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TIME DISCRETIZATION OF A NONLOCAL PHASE-FIELD SYSTEM WITH INERTIAL TERM

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Time discretizations of phase-field systems have been studied. For example, a time discretization, an error estimate for a parabolic-parabolic phase-field system have been studied by Colli–K. [Commun. Pure Appl. Anal. 18 (2019)]. Also, a time discretization and an error estimate for a simultaneous abstract evolution equation applying parabolic-hyperbolic phase field systems and the linearized equations of coupled sound and heat flow have been studied (see K. [ESAIM Math. Model. Numer. Anal.54 (2020), Electron. J. Differential Equations 2020, Paper No. 96]). On the other hand, although existence, continuous dependence estimates, behavior of solutions to nonlocal phase-field systems with inertial terms have been studied by Grasselli–Petzeltová–Schimperna [Quart. Appl. Math. 65 (2007)], time discretizations of these systems seem to be not studied yet. In this paper we focus on employing a time discretization scheme for a nonlocal phase-field system with inertial term and establishing an error estimate for the difference between continuous and discrete solutions.

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1. Introduction

Time discretizations of phase-field systems have been studied. For example, for the classical phase-field model proposed by Caginalp (cf. [2, 4]; one may also see the monographs [1, 5, 16])

$$\begin{cases} \theta_t + \varphi_t - \Delta\theta = f & \text{in } \Omega \times (0,T), \\ \varphi_t - \Delta\varphi + \beta(\varphi) + \pi(\varphi) = \theta & \text{in } \Omega \times (0,T), \end{cases} \tag{E1}$$

Colli–K. [3] have studied a time discretization and an error estimate, where Ω is a domain in $\mathbb{R}^d$ ($d \in \mathbb{N}$), $T > 0$, $\beta : \mathbb{R} \to \mathbb{R}$ is a maximal monotone function, $\pi : \mathbb{R} \to \mathbb{R}$ is an anti-monotone function, $f : \Omega \times (0,T) \to \mathbb{R}$ is a given function. Also, for a simultaneous abstract evolution equation applying the parabolic-hyperbolic phase-field system (see e.g., [6–8, 17, 18])

$$\begin{cases} \theta_t + \varphi_t - \Delta\theta = f & \text{in } \Omega \times (0,T), \\ \varphi_{tt} + \varphi_t - \Delta\varphi + \beta(\varphi) + \pi(\varphi) = \theta & \text{in } \Omega \times (0,T), \end{cases} \tag{E2}$$

a time discretization scheme has been employed and an error estimate has been derived (see [11]). Moreover, for a simultaneous abstract evolution equation applying (E2) (in the case that $f = 0$) and the linearized equations of coupled sound and heat flow (see e.g, Matsubara–Yokota [13])

$$\begin{cases} \theta_t + (\gamma - 1)\varphi_t - \sigma\Delta\theta = 0 & \text{in } \Omega \times (0,T), \\ \varphi_{tt} - c^2\Delta\varphi - m^2\varphi = -c^2\Delta\theta & \text{in } \Omega \times (0,T), \end{cases} \tag{E3}$$

a time discretization and an error estimate have been studied, where $c > 0$, $\sigma > 0$, $m \in \mathbb{R}$, $\gamma > 1$ are constants (see [12]). On the other hand, Grasselli–Petzeltová–Schimperna [9] have derived existence, a continuous dependence estimate and behavior of solutions to the nonlocal phase-field system

$$\begin{cases} \theta_t + \varphi_t - \Delta\theta = f & \text{in } \Omega \times (0,T), \\ \varphi_{tt} + \varphi_t + a(\cdot)\varphi - J * \varphi + \beta(\varphi) + \pi(\varphi) = \theta & \text{in } \Omega \times (0,T), \end{cases} \tag{E4}$$

where $a(x) := \int_\Omega J(x-y)\,dy$ for $x \in \Omega$, $(J * \varphi)(x) := \int_\Omega J(x-y)\varphi(y)\,dy$ for $x \in \Omega$, $J : \mathbb{R}^d \to \mathbb{R}$ is a given function. However, time discretizations of (E4) seem to be not studied yet.

In this paper, for the nonlocal phase-field system with inertial term

$$\begin{cases} \theta_t + \varphi_t - \Delta\theta = f & \text{in } \Omega \times (0,T), \\ \varphi_{tt} + \varphi_t + a(\cdot)\varphi - J * \varphi + \beta(\varphi) + \pi(\varphi) = \theta & \text{in } \Omega \times (0,T), \\ \partial_\nu \theta = 0 & \text{on } \partial\Omega \times (0,T), \\ \theta(0) = \theta_0,\ \varphi(0) = \varphi_0,\ \varphi_t(0) = v_0 & \text{in } \Omega, \end{cases} \tag{P}$$

we employ the following time discretization scheme: find $(\theta_{n+1}, \varphi_{n+1})$ such that

$$\begin{cases} \frac{\theta_{n+1}-\theta_n}{h} + \frac{\varphi_{n+1}-\varphi_n}{h} - \Delta\theta_{n+1} = f_{n+1} & \text{in } \Omega, \\ z_{n+1} + v_{n+1} + a(\cdot)\varphi_n - J * \varphi_n + \beta(\varphi_{n+1}) + \pi(\varphi_{n+1}) = \theta_{n+1} & \text{in } \Omega, \\ z_{n+1} = \frac{v_{n+1}-v_n}{h},\ v_{n+1} = \frac{\varphi_{n+1}-\varphi_n}{h} & \text{in } \Omega, \\ \partial_\nu \theta_{n+1} = 0 & \text{on } \partial\Omega \end{cases} \tag{P$_n$}$$

for $n = 0, ..., N-1$, where $h = \frac{T}{N}$, $N \in \mathbb{N}$ and $f_k := \dfrac{1}{h}\displaystyle\int_{(k-1)h}^{kh} f(s)\,ds$ for $k = 1, ..., N$. Here $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ is a bounded domain with smooth boundary $\partial\Omega$, ∂_ν denotes differentiation with respect to the outward normal of $\partial\Omega$, $\theta_0 : \Omega \to \mathbb{R}$, $\varphi_0 : \Omega \to \mathbb{R}$ and $v_0 : \Omega \to \mathbb{R}$ are given functions. Moreover, in this paper we assume that

(A1) $J(-x) = J(x)$ for all $x \in \mathbb{R}^d$ and $\displaystyle\sup_{x\in\Omega} \int_\Omega |J(x-y)|\,dy < +\infty$.

(A2) $\beta : \mathbb{R} \to \mathbb{R}$ is a single-valued maximal monotone function such that there exists a proper lower semicontinuous convex function $\hat{\beta} : \mathbb{R} \to [0, +\infty)$ satisfying that $\hat{\beta}(0) = 0$ and $\beta = \partial\hat{\beta}$, where $\partial\hat{\beta}$ is the subdifferential of $\hat{\beta}$. Moreover, $\beta : \mathbb{R} \to \mathbb{R}$ is local Lipschitz continuous.

(A3) $\pi : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function.

(A4) $f \in L^2(\Omega \times (0, T))$, $\theta_0 \in H^1(\Omega)$, $\varphi_0, v_0 \in L^\infty(\Omega)$.

In the case that $\beta(r) = ar^3$, $\widehat{\beta}(r) = \frac{a}{4}r^4$, $\pi(r) = br + c$ for $r \in \mathbb{R}$, where $a > 0$, $b, c \in \mathbb{R}$ are some constants, the conditions (A2) and (A3) hold.

Remark 1.1. We see from (A2), (A4) and the definition of the subdifferential that

$$0 \le \widehat{\beta}(\varphi_0) \le \beta(\varphi_0)\varphi_0 \in L^\infty(\Omega).$$

Let us define the Hilbert spaces

$$H := L^2(\Omega), \quad V := H^1(\Omega)$$

with inner products

$$(u_1, u_2)_H := \int_\Omega u_1 u_2\,dx \quad (u_1, u_2 \in H),$$
$$(v_1, v_2)_V := \int_\Omega \nabla v_1 \cdot \nabla v_2\,dx + \int_\Omega v_1 v_2\,dx \quad (v_1, v_2 \in V),$$

respectively, and with the related Hilbertian norms. Moreover, we use the notation

$$W := \{z \in H^2(\Omega) \mid \partial_\nu z = 0 \quad \text{a.e. on } \partial\Omega\}.$$

We define solutions of (P) as follows.

