\}}/\delta & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (X-z)^{h-\min\{k,h\}}\bar{x}\delta/\gamma & (X-z)^{k-\min\{k,h\}}\bar{y}\delta/\gamma \\ 0 & 0 \end{pmatrix}$ is a factorization in $M_2(D)$ and, by (1), $\begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix}$ is a product of idempotent matrices if such is $\mathbf{S} = \begin{pmatrix} (X-z)^{h-\min\{k,h\}}\bar{x}\delta/\gamma & (X-z)^{k-\min\{k,h\}}\bar{y}\delta/\gamma \\ 0 & 0 \end{pmatrix}$. Let us remark that the elements of the first row of $\mathbf{S}$, $(X - z)^{h-\min\{k,h\}}\bar{x}\delta/\gamma$ and $(X - z)^{k-\min\{k,h\}}\bar{y}\delta/\gamma$, are now weakly comaximal.

If $h \geq k$, then $\mathbf{S} = \begin{pmatrix} (X-z)^{h-k}\bar{x}\delta/\gamma & \bar{y}\delta/\gamma \\ 0 & 0 \end{pmatrix}$. Since $\bar{y}\delta \in \Gamma$, then $\bar{y}\delta/\gamma \geq 0$ or $\bar{y}\delta/\gamma \leq 0$. In the first case we conclude by applying Theorem 3.3; in the second, by applying the analogous of Theorem 3.3 stated in remark 3.4.

If $k > h$, we get $\mathbf{S} = \begin{pmatrix} \bar{x}\delta/\gamma & (X-z)^{k-h}\bar{y}\delta/\gamma \\ 0 & 0 \end{pmatrix}$. If h is even, then $\deg \bar{x}$ is odd, since $\deg x = h + \deg \bar{x}$ is odd. Moreover also $k-h$ is odd. It follows that $\bar{x}\delta/\gamma$ and $(X-z)^{k-h}\bar{y}\delta/\gamma$ are two weakly comaximal element of D with odd degree and, being $\bar{y}$ and δ elements of Γ, $(X-z)^{k-h}\bar{y}\delta/\gamma$ has a unique root $z \in \mathbb{R}$. Therefore, from the first part of the proof, we conclude that $\mathbf{S}$ is a product of idempotent matrices. If h is odd, being $k-h$ even, $(X-z)^{k-h}\bar{y}\delta$ is always ≥ 0 or ≤ 0 and we conclude by applying Theorem 3.3 (or its analogous in remark 3.4).

All possible cases have been examined. □

Lemma 3.9. *Let x, y be polynomials in $\mathbb{R}[X]$ without common roots, such that $\deg x$ is even, $\deg y$ is odd, $\deg x > \deg y$, y has a unique root y_1 and there exist $x_1, x_2 \in \mathbb{R}$ roots of x such that $y(x_1)y(x_2) < 0$. Then, there exists a suitable $r \in \mathbb{R}$, such that $rx + y$ has exactly two distinct roots $z_1, z_2 \in \mathbb{R}$. Moreover, if the sign of $x(y_1)$ and that of the leading coefficient of x are the same (resp. opposite), then $x(z_1)x(z_2) > 0$ (resp. $x(z_1)x(z_2) < 0$).*

Proof. We assume that $\lim_{X\to\pm\infty} x = +\infty$ and $\lim_{X\to\pm\infty} y = \pm\infty$. If the leading coefficients l.c.(x) and l.c.(y) have opposite signs or they are both negative, the proof can be easily adapted.

Up to a suitable translation we can assume $y(0) = 0$, $x_1 < 0$ and $x_2 > 0$. Let k be the (odd) multiplicity of 0 as root of y. Let $I_0 = (-\varepsilon, \varepsilon)$ be a sufficiently small neighborhood of 0 such that $x < 0$ in I_0 and, in $(0, \varepsilon]$, $y^{(i)} > 0$ for $1 \leq i \leq k-1$, $y^{(k)}$ does not change sign and $x^{(j)}$ is either zero or does not change sign for $1 \leq j \leq k$. The case $x > 0$ in I_0 can be similarly treated in the neighborhood $[-\varepsilon, 0)$. Under the above assumptions, by Lemma 3.5 there exists a real number $r_0 > 0$ such that, for any assigned $r \in (0, r_0]$, $rx + y$ has a unique root in $[\,0,\ \varepsilon\,]$. Let us observe that in $[-\varepsilon,\ 0\,]$ $rx + y < 0$ for every positive r.

Take $M \in \mathbb{R}^+$ such that $x, y > 0$ in the interval $[M, +\infty)$. Clearly, for every $r > 0$, $rx + y$ has no roots in $(0, M)$.

