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A STUDY ON FUZZY DIFFERENTIAL GAME

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In this paper we study a fuzzy differential game problem, in which the information obtained by any player may contain some sort of uncertainties, which are usually difficult to characterize either determinstically or stochastically. A necessary condition for optimal strategy of an open loop Nash-equilibrium solution for fuzzy differential game is derived and an illustrative example is presented to clarify the developed result.

1. Introduction.

For a continuous differential game problems of N-Players, the dynamical system may be described by

$$(1.1')\qquad \begin{cases} \min\limits_{(u_1,\cdots,u_N)} \; J_i(u_1,\cdots,u_N) = \varphi_i(x(t_f)) + \\ \qquad + \displaystyle\int_{t_\circ}^{t_f} I_i(x(t), u_1, u_2, \cdots, u_N, t)\,dt & (1.1) \\ \text{subject to} \\ \dot{x}(t) = f(x(t), u_1, \cdots, u_N, t) & (1.2) \\ h(u_1, \cdots, u_N) \geq 0 & (1.3) \\ x(t_\circ) = x_\circ & (1.4) \end{cases}$$

where I_i is C^1 on $\mathbb{R}^n \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \cdots \times \mathbb{R}^{m_N}$, and φ_i is C^1 on $\mathbb{R}^n$ $i = 1, 2, \cdots, N$, $x(t) \in \mathbb{R}^n$ is the state vector of the system at time $t \in [t_\circ, t_f]$, f is

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C^1 on $\mathbb{R}^n \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \cdots \times \mathbb{R}^{m_N}$, J_i is the cost for each Player i, and the control variables $u_i, i = 1, 2, \cdots, N$ are constrained by $h(u_1, u_2, \cdots, u_N, t) \geq 0, x(t_\circ)$ is the initial state known by all Players, $t = t_\circ$ is the starting of the game and $t = t_f$ is the end of the game; (see [1],[2],[8]).

In many pratical problems, the information obtained by any Player may contain some sort of uncertainties, which is treated, in this paper, as fuzzy information (see [6]). The considered differential game problem with fuzzy information is a fuzzy differential game. a fuzzy differential game can arise in many life problems, for example, the problem of guarding a territory (see [4], [5]).

2. Problem Formulation.

Assuming that each player i has fuzzy goal in the continuous differential game problem, then the rigid requirements of the continuous differential game may be softened into the following fuzzy version.

$$(2.1')\qquad \begin{cases} \widetilde{\text{minimize}}\ \ J_i(u_1, \cdots, u_N) = \varphi_i(x(t_f)) + \\ \qquad + \displaystyle\int_{t_\circ}^{t_f} I_i(u_1, \cdots, u_N, x(t))\, dt & (2.1) \\ \text{subject to} \\ \dot{x}(t) = f(x(t), u_1, \cdots, u_N, t) & (2.2) \\ h(u_1, \cdots, u_N) \geq 0 & (2.3) \\ x(t_\circ) = x_\circ & (2.4) \end{cases}$$

where the symbol "$\widetilde{\text{minimize}}$" denotes fuzzy version of "minimize" i.e., the cost of each player should be minimized as much as possible under the given constraints (see [9]), such fuzzy requirements for each player can be quantified by eliciting membership function $\mu_i(J_i), i = 1, 2, \cdots, N$ from the objective function J_i, for each player $i = 1, 2, \cdots, N$

where $\mu_i(J_i)$ defined by

$$(2.5)\qquad \mu_i(J_i) = \begin{cases} 1 \quad , & J_i \leq J_i^1 \\ \dfrac{J_i - J_i^\circ}{J_i^1 - J_i^\circ} \quad , & J_i^1 \leq J_i \leq J_i^\circ \\ 0 \quad , & J_i \geq J_i^\circ \end{cases}$$

where J_i° and J_i^1 denotes the value of the objective function J_i such that the degree of membership function is 0 and 1 respectively, i.e., J_i° is undesirable