Definition 1.2. A pair (θ, φ) with

$$\theta \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W),$$
$$\varphi \in W^{2,\infty}(0,T;H) \cap W^{2,2}(0,T;L^\infty(\Omega)) \cap W^{1,\infty}(0,T;L^\infty(\Omega))$$

is called a *solution* of (P) if (θ, φ) satisfies

$$\theta_t + \varphi_t - \Delta\theta = f \quad \text{a.e. on } \Omega \times (0,T),$$
$$\varphi_{tt} + \varphi_t + a(\cdot)\varphi - J * \varphi + \beta(\varphi) + \pi(\varphi) = \theta \quad \text{a.e. on } \Omega \times (0,T),$$
$$\theta(0) = \theta_0,\ \varphi(0) = \varphi_0,\ \varphi_t(0) = v_0 \quad \text{a.e. on } \Omega.$$

The first main result asserts existence and uniqueness of solutions to $(\mathrm{P})_n$ for $n = 0, ..., N-1$.

Theorem 1.3. *Assume that* (A1)-(A4) *hold. Then there exists* $h_0 \in (0,1]$ *such that for all* $h \in (0,h_0)$ *there exists a unique solution* $(\theta_{n+1}, \varphi_{n+1})$ *of* $(\mathrm{P})_n$ *satisfying*

$$\theta_{n+1} \in W,\ \varphi_{n+1} \in L^\infty(\Omega) \quad \textit{for } n = 0, ..., N-1.$$

Here, setting

$$\hat{\theta}_h(t) := \theta_n + \frac{\theta_{n+1} - \theta_n}{h}(t - nh), \tag{1.1}$$

$$\hat{\varphi}_h(t) := \varphi_n + \frac{\varphi_{n+1} - \varphi_n}{h}(t - nh), \tag{1.2}$$

$$\hat{v}_h(t) := v_n + \frac{v_{n+1} - v_n}{h}(t - nh) \tag{1.3}$$

for $t \in [nh, (n+1)h]$, $n = 0, ..., N-1$, and

$$\overline{\theta}_h(t) := \theta_{n+1},\ \overline{\varphi}_h(t) := \varphi_{n+1},\ \underline{\varphi}_h(t) := \varphi_n, \tag{1.4}$$

$$\overline{v}_h(t) := v_{n+1},\ \overline{z}_h(t) := z_{n+1},\ \overline{f}_h(t) := f_{n+1} \tag{1.5}$$

for $t \in (nh,(n+1)h]$, $n = 0,...,N-1$, we can rewrite $(\mathrm{P})_n$ as

$$\begin{cases} (\hat{\theta}_h)_t + (\hat{\varphi}_h)_t - \Delta\overline{\theta}_h = \overline{f}_h & \text{in } \Omega\times(0,T), \\ \overline{z}_h + \overline{v}_h + a(\cdot)\underline{\varphi}_h - J * \underline{\varphi}_h + \beta(\overline{\varphi}_h) + \pi(\overline{\varphi}_h) = \overline{\theta}_h & \text{in } \Omega\times(0,T), \\ \overline{z}_h = (\hat{v}_h)_t,\ \overline{v}_h = (\hat{\varphi}_h)_t & \text{in } \Omega\times(0,T), \\ \partial_\nu \overline{\theta}_h = 0 & \text{on } \partial\Omega\times(0,T), \\ \hat{\theta}_h(0) = \theta_0,\ \hat{\varphi}_h(0) = \varphi_0,\ \hat{v}_h(0) = v_0 & \text{in } \Omega. \end{cases} \qquad (\mathrm{P})_h$$

We can prove the following theorem by passing to the limit in $(\mathrm{P})_h$ as $h \searrow 0$ (see Section 4).

Theorem 1.4. *Assume that* (A1)-(A4) *hold. Then there exists a unique solution* (θ,φ) *of* (P).

The following theorem is concerned with the error estimate between the solution of (P) and the solution of $(\mathrm{P})_h$.

Theorem 1.5. *Let h_0 be as in Theorem 1.3. Assume that* (A1)-(A4) *hold. Assume further that $f \in W^{1,1}(0,T;H)$. Then there exist constants $M > 0$ and $h_{00} \in (0,h_0)$ depending on the data such that*

$$\|\widehat{v}_h - v\|_{C([0,T];H)} + \|\overline{v}_h - v\|_{L^2(0,T;H)} + \|\widehat{\varphi}_h - \varphi\|_{C([0,T];H)} + \|\widehat{\theta}_h - \theta\|_{C([0,T];H)} + \|\nabla(\overline{\theta}_h - \theta)\|_{L^2(0,T;H)} \le Mh^{1/2}$$

for all $h \in (0,h_{00})$, where $v = \varphi_t$.

Remark 1.6. From (1.1)-(1.5) we can obtain directly the following properties:

$$\|\widehat{\theta}_h\|_{L^\infty(0,T;V)} = \max\{\|\theta_0\|_V,\ \|\overline{\theta}_h\|_{L^\infty(0,T;V)}\}, \tag{1.6}$$

$$\|\widehat{\varphi}_h\|_{L^\infty(0,T;L^\infty(\Omega))} = \max\{\|\varphi_0\|_{L^\infty(\Omega)},\ \|\overline{\varphi}_h\|_{L^\infty(0,T;L^\infty(\Omega))}\}, \tag{1.7}$$

$$\|\widehat{v}_h\|_{L^\infty(0,T;L^\infty(\Omega))} = \max\{\|v_0\|_{L^\infty(\Omega)},\ \|\overline{v}_h\|_{L^\infty(0,T;L^\infty(\Omega))}\}, \tag{1.8}$$

$$\|\overline{\theta}_h - \widehat{\theta}_h\|^2_{L^2(0,T;H)} = \frac{h^2}{3}\|(\widehat{\theta}_h)_t\|^2_{L^2(0,T;H)}, \tag{1.9}$$

$$\|\overline{\varphi}_h - \widehat{\varphi}_h\|_{L^\infty(0,T;L^\infty(\Omega))} = h\|(\widehat{\varphi}_h)_t\|_{L^\infty(0,T;L^\infty(\Omega))} = h\|\overline{v}_h\|_{L^\infty(0,T;L^\infty(\Omega))}, \tag{1.10}$$

$$\|\overline{v}_h - \widehat{v}_h\|_{L^\infty(0,T;H)} = h\|(\widehat{v}_h)_t\|_{L^\infty(0,T;H)} = h\|\overline{z}_h\|_{L^\infty(0,T;H)}, \tag{1.11}$$

$$h(\widehat{\varphi}_h)_t = \overline{\varphi}_h - \underline{\varphi}_h. \tag{1.12}$$

Remark 1.7. Unlike in the case of local parabolic-hyperbolic phase-field systems, we cannot establish the $L^p(0,T;V)$-estimate ($1 \le p \le \infty$) for $\{\widehat{\varphi}_h\}_h$ and cannot apply the Aubin–Lions lemma (see e.g., [15, Section 8, Corollary 4]) for $\{\widehat{\varphi}_h\}_h$. Thus, since $\pi : \mathbb{R} \to \mathbb{R}$ is not monotone, to obtain the strong convergence of $\{\widehat{\varphi}_h\}_h$ in $L^\infty(0,T;H)$, which is necessary to verify that $\pi(\overline{\varphi}_h) \to \pi(\varphi)$ strongly in $L^\infty(0,T;H)$ as $h = h_j \searrow 0$ by the Lipschitz continuity of π and the property (1.10), we will try to confirm Cauchy's criterion for solutions of (P)$_h$ (see Lemma 3.8).

This paper is organized as follows. In Section 2 we prove existence and uniqueness of solutions to (P)$_n$ for $n = 0,...,N-1$. In Section 3 we derive a priori estimates and Cauchy's criterion for solutions of (P)$_h$. Section 4 is devoted to the proofs of existence and uniqueness of solutions to (P) and an error estimate between the solution of (P) and the solution of (P)$_h$.

2. Existence and uniqueness for the discrete problem

In this section we will show Theorem 1.3.