Now we choose $N \in \mathbb{R}^+$ such that $x > 0$, $y < 0$ for $X \leq -N$ and $x' > 0$ and $y' < 0$ in the interval $(-\infty, -N)$. Under these assumptions, there exists a real number $r_1 > 0$ such that, for any assigned $r \in (0, r_1]$, $rx' + y' < 0$ and $(rx + y)(-N) < 0$. Therefore, $rx + y$ has a unique root in $(-\infty, -N]$ for every $r \in (0, r_1]$.

Let us consider the closed intervals $I_1 = [\,-N,\ -\varepsilon\,]$ and $I_2 = [\,\varepsilon,\ M\,]$. Let $M_x = \max_{I_1 \cup I_2} |x| > 0$ and $m_y = \min_{I_1 \cup I_2} |y| > 0$ (y has no roots other than 0).

By choosing $0 < r < m_y/M_x$ the polynomial $rx + y$ has no zeroes in $I_1 \cup I_2$.

We can conclude by choosing any $0 < r < \min\{r_0, r_1, m_y/M_x\}$.

The last statement of the theorem follows immediately by construction. □

Theorem 3.10. *Let p and q be two weakly comaximal elements of D. If $\deg p$ is even, $\deg q$ is odd and either p has a unique root or q has a unique root u such that $p(u)$ has the same sign of the leading coefficient of p, then the matrix $\begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix}$ is a product of idempotent matrices.*

Proof. We start assuming that p has a unique root. Since p has even degree it is not restrictive to assume that $p \geq 0$. Then we conclude by Theorem 3.3.

Assume now that q has a unique root u and that $p(u)$ has the same sign of the leading coefficient of p. We distinguish two cases.

If $\deg q > \deg p$, since q has a unique root and p and q do not have common factors, we conclude by applying Th. 3.2 (ii).

If $\deg p > \deg q$ we have two possibilities. If q does not change sign on the roots of p, we are done by Theorem 3.2 (i). Otherwise, by Lemma 3.9, it is always possible to find a suitable $r \in \mathbb{R}$ such that $rp + q$ has exactly two roots z_1, z_2 such that $x(z_1)x(z_2) > 0$. Therefore, by Theorem 3.2 (ii), the matrix $\begin{pmatrix} p & rp+q \\ 0 & 0 \end{pmatrix}$, similar to $\begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix}$, is a product of idempotent matrices. □

Remark 3.11. Let $p = x/\gamma$ and $q = y/\gamma$ be elements of D such that $\max\{\deg p, \deg q\} = 0$. If p and q have a common factor $M \notin \Gamma$, whenever the degree of M is odd and $\deg \delta \geq 1 + \deg M$, the decomposition

$$\begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} M/\delta & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x\delta/M\gamma & y\delta/M\gamma \\ 0 & 0 \end{pmatrix}$$

is not a factorization in D since $\max\{\deg(x\delta/M\gamma), \deg(y\delta/M\gamma)\} \geq 1$. For this reason, we cannot generalize Theorem 3.10 to the non-comaximal case as we have done in Theorem 3.8.

However, under the additional hypothesis that $\max\{\deg p, \deg q\} < 0$, the following corollary holds.

Corollary 3.12. *Let $p = (X - z)^k \bar{x}/\gamma$ and $q = (X - z)^h \bar{y}/\gamma$, with $k, h \in \mathbb{N}^+$, $\bar{x}, \bar{y} \in \mathbb{R}[X]$, $\gamma \in \Gamma^+$, $\bar{x}(z) \neq 0$, $\bar{y}(z) \neq 0$ be two elements of D such that $\max\{\deg p, \deg q\} < 0$. If $\deg p$ is even, $\deg q$ is odd and either p has z as unique root and $\mathrm{sgn}(\bar{y}(z)) = \mathrm{sgn}(\mathrm{l.c.}(\bar{y}))$ or q has z as unique root and $\mathrm{sgn}(\bar{x}(z)) = \mathrm{sgn}(\mathrm{l.c.}(\bar{x}))$, then the matrix $\begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix}$ is a product of idempotent matrices.*

Proof. We skip the details of the proof since it is analogous to the second part of the proof of Theorem 3.8. We reach our conclusion by properly applying Theorems 3.3 and 3.10 and using (1). □

Remark 3.13. It is worth noting that the pairs $(p,q) \in D^2$ such that $\begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix}$ is a product of idempotent matrices can generate both principal and non-principal ideals of D. Thus, this characterization of the elements p and q is not related to the idempotent factorization of $\begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix}$. The same fact can be observed in [5] for the factorization into idempotent factors of matrices of the form $\begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix}$ over real quadratic integer rings.