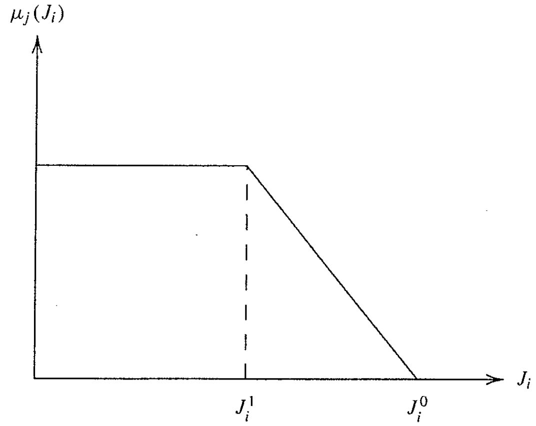

Figure 1. Linear membership function

cost and J_i^1 is desirable cost for each player $i = 1, \cdots, N$. Figure 1 illustrates the shape of the linear membership function.
The problem now is to find u^* such that $\mu_D(J_i(u^*)) = \max_u \min_{i=1,\cdots,N} \mu_i(J_i(u))$, where μ_D is the membership degree of fuzzy decision making and $\mu_i(J_i)$ is the membership degree of fuzzy goal for each player under the given constraints. By introducing the auxiliary variable λ, this problem can be transformed into the following equivalent problem.

$$(2.6') \quad \begin{cases} \underset{u}{\text{maximize}}\ \lambda & (2.6) \\ \text{subject to} & \\ \lambda \le \mu_i(J_i(u_1, u_2, \cdots, u_N)) & (2.7) \\ \dot{x}(t) = f(x(t), u_1, \cdots, u_N, t) & (2.8) \\ x(t_\circ) = x_\circ & (2.9) \end{cases}$$

3. Nash-Equilibrium Fuzzy Continuous Differential Game.

In this section we shall discuss the Nash-equilibrium solution for N-players fuzzy continuous differential game.

3.1. Definition (Nash-Equilibrium fuzzy strategy)

If $J_1(u_1, u_2, \cdots, u_N), \cdots, J_N(u_1, \cdots, u_N)$ are cost functions with membership function $\mu_1(J_1), \cdots, \mu_N(J_N)$ for players $1, 2, \cdots, N$, the control N-tuplue $(u_1^*, \cdots, u_N^*)$ is Nash-equilibrium fuzzy strategy if for $i = 1, 2, \cdots, N$

$$\mu_i(J_i(u_1^*, \cdots, u_{i-1}^*, u_i^*, u_{i+1}^*, \cdots, u_N^*)) \ge$$

$$\mu_i(J_i(u_1^*, \cdots, u_{i-1}^*, u_i, u_{i+1}^*, \cdots, u_N^*).$$

Intuitively, the Nash-equilibrium concept means that if one players tries to alter his strategy unilaterally, he can not improve his own optimization criterion.

3.2 Formulation of Nash-Equilibrium Continuous Differential Game

The Nash-Equilibrium fuzzy continuous differential game problem can be formulated as follows. Find $u^* = (u_i^*, v^*)$ that solves the problem

$$(3.2') \quad \begin{cases} \max\limits_{u_i} \lambda & (3.2.1) \\ \text{subject to} & \\ \lambda \leq \mu_i(J_i(u_i, v^*)) & (3.2.2) \\ \dot{x}(t) = f(x, u_i, v^*) & (3.2.3) \\ h(u_i(t), v^*) \geq 0. & (3.2.4) \\ x(t_\circ) = x_\circ & \end{cases}$$

where

$$\mu_i(J_i(u_i, v^*)) = \begin{cases} 1 & , \quad J_i \leq J_i^1 \\ \dfrac{J_i(u_i, v^*) - J_i^\circ}{J_i^1 - J_i^\circ} & , \quad J_i^1 \leq J_i \leq J_i^\circ \\ 0 & , \quad J_i \geq J_i^\circ, \end{cases}$$

J_i° is determined by the player i based on his strategy,

$$J_i(u_i, v^*) = \varphi_i(x(t_f)) + \int_{t_\circ}^{t_f} I_i(x(t), u_i, v^*, t)\, dt,$$

$$f : [t_\circ, t_f] \times \mathbb{R}^n \times \mathbb{R}^s \longrightarrow \mathbb{R}^n \text{ is } C^1,\, S = \sum_{j=1}^{N} S_j,\, i \neq j,$$

where S_j is the dimension of U_i, $u_i \in U_i \subset \mathbb{R}^{S_i}$.

$$I_i : [t_\circ, t_f] \times \mathbb{R}^n \times \mathbb{R}^s \longrightarrow \mathbb{R} \text{ is } C^1,\ i = 1, 2, \cdots, N$$

$$h(\cdot) : [t_\circ, t_f] \times \mathbb{R}^s \longrightarrow \mathbb{R}^q \text{ is } C^1,$$

denotes the control or decision of player i, which is taken to be picewise continuous function of time for all i, and v^* is the composite control for the remaining players, $x(t) = (x_1, x_2, \cdots, x_N) \in \mathbb{R}^n$ is the state vector of the system at time t, $t \in [t_\circ, t_f]$, J_i is the cost for each player i, and the control variables is constraint by $h(u_i, v^*) \geq 0$, $x(t_\circ)$ is the initial state known by all players, $t = t_\circ$ is the starting of the game and $t = t_f$ is the end of the game.