Lemma 2.1. *For all $g \in H$ and all $h \in (0, \frac{1}{\|\pi'\|_{L^\infty(\mathbb{R})}})$ there exists a unique solution $\varphi \in H$ of the equation*

$$\varphi + h\varphi + h^2\beta(\varphi) + h^2\pi(\varphi) = g \quad \text{a.e. on } \Omega. \tag{2.1}$$

Proof. We set the operator $\Phi : D(\Phi) \subset H \to H$ as

$$\Phi z := h^2\beta(z) \quad \text{for } z \in D(\Phi) := \{z \in H \mid \beta(z) \in H\}.$$

Then this operator is maximal monotone. Also, we define the operator $\Psi : H \to H$ as

$$\Psi(z) := hz + h^2\pi(z) \quad \text{for } z \in H.$$

Then this operator is Lipschitz continuous, monotone for all $h \in (0, \frac{1}{\|\pi'\|_{L^\infty(\mathbb{R})}})$. Thus the operator $\Phi + \Psi : D(\Phi) \subset H \to H$ is maximal monotone (see e.g., [14, Lemma IV.2.1 (p.165)]) and then for all $g \in H$ and all $h \in (0, \frac{1}{\|\pi'\|_{L^\infty(\mathbb{R})}})$ there exists a unique solution $\varphi \in D(\Phi)$ of the equation (2.1). $\square$

Proof of Theorem 1.1. We can rewrite (P)$_n$ as

$$\begin{cases} \theta_{n+1} - h\Delta\theta_{n+1} = hf_{n+1} + \varphi_n - \varphi_{n+1} + \theta_n, \\ \varphi_{n+1} + h\varphi_{n+1} + h^2\beta(\varphi_{n+1}) + h^2\pi(\varphi_{n+1}) \\ = h^2\theta_{n+1} + \varphi_n + hv_n + h\varphi_n - h^2a(\cdot)\varphi_n + h^2J*\varphi_n. \end{cases} \tag{Q$_n$}$$

It is enough for the proof of Theorem 1.3 to establish existence and uniqueness of solutions to $(\mathrm{Q})_n$ in the case that $n = 0$. Let $h \in (0, \frac{1}{\|\pi'\|_{L^\infty(\mathbb{R})}})$. Then for all $\varphi \in H$ there exists a unique function $\overline{\theta} \in W$ such that

$$\overline{\theta} - h\Delta\overline{\theta} = hf_1 + \varphi_0 - \varphi + \theta_0. \tag{2.2}$$

Also, owing to (A1), (A4) and Lemma 2.1, for all $\theta \in H$ there exists a unique solution $\overline{\varphi} \in H$ of the equation

$$\begin{aligned} &\overline{\varphi} + h\overline{\varphi} + h^2\beta(\overline{\varphi}) + h^2\pi(\overline{\varphi}) \\ &= h^2\theta + \varphi_0 + hv_0 + h\varphi_0 - h^2a(\cdot)\varphi_0 + h^2J * \varphi_0. \end{aligned} \tag{2.3}$$

Thus we can set $\mathcal{T} : H \to H, \mathcal{U} : H \to H$ and $\mathcal{S} : H \to H$ as

$$\mathcal{T}\varphi = \overline{\theta},\ \mathcal{U}\theta = \overline{\varphi} \quad \text{for } \varphi, \theta \in H$$

and

$$\mathcal{S} = \mathcal{U} \circ \mathcal{T},$$

respectively. Now we let $\varphi, \widetilde{\varphi} \in H$. Then it follows from (2.2) that

$$\begin{aligned} \|\mathcal{T}\varphi - \mathcal{T}\widetilde{\varphi}\|_H^2 + h\|\nabla(\mathcal{T}\varphi - \mathcal{T}\widetilde{\varphi})\|_H^2 &= -(\varphi - \widetilde{\varphi}, \mathcal{T}\varphi - \mathcal{T}\widetilde{\varphi})_H \\ &\le \|\varphi - \widetilde{\varphi}\|_H \|\mathcal{T}\varphi - \mathcal{T}\widetilde{\varphi}\|_H \end{aligned}$$

and then it holds that

$$\|\mathcal{T}\varphi - \mathcal{T}\widetilde{\varphi}\|_H \le \|\varphi - \widetilde{\varphi}\|_H. \tag{2.4}$$

Also we use (2.3) and (A3) to have that

$$\begin{aligned} &(1+h)\|\mathcal{S}\varphi - \mathcal{S}\widetilde{\varphi}\|_H^2 + h^2(\beta(\mathcal{S}\varphi) - \beta(\mathcal{S}\widetilde{\varphi}), \mathcal{S}\varphi - \mathcal{S}\widetilde{\varphi})_H \\ &= h^2(\mathcal{T}\varphi - \mathcal{T}\widetilde{\varphi}, \mathcal{S}\varphi - \mathcal{S}\widetilde{\varphi})_H - h^2(\pi(\mathcal{S}\varphi) - \pi(\mathcal{S}\widetilde{\varphi}), \mathcal{S}\varphi - \mathcal{S}\widetilde{\varphi})_H \\ &\le h^2\|\mathcal{T}\varphi - \mathcal{T}\widetilde{\varphi}\|_H \|\mathcal{S}\varphi - \mathcal{S}\widetilde{\varphi}\|_H + \|\pi'\|_{L^\infty(\mathbb{R})} h^2 \|\mathcal{S}\varphi - \mathcal{S}\widetilde{\varphi}\|_H^2, \end{aligned}$$

whence the monotonicity of β leads to the inequality

$$\|\mathcal{S}\varphi - \mathcal{S}\widetilde{\varphi}\|_H \le \frac{h^2}{1 + h - \|\pi'\|_{L^\infty(\mathbb{R})} h^2} \|\mathcal{T}\varphi - \mathcal{T}\widetilde{\varphi}\|_H. \tag{2.5}$$

Therefore combining (2.4) and (2.5) implies that

$$\|\mathcal{S}\varphi - \mathcal{S}\widetilde{\varphi}\|_H \le \frac{h^2}{1 + h - \|\pi'\|_{L^\infty(\mathbb{R})} h^2} \|\varphi - \widetilde{\varphi}\|_H,$$

and hence there exists $h_{00} \in (0, \min\{1, \frac{1}{\|\pi'\|_{L^\infty(\mathbb{R})}}\})$ such that

$$0 < \frac{h^2}{1 + h - \|\pi'\|_{L^\infty(\mathbb{R})} h^2} < 1$$

for all $h \in (0, h_{00})$. Thus $\mathcal{S} : H \to H$ is a contraction mapping in H for all $h \in (0, h_{00})$ and then the Banach fixed-point theorem means that for all $h \in (0, h_{00})$ there exists a unique function $\varphi_1 \in H$ such that $\varphi_1 = \mathcal{S}\varphi_1$. Hence, for all $h \in (0, h_{00})$, by putting $\theta_1 := \mathcal{T}\varphi_1 \in W$, there exists a unique pair $(\theta_1, \varphi_1) \in H \times H$ satisfying $(\mathrm{Q})_n$ in the case that $n = 0$. Next we verify that $\varphi_1 \in L^\infty(\Omega)$. Let $h \in (0, h_{00})$. Then, noting that $g_1 := h^2\theta_1 + \varphi_0 + hv_0 + h\varphi_0 - h^2 a(\cdot)\varphi_0 + h^2 J * \varphi_0 \in L^\infty(\Omega)$ by $\theta_1 \in W$, $W \subset L^\infty(\Omega)$, (A4) and (A1), we can obtain that

$$\begin{aligned}
&|\varphi_1(x)|^2 + h|\varphi_1(x)|^2 + h^2\beta(\varphi_1(x))\varphi_1(x) \\
&= g_1(x)\varphi_1(x) - h^2(\pi(\varphi_1(x)) - \pi(0))\varphi_1(x) - h^2\pi(0)\varphi_1(x) \\
&\le \frac{1}{2}\|g_1\|^2_{L^\infty(\Omega)} + \frac{1}{2}|\varphi_1(x)|^2 + h^2\|\pi'\|_{L^\infty(\mathbb{R})}|\varphi_1(x)|^2 + \frac{1}{2}h^2|\varphi_1(x)|^2 + \frac{1}{2}h^2|\pi(0)|^2
\end{aligned}$$

by multiplying the second equation in $(\mathrm{Q})_0$ by $\varphi_1(x)$ and by using the Young inequality and (A3). Therefore, by the monotonicity of β, there exists $h_0 \in (0, h_{00})$ such that for all $h \in (0, h_0)$ there exists a constant $C_1 = C_1(h) > 0$ such that $|\varphi_1(x)| \le C_1$ for a.a. $x \in \Omega$. □

3. Uniform estimates and Cauchy's criterion

In this section we will derive a priori estimates and Cauchy's criterion for solutions of $(\mathrm{P})_h$.