Theorems 3.3, 3.8, 3.10 and Corollary 3.12 contribute to narrow down the class of singular dimension 2 matrices over D that might not admit an idempotent factorization. We provide an explicit example here below.

Example 3.14. The simplest example of 2×2 singular matrix over D to which the above results do not apply and for which we cannot prove or disprove the existence of an idempotent factorization is the matrix

$$\mathbf{M} = \begin{pmatrix} (X^2-1)/(1+X^2) & X/(1+X^2) \\ 0 & 0 \end{pmatrix}.$$

Nevertheless, it can bee seen without too much effort that $\mathbf{M}$ does not admit "easy" idempotent decompositions.

First of all, $\mathbf{M}$ cannot factor in $M_2(D)$ as $\mathbf{M} = \begin{pmatrix} p' & 0 \\ 0 & 0 \end{pmatrix}\mathbf{T}$, with $\mathbf{T}$ idempotent. Let $p' = x'/\eta$, $a = a'/\delta$, $b = b'/\delta$, $c = c'/\delta \in D$, $a(1-a) = bc$ and assume by contradiction that

$$\mathbf{M} = \begin{pmatrix} p' & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}. \tag{2}$$

The matrix equation (2) leads to the equalities

$$(X^2-1)/X = a'/b' = c'/(\delta - a').$$

It follows that there exist $t, s \in \mathbb{R}[X]$ such that $a' = (X^2-1)t$, $b' = Xt$, $c' = (X^2-1)s$ and $\delta - a' = Xs$. Therefore,

$$(X^2-1)t + Xs = \delta. \tag{3}$$

The assumption that $a,b,c \in D$ also implies that $\deg t + 2, \deg s + 2 \leq \deg \delta$ hence, by (3), $\deg \delta = \deg t + 2 < \deg s + 1$ and $\deg t$ is even. Moreover, being δ a monic polynomial in Γ^+, t is monic as well and, since $\delta(0) = -t(0) > 0$, there exist $t_1, t_2 \in \mathbb{R}$ roots of t such that t is negative in (t_1, t_2). Now, from the product in (2), we have that $(X^2-1)/(1+X^2) = p'a$, i.e., $x't = \eta\delta/(1+X^2) \in \Gamma^+$. But this is impossible since $t \notin \Gamma$. Analogous arguments show that $\mathbf{M}$ cannot factor in $M_2(D)$ as $\mathbf{M} = \begin{pmatrix} 0 & q' \\ 0 & 0 \end{pmatrix} \mathbf{T}$, with $\mathbf{T}$ idempotent.

Moreover, it is also easy to show that $\mathbf{M}$ cannot be written as a product of two idempotent matrices. If we assume by contradiction that this happens, by Lemma 2.2,

$$\mathbf{M} = \begin{pmatrix} 1 & q' \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix} \tag{4}$$

with $q', a, b, c \in D$ and $a(1-a) = bc$. Set $q' = y'/\eta$, $a = a'/\delta$, $b = b'/\delta$, $c = c'/\delta$. Arguing as in the previous case, the matrix equation (4) implies that there exist $t, s \in \mathbb{R}[X]$ such that

$$(X^2-1)t + Xs = \delta$$

and

$$\eta t + y's = \eta\delta/(1+X^2).$$

Evaluating the first equality in $+1$ and -1, we get $s(-1)s(1) < 0$. It follows that there exists a root of s $s_1 \in (-1,1)$ and that $t(s_1) = \delta(s_1)/(s_1^2-1) < 0$. Evaluating the second equality in s_1 we obtain the contradiction $t(s_1) = \delta(s_1)/(s_1^2+1) > 0$.

Remark 3.15. As recalled in the introduction, Salce and Zanardo conjectured in [11] that every integral domain R satisfying the property (ID_2) should be a Bézout domain. The conjecture, suggested by previous results by Laffey [9], Ruitenburg [10] and Bhaskara Rao [1], is motivated by many positive cases. Unique factorization domains, projective-free domains, local domains and PRINC domains (introduced in [11]) turn to be Bézout whenever they satisfy property (ID_2). In [3] it is proved that if every singular 2×2 matrix over R is a product of idempotent matrices, then R is a Prüfer domain such that every invertible 2×2 matrix over R is a product of elementary matrices. Also, interesting examples of Prüfer non Bézout domains not satisfying (ID_2) were provided. On the other hand, the recent paper [5] raised some doubts on the general validity of the conjecture. In fact, the authors showed that the large family of dimension 2 column-row matrices over a real quadratic integer ring $\mathcal{O}$ factorize as products of idempotent matrices, even when $\mathcal{O}$ is not a Bézout domain. The failure of property (ID_2) for the minimal Dress ring D of $\mathbb{R}(X)$ should be proved using matrices similar to that in the above example.

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