3.3 Theorem. *Let*

$$f(x(t), u_i(t), v^*(t), t), \hat{I}_i(x(t), u_i(t), v^*(t), t)$$

and $h(u_i(t), v^*(t), t)$ $i = 1, \cdots, N$ *be continuously differentiable on* $\mathbb{R}^n$. *If* $u^* = (u_i^*, v^*)$ *is an optimal control, with the state trajectory* $\{x^*(t), t \in [t_\circ, t_f]\}$ *for problem (3.2'), then there exist N-costate vectors* $P_i(t)$ *and* N*-Hamiltonian functions* $\hat{H}_i$ *defined by* $P_i(t) : [t_\circ, t_f] \longrightarrow \mathbb{R}^n$,

$$\hat{H}_i(x(t), u_i(t), v^*(t), P_i(t), t) = \hat{I}_i(x(t), u_i(t), v^*(t), t) + P_i f(x(t), u_i(t), v^*(t), t) + Q(i)h(u_i(t), v^*(t)) \tag{3.3.1}$$

such that

$$\dot{x}^*(t) = f(x^*(t), u_i^*, v^*(t), t), x^*(t_\circ) = x_\circ \tag{3.3.2}$$

$$P_i(t) = -P_i(t)\frac{\partial f(x^*, u_i^*, v^*, t)}{\partial x} - \frac{\partial \hat{I}_i(x^*, u_i^*, v^*, t)}{\partial x}, \tag{3.3.3}$$

$$\frac{\partial \hat{H}_i(x(t), u_i(t), v^*(t), p_i(t), t)}{\partial u_i} = 0 \tag{3.3.4}$$

$$P_i(t_f) = \frac{\partial \hat{\varphi}_i(x(t_f))}{\partial x(t_f)}, \tag{3.3.5}$$

$$h_i(u_i^*, v^*) \geq 0, \tag{3.3.6}$$

and

$$Q(i)h_i(u_i^*, v^*, t) = 0, \quad Q(i) \geq 0, \tag{3.3.7}$$

where

$$\hat{\varphi}_i = \frac{\varphi_i(x(t_f)) - J_i^\circ}{J_i^1 - J_i^\circ},$$

$$\hat{I}_i = \frac{I_i(x(t), u_i(t), v^*, t)}{J_i^1 - J_i^\circ}$$

Proof. Since the functions f and $\hat{I}_i$ are continuous differentiable, then there exists a solution $P(t)$ for the equation

$$\text{(3.3.8)} \qquad \dot{P}(t) = -P(t)\frac{\partial f}{\partial x} - \frac{\partial \hat{I}}{\partial x}, \quad P(t_f) = \frac{\partial \hat{\varphi}(x(t_f))}{\partial x(t_f)}$$

The adjoint equation of the above equation is

$$\text{(3.3.9)} \quad P(t)\delta\dot{x}(t) = \left[\frac{\partial \hat{I}_i}{\partial x} + pf_x\right]\delta x + \Big[I_i(x^*, u_i^*, v^*, t) + pf(x^*, u_i^*, v^*, t) + Q(i)h(u_i^*, v^*)\Big] - \Big[I_i(x^*, u_i, v^*, t) + pf(x^*, u_i, v^*, t) + Q(i)h(u_i, v^*)\Big]$$

which has the solution $\delta x(t)$ with initial condition $\delta x(t_\circ) = 0$. Since Theorem 10.1 in ([3]), states that