Lemma 3.1. *Let h_0 be as in Theorem 1.3. Then there exist constants $C > 0$ and $h_1 \in (0, h_0)$ depending on the data such that*

$$\|\overline{v}_h\|^2_{L^\infty(0,T;H)} + \|\overline{\varphi}_h\|^2_{L^\infty(0,T;H)} + \|(\widehat{\theta}_h)_t\|^2_{L^2(0,T;H)} + \|\overline{\theta}_h\|^2_{L^\infty(0,T;V)} \le C$$

for all $h \in (0, h_1)$.

Proof. We test the first equation in $(\mathrm{P})_n$ by $\theta_{n+1} - \theta_n$, integrate over Ω and use the identities $a(a-b) = \frac{1}{2}a^2 - \frac{1}{2}b^2 + \frac{1}{2}(a-b)^2$ $(a, b \in \mathbb{R})$ and $v_{n+1} = \frac{\varphi_{n+1} - \varphi_n}{h}$, the Young inequality to infer that

$$\begin{aligned}
&h\left\|\frac{\theta_{n+1} - \theta_n}{h}\right\|_H^2 + \frac{1}{2}\|\theta_{n+1}\|_V^2 - \frac{1}{2}\|\theta_n\|_V^2 + \frac{1}{2}\|\theta_{n+1} - \theta_n\|_V^2 \\
&= h\Big(f_{n+1}, \frac{\theta_{n+1} - \theta_n}{h}\Big)_H - h\Big(v_{n+1}, \frac{\theta_{n+1} - \theta_n}{h}\Big)_H + h\Big(\theta_{n+1}, \frac{\theta_{n+1} - \theta_n}{h}\Big)_H \\
&\le \frac{3}{2}h\|f_{n+1}\|_H^2 + \frac{3}{2}h\|v_{n+1}\|_H^2 + \frac{3}{2}h\|\theta_{n+1}\|_V^2 + \frac{1}{2}h\left\|\frac{\theta_{n+1} - \theta_n}{h}\right\|_H^2. \qquad (3.1)
\end{aligned}$$

Multiplying the second equation in $(\mathrm{P})_n$ by hv_{n+1}, integrating over Ω and applying the identity $a(a-b)=\frac{1}{2}a^2-\frac{1}{2}b^2+\frac{1}{2}(a-b)^2$ ($a,b\in\mathbb{R}$), we see from (A1), (A3) and the Young inequality that there exists a constant $C_1>0$ such that

$$\begin{aligned}
&\frac{1}{2}\|v_{n+1}\|_H^2-\frac{1}{2}\|v_n\|_H^2+\frac{1}{2}\|v_{n+1}-v_n\|_H^2+h\|v_{n+1}\|_H^2+(\beta(\varphi_{n+1}),\varphi_{n+1}-\varphi_n)_H\\
&=h(\theta_{n+1},v_{n+1})_H-h(a(\cdot)\varphi_n-J*\varphi_n,v_{n+1})_H-h(\pi(\varphi_{n+1})-\pi(0),v_{n+1})_H\\
&\quad-h(\pi(0),v_{n+1})_H\\
&\le\frac{1}{2}h\|\theta_{n+1}\|_V^2+C_1h\|\varphi_n\|_H^2+\frac{\|\pi'\|_{L^\infty(\mathbb{R})}^2}{2}h\|\varphi_{n+1}\|_H^2\\
&\quad+2h\|v_{n+1}\|_H^2+\frac{\|\pi(0)\|_H^2}{2}h \qquad (3.2)
\end{aligned}$$

for all $h\in(0,h_0)$. Here it follows from (A2) and the definition of the subdifferential that

$$(\beta(\varphi_{n+1}),\varphi_{n+1}-\varphi_n)_H\ge\int_\Omega\widehat{\beta}(\varphi_{n+1})-\int_\Omega\widehat{\beta}(\varphi_n) \qquad (3.3)$$

and we have from the identities $a(a-b)=\frac{1}{2}a^2-\frac{1}{2}b^2+\frac{1}{2}(a-b)^2$ ($a,b\in\mathbb{R}$) and $hv_{n+1}=\varphi_{n+1}-\varphi_n$, the Young inequality that

$$\begin{aligned}
&\frac{1}{2}\|\varphi_{n+1}\|_H^2-\frac{1}{2}\|\varphi_n\|_H^2+\frac{1}{2}\|\varphi_{n+1}-\varphi_n\|_H^2\\
&=(\varphi_{n+1},\varphi_{n+1}-\varphi_n)_H=h(\varphi_{n+1},v_{n+1})_H\le\frac{1}{2}h\|\varphi_{n+1}\|_H^2+\frac{1}{2}h\|v_{n+1}\|_H^2. \qquad (3.4)
\end{aligned}$$

Hence, owing to (3.1)-(3.4) and summing over $n=0,...,m-1$ with $1\le m\le N$, it holds that

$$\begin{aligned}
&\frac{1}{2}h\sum_{n=0}^{m-1}\left\|\frac{\theta_{n+1}-\theta_n}{h}\right\|_H^2+\frac{1}{2}\|\theta_m\|_V^2+\frac{1}{2}h^2\sum_{n=0}^{m-1}\left\|\frac{\theta_{n+1}-\theta_n}{h}\right\|_V^2+\frac{1}{2}\|v_m\|_H^2\\
&+\frac{1}{2}h^2\sum_{n=0}^{m-1}\|z_{n+1}\|_H^2+h\sum_{n=0}^{m-1}\|v_{n+1}\|_H^2+\int_\Omega\widehat{\beta}(\varphi_m)+\frac{1}{2}\|\varphi_m\|_H^2+\frac{1}{2}h^2\sum_{n=0}^{m-1}\|v_{n+1}\|_H^2\\
&\le\frac{1}{2}\|\theta_0\|_V^2+\frac{1}{2}\|v_0\|_H^2+\int_\Omega\widehat{\beta}(\varphi_0)+\frac{1}{2}\|\varphi_0\|_H^2+\frac{3}{2}h\sum_{n=0}^{m-1}\|f_{n+1}\|_H^2\\
&\quad+4h\sum_{n=0}^{m-1}\|v_{n+1}\|_H^2+2h\sum_{n=0}^{m-1}\|\theta_{n+1}\|_V^2\\
&\quad+C_1h\sum_{n=0}^{m-1}\|\varphi_n\|_H^2+\frac{\|\pi'\|_{L^\infty(\mathbb{R})}^2+1}{2}h\sum_{n=0}^{m-1}\|\varphi_{n+1}\|_H^2+\frac{\|\pi(0)\|_H^2}{2}T
\end{aligned}$$

for all $h \in (0, h_0)$ and $m = 1, \ldots, N$. Then the inequality

$$\begin{aligned}
&\frac{1}{2}h\sum_{n=0}^{m-1}\left\|\frac{\theta_{n+1}-\theta_n}{h}\right\|_H^2 + \frac{1-4h}{2}\|\theta_m\|_V^2 + \frac{1}{2}h^2\sum_{n=0}^{m-1}\left\|\frac{\theta_{n+1}-\theta_n}{h}\right\|_V^2 \\
&+ \frac{1-8h}{2}\|v_m\|_H^2 + \frac{1}{2}h^2\sum_{n=0}^{m-1}\|z_{n+1}\|_H^2 + h\sum_{n=0}^{m-1}\|v_{n+1}\|_H^2 + \int_\Omega \widehat{\beta}(\varphi_m) \\
&+ \frac{1-(\|\pi'\|_{L^\infty(\mathbb{R})}^2+1)h}{2}\|\varphi_m\|_H^2 + \frac{1}{2}h^2\sum_{n=0}^{m-1}\|v_{n+1}\|_H^2 \\
&\le \frac{1}{2}\|\theta_0\|_V^2 + \frac{1}{2}\|v_0\|_H^2 + \int_\Omega \widehat{\beta}(\varphi_0) + \frac{1}{2}\|\varphi_0\|_H^2 + \frac{3}{2}h\sum_{n=0}^{m-1}\|f_{n+1}\|_H^2 \\
&\quad + 4h\sum_{j=0}^{m-1}\|v_j\|_H^2 + 2h\sum_{j=0}^{m-1}\|\theta_j\|_V^2 + \frac{2C_1+\|\pi'\|_{L^\infty(\mathbb{R})}^2+1}{2}h\sum_{j=0}^{m-1}\|\varphi_j\|_H^2 \\
&\quad + \frac{\|\pi(0)\|_H^2}{2}T
\end{aligned}$$

holds for all $h \in (0, h_0)$ and $m = 1, \ldots, N$. Thus there exist constants $C_2 > 0$ and $h_1 \in (0, h_0)$ such that

$$\begin{aligned}
&h\sum_{n=0}^{m-1}\left\|\frac{\theta_{n+1}-\theta_n}{h}\right\|_H^2 + \|\theta_m\|_V^2 + h^2\sum_{n=0}^{m-1}\left\|\frac{\theta_{n+1}-\theta_n}{h}\right\|_V^2 \\
&+ \|v_m\|_H^2 + h^2\sum_{n=0}^{m-1}\|z_{n+1}\|_H^2 + h\sum_{n=0}^{m-1}\|v_{n+1}\|_H^2 + \int_\Omega \widehat{\beta}(\varphi_m) \\
&+ \|\varphi_m\|_H^2 + h^2\sum_{n=0}^{m-1}\|v_{n+1}\|_H^2 \\
&\le C_2 + C_2h\sum_{j=0}^{m-1}\|v_j\|_H^2 + C_2h\sum_{j=0}^{m-1}\|\theta_j\|_V^2 + C_2h\sum_{j=0}^{m-1}\|\varphi_j\|_H^2
\end{aligned}$$

for all $h \in (0, h_1)$ and $m = 1, \ldots, N$. Therefore, thanks to the discrete Gronwall lemma (see e.g., [10, Prop. 2.2.1]), we can obtain that there exists a constant $C_3 > 0$ such that

$$\begin{aligned}
&h\sum_{n=0}^{m-1}\left\|\frac{\theta_{n+1}-\theta_n}{h}\right\|_H^2 + \|\theta_m\|_V^2 + h^2\sum_{n=0}^{m-1}\left\|\frac{\theta_{n+1}-\theta_n}{h}\right\|_V^2 \\
&+ \|v_m\|_H^2 + h^2\sum_{n=0}^{m-1}\|z_{n+1}\|_H^2 + h\sum_{n=0}^{m-1}\|v_{n+1}\|_H^2 + \int_\Omega \widehat{\beta}(\varphi_m) \\
&+ \|\varphi_m\|_H^2 + h^2\sum_{n=0}^{m-1}\|v_{n+1}\|_H^2 \le C_3
\end{aligned}$$

for all $h \in (0, h_1)$ and $m = 1, ..., N$. □

Lemma 3.2. *Let h_1 be as in Lemma 3.1. Then there exists a constant $C > 0$ depending on the data such that*

$$\|\overline{\theta}_h\|_{L^2(0,T;W)} \leq C$$

for all $h \in (0, h_1)$.

Proof. We have from the first equation in $(\mathrm{P})_h$ and Lemma 3.1 that there exists a constant $C_1 > 0$ such that

$$\| -\Delta\overline{\theta}_h\|_{L^2(0,T;H)} \leq C_1 \tag{3.5}$$

for all $h \in (0, h_1)$. Thus we can prove Lemma 3.2 by Lemma 3.1, (3.5) and the elliptic regularity theory. □

Lemma 3.3. *Let h_1 be as in Lemma 3.1. Then there exist constants $C > 0$ and $h_2 \in (0, h_1)$ depending on the data such that*

$$\|\overline{v}_h\|^2_{L^\infty(\Omega\times(0,T))} + \|\overline{\varphi}_h\|^2_{L^\infty(\Omega\times(0,T))} \leq C$$

for all $h \in (0, h_2)$.

Proof. We derive from the identities $a(a-b) = \frac{1}{2}a^2 - \frac{1}{2}b^2 + \frac{1}{2}(a-b)^2$ $(a, b \in \mathbb{R})$ and $hv_{n+1} = \varphi_{n+1} - \varphi_n$, the Young inequality that

$$\begin{aligned}
&\frac{1}{2}|\varphi_{n+1}(x)|^2 - \frac{1}{2}|\varphi_n(x)|^2 + \frac{1}{2}|\varphi_{n+1}(x) - \varphi_n(x)|^2 \\
&= \varphi_{n+1}(x)(\varphi_{n+1}(x) - \varphi_n(x)) \\
&= h\varphi_{n+1}(x)v_{n+1}(x) \\
&\leq \frac{1}{2}h\|\varphi_{n+1}\|^2_{L^\infty(\Omega)} + \frac{1}{2}h\|v_{n+1}\|^2_{L^\infty(\Omega)}.
\end{aligned} \tag{3.6}$$

Testing the second equation in $(\mathrm{P})_h$ by $hv_{n+1}(x)$ and using (A1), (A3), the Young inequality mean that there exists a constant $C_1 > 0$ such that

$$\begin{aligned}
&\frac{1}{2}|v_{n+1}(x)|^2 - \frac{1}{2}|v_n(x)|^2 + \frac{1}{2}|v_{n+1}(x) - v_n(x)|^2 \\
&\quad + \beta(\varphi_{n+1}(x))(\varphi_{n+1}(x) - \varphi_n(x)) \\
&= h\big(\theta_{n+1}(x) - a(x)\varphi_n(x) + (J * \varphi_n)(x) + \pi(0) - \pi(\varphi_{n+1}(x)) - \pi(0)\big)v_{n+1}(x) \\
&\le \frac{1}{2}h\|\theta_{n+1}\|^2_{L^\infty(\Omega)} + \frac{1}{2}h\| - a(\cdot)\varphi_n + J * \varphi_n\|^2_{L^\infty(\Omega)} \\
&\quad + \frac{\|\pi'\|^2_{L^\infty(\mathbb{R})}}{2}h\|\varphi_{n+1}\|^2_{L^\infty(\Omega)} + \frac{|\pi(0)|^2}{2}h + 2h\|v_{n+1}\|^2_{L^\infty(\Omega)} \\
&\le \frac{1}{2}h\|\theta_{n+1}\|^2_{L^\infty(\Omega)} + C_1 h\|\varphi_n\|^2_{L^\infty(\Omega)} \\
&\quad + \frac{\|\pi'\|^2_{L^\infty(\mathbb{R})}}{2}h\|\varphi_{n+1}\|^2_{L^\infty(\Omega)} + \frac{|\pi(0)|^2}{2}h + 2h\|v_{n+1}\|^2_{L^\infty(\Omega)} \qquad (3.7)
\end{aligned}$$

for all $h \in (0, h_1)$ and a.a. $x \in \Omega$. Here the condition (A2) and the definition of the subdifferential lead to the inequality

$$\beta(\varphi_{n+1}(x))(\varphi_{n+1}(x) - \varphi_n(x)) \ge \widehat{\beta}(\varphi_{n+1}(x)) - \widehat{\beta}(\varphi_n(x)). \qquad (3.8)$$

Thus it follows from (3.6)-(3.8), summing over $n = 0, ..., m-1$ with $1 \le m \le N$ and Remark 1.1 that

$$\begin{aligned}
&\frac{1}{2}|\varphi_m(x)|^2 + \frac{1}{2}|v_m(x)|^2 + \widehat{\beta}(\varphi_m(x)) \\
&\le \frac{1}{2}\|\varphi_0\|^2_{L^\infty(\Omega)} + \frac{1}{2}\|v_0\|^2_{L^\infty(\Omega)} + \|\widehat{\beta}(\varphi_0)\|_{L^\infty(\Omega)} \\
&\quad + \frac{1}{2}h\sum_{n=0}^{m-1}\|\theta_{n+1}\|^2_{L^\infty(\Omega)} + C_1 h\sum_{n=0}^{m-1}\|\varphi_n\|^2_{L^\infty(\Omega)} \\
&\quad + \frac{\|\pi'\|^2_{L^\infty(\mathbb{R})} + 1}{2}h\sum_{n=0}^{m-1}\|\varphi_{n+1}\|^2_{L^\infty(\Omega)} + \frac{5}{2}h\sum_{n=0}^{m-1}\|v_{n+1}\|^2_{L^\infty(\Omega)} + \frac{|\pi(0)|^2}{2}T
\end{aligned}$$

for all $h \in (0, h_1)$, $m = 1, ..., N$ and a.a. $x \in \Omega$, which implies that

$$\begin{aligned}
&\frac{1}{2}\|\varphi_m\|^2_{L^\infty(\Omega)} + \frac{1}{2}\|v_m\|^2_{L^\infty(\Omega)} \\
&\le \frac{1}{2}\|\varphi_0\|^2_{L^\infty(\Omega)} + \frac{1}{2}\|v_0\|^2_{L^\infty(\Omega)} + \|\widehat{\beta}(\varphi_0)\|_{L^\infty(\Omega)} \\
&\quad + \frac{1}{2}h\sum_{n=0}^{m-1}\|\theta_{n+1}\|^2_{L^\infty(\Omega)} + C_1 h\sum_{n=0}^{m-1}\|\varphi_n\|^2_{L^\infty(\Omega)} \\
&\quad + \frac{\|\pi'\|^2_{L^\infty(\mathbb{R})} + 1}{2}h\sum_{n=0}^{m-1}\|\varphi_{n+1}\|^2_{L^\infty(\Omega)} + \frac{5}{2}h\sum_{n=0}^{m-1}\|v_{n+1}\|^2_{L^\infty(\Omega)} + \frac{|\pi(0)|^2}{2}T
\end{aligned}$$

for all $h \in (0,h_1)$ and $m = 1,\dots,N$. Then the inequality

$$\begin{aligned}
&\frac{1-(\|\pi'\|^2_{L^\infty(\mathbb{R})}+1)h}{2}\|\varphi_m\|^2_{L^\infty(\Omega)} + \frac{1-5h}{2}\|v_m\|^2_{L^\infty(\Omega)} \\
&\le \frac{1}{2}\|\varphi_0\|^2_{L^\infty(\Omega)} + \frac{1}{2}\|v_0\|^2_{L^\infty(\Omega)} + \|\widehat{\beta}(\varphi_0)\|_{L^\infty(\Omega)} \\
&\quad + \frac{1}{2}h\sum_{n=0}^{m-1}\|\theta_{n+1}\|^2_{L^\infty(\Omega)} + \frac{2C_1+\|\pi'\|^2_{L^\infty(\mathbb{R})}+1}{2}h\sum_{j=0}^{m-1}\|\varphi_j\|^2_{L^\infty(\Omega)} \\
&\quad + \frac{5}{2}h\sum_{j=0}^{m-1}\|v_j\|^2_{L^\infty(\Omega)} + \frac{|\pi(0)|^2}{2}T
\end{aligned} \tag{3.9}$$

holds for all $h \in (0,h_1)$ and $m = 1,\dots,N$. Here we see from the continuity of the embedding $W \hookrightarrow L^\infty(\Omega)$ and Lemma 3.2 that there exist constants $C_2, C_3 > 0$ such that

$$h\sum_{n=0}^{N-1}\|\theta_{n+1}\|^2_{L^\infty(\Omega)} = \|\overline{\theta}_h\|^2_{L^2(0,T;L^\infty(\Omega))} \le C_2\|\overline{\theta}_h\|^2_{L^2(0,T;W)} \le C_3 \tag{3.10}$$

for all $h \in (0,h_1)$. Therefore we have from (3.9) and (3.10) that there exist constants $C_4 > 0$ and $h_2 \in (0,h_1)$ such that

$$\begin{aligned}
&\|\varphi_m\|^2_{L^\infty(\Omega)} + \|v_m\|^2_{L^\infty(\Omega)} \\
&\le C_4 + C_4 h\sum_{j=0}^{m-1}\|\varphi_j\|^2_{L^\infty(\Omega)} + C_4 h\sum_{j=0}^{m-1}\|v_j\|^2_{L^\infty(\Omega)}
\end{aligned}$$

for all $h \in (0,h_2)$ and $m = 1,\dots,N$. Then we can obtain that there exists a constant $C_5 > 0$ such that

$$\|\varphi_m\|^2_{L^\infty(\Omega)} + \|v_m\|^2_{L^\infty(\Omega)} \le C_5$$

for all $h \in (0,h_2)$ and $m = 1,\dots,N$ by the discrete Gronwall lemma (see e.g., [10, Prop. 2.2.1]). □

Lemma 3.4. *Let h_2 be as in Lemma 3.3. Then there exists a constant $C > 0$ depending on the data such that*

$$\|\underline{\varphi}_h\|^2_{L^\infty(\Omega\times(0,T))} \le C$$

for all $h \in (0,h_2)$.

Proof. We can verify this lemma by Lemma 3.3 and (A4). □

Lemma 3.5. *Let h_2 be as in Lemma 3.3. Then there exists a constant $C > 0$ depending on the data such that*

$$\|\bar{z}_h\|^2_{L^2(0,T;L^\infty(\Omega))} \le C$$

for all $h \in (0, h_2)$.

Proof. Since it follows from Lemma 3.3 and the continuity of β that there exists a constant $C_1 > 0$ such that

$$\|\beta(\overline{\varphi}_h)\|_{L^\infty(\Omega\times(0,T))} \le C_1$$

for all $h \in (0, h_2)$, we can confirm that Lemma 3.5 holds by the second equation in (P)$_h$, (A1), (A3), Lemmas 3.3, 3.4, the continuity of the embedding $W \hookrightarrow L^\infty(\Omega)$ and Lemma 3.2. □

Lemma 3.6. *Let h_2 be as in Lemma 3.3. Then there exist constants $C > 0$ and $h_3 \in (0, h_2)$ depending on the data such that*

$$\|\bar{z}_h\|^2_{L^\infty(0,T;H)} \le C$$

for all $h \in (0, h_3)$.

Proof. Since the second equation in (P)$_n$ leads to the identity

$$z_1 + hz_1 + a(\cdot)\varphi_0 - J * \varphi_0 + \beta(\varphi_1) + \pi(\varphi_1) = \theta_1,$$

it holds that

$$\begin{aligned}&\|z_1\|^2_H + h\|z_1\|^2_H \\ &= -(a(\cdot)\varphi_0 - J * \varphi_0, z_1)_H - (\beta(\varphi_1), z_1)_H - (\pi(\varphi_1), z_1)_H + (\theta_1, z_1)_H.\end{aligned}$$

Thus we deduce from the Young inequality, (A1), the continuity of β, (A3), Lemmas 3.1, 3.3 that there exists a constant $C_1 > 0$ such that

$$\|z_1\|^2_H \le C_1 \tag{3.11}$$

for all $h \in (0, h_2)$. Now we let $n \in \{1, ..., N-1\}$. Then we have from the second equation in (P)$_n$ that

$$\begin{aligned}&z_{n+1} - z_n + hz_{n+1} + ha(\cdot)v_n - hJ * v_n + \beta(\varphi_{n+1}) - \beta(\varphi_n) \\ &+ \pi(\varphi_{n+1}) - \pi(\varphi_n) = \theta_{n+1} - \theta_n.\end{aligned} \tag{3.12}$$

Moreover, we test (3.12) by z_{n+1}, integrate over Ω, recall (A1), Lemma 3.3, the local Lipschitz continuity of β, (A3), and use the Young inequality to infer that there exist constants $C_2, C_3 > 0$ such that

$$\begin{aligned}
&\frac{1}{2}\|z_{n+1}\|_H^2 - \frac{1}{2}\|z_n\|_H^2 + \frac{1}{2}\|z_{n+1} - z_n\|_H^2 + h\|z_{n+1}\|_H^2 \\
&= -h(a(\cdot)v_n - J * v_n, z_{n+1})_H - h\Big(\frac{\beta(\varphi_{n+1}) - \beta(\varphi_n)}{h}, z_{n+1}\Big)_H \\
&\quad - h\Big(\frac{\pi(\varphi_{n+1}) - \pi(\varphi_n)}{h}, z_{n+1}\Big)_H + h\Big(\frac{\theta_{n+1} - \theta_n}{h}, z_{n+1}\Big)_H \\
&\le C_2 h\|v_n\|_H\|z_{n+1}\|_H + C_2 h\|v_{n+1}\|_H\|z_{n+1}\|_H + h\Big\|\frac{\theta_{n+1} - \theta_n}{h}\Big\|_H\|z_{n+1}\|_H \\
&\le C_3 h\|z_{n+1}\|_H + h\Big\|\frac{\theta_{n+1} - \theta_n}{h}\Big\|_H\|z_{n+1}\|_H \\
&\le h\|z_{n+1}\|_H^2 + \frac{1}{2}h\Big\|\frac{\theta_{n+1} - \theta_n}{h}\Big\|_H^2 + \frac{C_3^2}{2}h
\end{aligned} \tag{3.13}$$

for all $h \in (0, h_2)$. Thus, summing (3.13) over $n = 1, ..., \ell - 1$ with $2 \le \ell \le N$, we see from (3.11) and Lemma 3.1 that there exists a constant $C_4 > 0$ such that

$$\begin{aligned}
\frac{1}{2}\|z_\ell\|_H^2 &\le \frac{1}{2}\|z_1\|_H^2 + h\sum_{n=1}^{\ell-1}\|z_{n+1}\|_H^2 + \frac{1}{2}h\sum_{n=1}^{\ell-1}\Big\|\frac{\theta_{n+1} - \theta_n}{h}\Big\|_H^2 + \frac{C_3^2}{2}T \\
&\le C_4 + h\sum_{n=1}^{\ell-1}\|z_{n+1}\|_H^2
\end{aligned}$$

for all $h \in (0, h_2)$ and $\ell = 2, ..., N$, whence we have from (3.11) that there exist constants $C_5 > 0$ and $h_3 \in (0, h_2)$ such that

$$\|z_m\|_H^2 \le C_5 + C_5 h\sum_{j=0}^{m-1}\|z_j\|_H^2$$

for all $h \in (0, h_3)$ and $m = 1, ..., N$. Therefore the discrete Gronwall lemma (see e.g., [10, Prop. 2.2.1]) implies that there exists a constant $C_6 > 0$ such that

$$\|z_m\|_H^2 \le C_6$$

for all $h \in (0, h_3)$ and $m = 1, ..., N$. □

Lemma 3.7. *Let h_3 be as in Lemma 3.6. Then there exists a constant $C > 0$ depending on the data such that*

$$\begin{aligned}
&\|\widehat{\theta}_h\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|\widehat{\varphi}_h\|_{W^{1,\infty}(0,T;L^\infty(\Omega))} \\
&+ \|\widehat{v}_h\|_{W^{1,\infty}(0,T;H)\cap W^{1,2}(0,T;L^\infty(\Omega))\cap L^\infty(0,T;L^\infty(\Omega))} \le C
\end{aligned}$$

for all $h \in (0, h_3)$.

Proof. This lemma can be proved by (1.6)-(1.8), Lemmas 3.1, 3.3, 3.5 and 3.6. □

The following lemma asserts Cauchy's criterion for solutions of $(\mathrm{P})_h$.

Lemma 3.8. *Let h_3 be as in Lemma 3.6. Then there exists a constant $C > 0$ depending on the data such that*

$$\begin{aligned}&\|\widehat{v}_h - \widehat{v}_\tau\|_{C([0,T];H)} + \|\overline{v}_h - \overline{v}_\tau\|_{L^2(0,T;H)} + \|\widehat{\varphi}_h - \widehat{\varphi}_\tau\|_{C([0,T];H)} \\ &+ \|\widehat{\theta}_h - \widehat{\theta}_\tau\|_{C([0,T];H)} + \|\nabla(\overline{\theta}_h - \overline{\theta}_\tau)\|_{L^2(0,T;H)} \\ &\le C(h^{1/2} + \tau^{1/2}) + C\|\overline{f}_h - \overline{f}_\tau\|_{L^2(0,T;H)}\end{aligned}$$

for all $h, \tau \in (0, h_3)$.

Proof. It holds that

$$\begin{aligned}&\frac{1}{2}\frac{d}{ds}\|\widehat{v}_h(s) - \widehat{v}_\tau(s)\|_H^2 = (\overline{z}_h(s) - \overline{z}_\tau(s), \widehat{v}_h(s) - \widehat{v}_\tau(s))_H \\ &= (\overline{z}_h(s) - \overline{z}_\tau(s), \widehat{v}_h(s) - \overline{v}_h(s))_H + (\overline{z}_h(s) - \overline{z}_\tau(s), \overline{v}_h(s) - \overline{v}_\tau(s))_H \\ &\quad + (\overline{z}_h(s) - \overline{z}_\tau(s), \overline{v}_\tau(s) - \widehat{v}_\tau(s))_H. \end{aligned} \tag{3.14}$$

Here we derive from the second equation in $(\mathrm{P})_h$ that

$$\begin{aligned}&(\overline{z}_h(s) - \overline{z}_\tau(s), \overline{v}_h(s) - \overline{v}_\tau(s))_H \\ &= -\|\overline{v}_h(s) - \overline{v}_\tau(s)\|_H^2 \\ &\quad + \big(-a(\cdot)(\underline{\varphi}_h(s) - \underline{\varphi}_\tau(s)) + J * (\underline{\varphi}_h(s) - \underline{\varphi}_\tau(s)), \overline{v}_h(s) - \overline{v}_\tau(s)\big)_H \\ &\quad - \big(\beta(\overline{\varphi}_h(s)) - \beta(\overline{\varphi}_\tau(s)), \overline{v}_h(s) - \overline{v}_\tau(s)\big)_H \\ &\quad - \big(\pi(\overline{\varphi}_h(s)) - \pi(\overline{\varphi}_\tau(s)), \overline{v}_h(s) - \overline{v}_\tau(s)\big)_H \\ &\quad + (\overline{\theta}_h(s) - \overline{\theta}_\tau(s), \overline{v}_h(s) - \overline{v}_\tau(s))_H. \end{aligned} \tag{3.15}$$

The property (1.12) means that

$$\begin{aligned}&\|\underline{\varphi}_h(s) - \underline{\varphi}_\tau(s)\|_H^2 \\ &= \|-h(\widehat{\varphi}_h)_t(s) + \tau(\widehat{\varphi}_\tau)_t(s) + \overline{\varphi}_h(s) - \overline{\varphi}_\tau(s)\|_H^2 \\ &\le 3h^2\|(\widehat{\varphi}_h)_t(s)\|_H^2 + 3\tau^2\|(\widehat{\varphi}_\tau)_t(s)\|_H^2 + 3\|\overline{\varphi}_h(s) - \overline{\varphi}_\tau(s)\|_H^2. \end{aligned} \tag{3.16}$$

We can obtain that

$$\begin{aligned}&\|\overline{\varphi}_h(s) - \overline{\varphi}_\tau(s)\|_H^2 \\ &= \|\overline{\varphi}_h(s) - \widehat{\varphi}_h(s) + \widehat{\varphi}_h(s) - \widehat{\varphi}_\tau(s) + \widehat{\varphi}_\tau(s) - \overline{\varphi}_\tau(s)\|_H^2 \\ &\le 3\|\overline{\varphi}_h(s) - \widehat{\varphi}_h(s)\|_H^2 + 3\|\widehat{\varphi}_h(s) - \widehat{\varphi}_\tau(s)\|_H^2 + 3\|\widehat{\varphi}_\tau(s) - \overline{\varphi}_\tau(s)\|_H^2 \end{aligned} \tag{3.17}$$

and

$$\frac{1}{2}\frac{d}{ds}\|\widehat{\varphi}_h(s)-\widehat{\varphi}_\tau(s)\|_H^2 = (\overline{v}_h(s)-\overline{v}_\tau(s),\widehat{\varphi}_h(s)-\widehat{\varphi}_\tau(s))_H. \tag{3.18}$$

It follows from the first equation in (P)$_h$ that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{ds}\|\widehat{\theta}_h(s)-\widehat{\theta}_\tau(s)\|_H^2 \\
&= -\big(-\Delta(\overline{\theta}_h(s)-\overline{\theta}_\tau(s)),\widehat{\theta}_h(s)-\widehat{\theta}_\tau(s)\big)_H - (\overline{v}_h(s)-\overline{v}_\tau(s),\widehat{\theta}_h(s)-\widehat{\theta}_\tau(s))_H \\
&\quad + (\overline{f}_h(s)-\overline{f}_\tau(s),\widehat{\theta}_h(s)-\widehat{\theta}_\tau(s))_H \\
&= -\|\nabla(\overline{\theta}_h(s)-\overline{\theta}_\tau(s))\|_H^2 - \big(-\Delta(\overline{\theta}_h(s)-\overline{\theta}_\tau(s)),\widehat{\theta}_h(s)-\overline{\theta}_h(s)\big)_H \\
&\quad - \big(-\Delta(\overline{\theta}_h(s)-\overline{\theta}_\tau(s)),\overline{\theta}_\tau(s)-\widehat{\theta}_\tau(s)\big)_H - (\overline{\theta}_h(s)-\overline{\theta}_\tau(s),\overline{v}_h(s)-\overline{v}_\tau(s))_H \\
&\quad - (\overline{v}_h(s)-\overline{v}_\tau(s),\widehat{\theta}_h(s)-\overline{\theta}_h(s))_H - (\overline{v}_h(s)-\overline{v}_\tau(s),\overline{\theta}_\tau(s)-\widehat{\theta}_\tau(s))_H \\
&\quad + (\overline{f}_h(s)-\overline{f}_\tau(s),\widehat{\theta}_h(s)-\widehat{\theta}_\tau(s))_H.
\end{aligned} \tag{3.19}$$

Therefore we have from (3.14)-(3.19), the integration over $(0,t)$, where $t \in [0,T]$, the Schwarz inequality, the Young inequality, (A1), Lemma 3.3, the local Lipschitz continuity of β, (A3), (1.9)-(1.11), Lemmas 3.2, 3.6 and 3.7 that there exists a constant $C_1 > 0$ such that

$$\begin{aligned}
&\|\widehat{v}_h(t)-\widehat{v}_\tau(t)\|_H^2 + \int_0^t \|\overline{v}_h(s)-\overline{v}_\tau(s)\|_H^2\,ds + \|\widehat{\varphi}_h(t)-\widehat{\varphi}_\tau(t)\|_H^2 \\
&+ \|\widehat{\theta}_h(t)-\widehat{\theta}_\tau(t)\|_H^2 + \int_0^t \|\nabla(\overline{\theta}_h(s)-\overline{\theta}_\tau(s))\|_H^2\,ds \\
&\le C_1(h+\tau) + C_1\|\overline{f}_h-\overline{f}_\tau\|_{L^2(0,T;H)}^2 + C_1\int_0^t \|\widehat{\varphi}_h(s)-\widehat{\varphi}_\tau(s)\|_H^2\,ds \\
&\quad + C_1\int_0^t \|\widehat{\theta}_h(s)-\widehat{\theta}_\tau(s)\|_H^2\,ds
\end{aligned}$$

for all $h,\tau \in (0,h_4)$. Then we can prove Lemma 3.8 by the Gronwall lemma. □

4. Existence and uniqueness for (P) and an error estimate

In this section we verify Theorems 1.4 and 1.5.

Proof of Theorem 1.2. Since $\overline{f}_h$ converges to f strongly in $L^2(0,T;H)$ as $h \searrow 0$ (see [3, Section 5]), we see from Lemmas 3.1-3.8, (1.9)-(1.12) that there exist some functions

$$\begin{aligned}
&\theta \in H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W), \\
&\varphi \in W^{2,\infty}(0,T;H)\cap W^{2,2}(0,T;L^\infty(\Omega))\cap W^{1,\infty}(0,T;L^\infty(\Omega))
\end{aligned}$$

such that

$$\widehat{\theta}_h \to \theta \quad \text{weakly}^* \text{ in } H^1(0,T;H) \cap L^\infty(0,T;V), \tag{4.1}$$

$$\overline{\theta}_h \to \theta \quad \text{weakly}^* \text{ in } L^\infty(0,T;V),$$

$$\overline{\theta}_h \to \theta \quad \text{weakly in } L^2(0,T;W), \tag{4.2}$$

$$\widehat{\theta}_h \to \theta \quad \text{strongly in } C([0,T];H), \tag{4.3}$$

$$\overline{\theta}_h \to \theta \quad \text{strongly in } L^2(0,T;V),$$

$$\widehat{\varphi}_h \to \varphi \quad \text{weakly}^* \text{ in } W^{1,\infty}(0,T;L^\infty(\Omega)), \tag{4.4}$$

$$\overline{\varphi}_h \to \varphi \quad \text{weakly}^* \text{ in } L^\infty(\Omega \times (0,T)),$$

$$\underline{\varphi}_h \to \varphi \quad \text{weakly}^* \text{ in } L^\infty(\Omega \times (0,T)),$$

$$\widehat{\varphi}_h \to \varphi \quad \text{strongly in } C([0,T];H), \tag{4.5}$$

and

$$\widehat{v}_h \to \varphi_t \quad \text{weakly}^* \text{ in } W^{1,\infty}(0,T;H) \cap W^{1,2}(0,T;L^\infty(\Omega)) \cap L^\infty(\Omega \times (0,T)),$$

$$\overline{v}_h \to \varphi_t \quad \text{weakly}^* \text{ in } L^\infty(\Omega \times (0,T)), \tag{4.6}$$

$$\widehat{v}_h \to \varphi_t \quad \text{strongly in } C([0,T];H), \tag{4.7}$$

$$\overline{v}_h \to \varphi_t \quad \text{strongly in } L^2(0,T;H),$$

$$\overline{z}_h \to \varphi_{tt} \quad \text{weakly}^* \text{ in } L^\infty(0,T;H) \cap L^2(0,T;L^\infty(\Omega)) \tag{4.8}$$

as $h = h_j \searrow 0$. Here we recall (1.10) and Lemma 3.3 to derive from (4.5) that

$$\begin{aligned} \|\overline{\varphi}_h - \varphi\|_{L^\infty(0,T;H)} &\le \|\overline{\varphi}_h - \widehat{\varphi}_h\|_{L^\infty(0,T;H)} + \|\widehat{\varphi}_h - \varphi\|_{L^\infty(0,T;H)} \\ &\le |\Omega|^{1/2} \|\overline{\varphi}_h - \widehat{\varphi}_h\|_{L^\infty(0,T;L^\infty(\Omega))} + \|\widehat{\varphi}_h - \varphi\|_{L^\infty(0,T;H)} \\ &= \frac{|\Omega|^{1/2}}{\sqrt{3}} h \|\overline{v}_h\|_{L^\infty(0,T;L^\infty(\Omega))} + \|\widehat{\varphi}_h - \varphi\|_{L^\infty(0,T;H)} \to 0 \end{aligned}$$

as $h = h_j \searrow 0$, and hence it holds that

$$\overline{\varphi}_h \to \varphi \quad \text{strongly in } L^\infty(0,T;H) \tag{4.9}$$

as $h = h_j \searrow 0$. Thus we infer from (1.12), (4.9) and Lemma 3.7 that

$$\underline{\varphi}_h \to \varphi \quad \text{strongly in } L^\infty(0,T;H) \tag{4.10}$$

as $h = h_j \searrow 0$. Therefore, owing to (4.1)-(4.10), (A1), Lemma 3.3, the local Lipschitz continuity of β, and (A3), we can establish existence of solutions to (P). Moreover, we can confirm uniqueness of solutions to (P) in a similar way to the proof of Lemma 3.8. □

Proof of Theorem 1.3. Since the inclusion $f \in L^2(0,T;H) \cap W^{1,1}(0,T;H)$ implies that there exists a constant $C_1 > 0$ such that

$$\|\overline{f}_h - f\|_{L^2(0,T;H)} \leq C_1 h^{1/2}$$

for all $h > 0$ (see [3, Section 5]), we can prove Theorem 1.5 by Lemma 3.8. $\square$

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