"Let $A(t)$ be an $n \times n$ matrix, $G(t)$ an n-dimensional vector of piecewise continuous functions defined on an interval $[t_\circ, t_1]$, and $y_\circ$ an n-dimensional vector. Then if $\tau \in [t_\circ, t_1]$ there is a unique piecewise continuously differentiable solution of the vector differential equation $\dot{y}(t) = A(t)y + G(t)$ on the interval $[t_\circ, t_1]$ which satisfies the condition $y(\tau) = y_\circ$",

then we get

$$\text{(3.3.10)} \qquad \frac{d}{dt}\big[P(t)\delta x(t)\big] =$$

$$\Big[\hat{I}_i(x^*(t), u_i^*(t), v^*(t), t) + Pf(x^*(t), u_i^*(t), v^*(t), t) + Q(i)h(u_i^*, v^*)\Big] - \Big[\hat{I}_i(x^*, u_i, v^*, t) + pf(x^*(t), u_i(t), v^*(t), t) + Q(i)h(u_i, v^*)\Big].$$

By integrating from $t_\circ$ to t_f

$$\int_{t_\circ}^{t_f} \frac{d(P(t)\delta x(t))}{dt}\,dt =$$

$$\int_{t_\circ}^{t_f} \Big[\big[I_i(x^*, u_i^*(t), v^*(t), t) + Pf(x^*, u_i^*(t), v^*(t), t) + Q(i)h(u_i^*, v^*)\big] - \big[I_i(x^*, u_i(t), v^*(t), t) + pf(x^*(t), u_i(t), v^*(t), t) + Q(i)h(u_i, v^*)\big]\Big]\,dt,$$

then

(3.3.11) $$P(t_f)\delta x(t_f) =$$

$$\int_{t_o}^{t_f}\Big[\big[\hat{I}_i(x^*(t), u_i^*(t), v^*(t), t) + Pf(x^*(t), u_i^*(t), v^*(t), t) + Q(i)h(u_i^*, v^*)\big]-$$

$$\big[\hat{I}_i(x^*(t), u_i(t), v^*(t), t) + pf(x^*(t), u_i(t), v^*, t) + Q(i)h(u_i, v^*)\big]\Big]\, dt,$$

Since

$$P(t_f) = \frac{\partial\hat{\varphi}(x(t_f))}{\partial x(t_f)}$$

and

$$\delta\mu(J(u_i, v)) = \frac{\partial\hat{\varphi}(x(t_f))}{\partial x(t_f)}\partial x(t_f),$$

then according to Theorem 11.1 in ([3]). equation (3.3.10) takes the form

(3.3.12) $$\delta\mu(J(u_i, v^*)) =$$

$$\int_{t_o}^{t_f}\Big[\big[\hat{I}_i(x^*, u_i^*(t), v^*(t), t) + Pf(x^*(t), u_i^*(t), v^*(t), t) + Q(i)h(u_i^*, v^*)\big]-$$

$$\big[\hat{I}_i(x^*(t), u_i(t), v^*(t), t) + pf(x^*(t), u_i(t), v^*(t), t) + Q(i)h(u_i, v^*)\big]\Big]\, dt.$$

Since U is convex, then from differentiability of f and $\hat{I}$ and according to Theorem 11.2 in ([3]). We have for each $u_i \in U$.

(3.3.13) $$\Big[\hat{I}_i(x^*, u_i^*, v^*, t) + Pf(x^*, u_i^*, v^*, t) + Q(i)h(u_i^*, v^*)\Big]$$

$$-\Big[\hat{I}_i(x^*(t), u_i(t), v^*(t), t) + Pf(x^*, u_i, v^*, t) + Q(i)h(u_i, v^*)\Big] \le 0$$

Hence

$$\hat{I}_i(x^*(t), u_i^*(t), v^*(t), t) + Pf(x^*, u_i^*, v^*, t) + Q(i)h(u_i^*, v^*)\Big]$$

$$= \min_{u_i \in U} \hat{I}_i(x^*(t), u_i(t), v^*(t), t) + Pf(x^*, u_i(t), v^*(t), t) + Q(i)h(u_i, v^*))$$

Example. Let the state equations given by the following

$$\dot{x}_1(t) = u_1(t) + u_2(t), \quad \dot{x}_2(t) = u_2(t) - u_1(t)$$

where the cost for each player is

$$J_1 = x_1(t_f) + \int_{t_\circ}^{t_f} (u_1 - 2)^2 dt, \quad J_2 = 2x_2(t_f) + \int_{t_\circ}^{t_f} (u_2 - 1)^2 dt$$

the time interval $[t_\circ, t_f]$ is prescribed.

Solution. The Hamiltonian function H_1 for each player i given by

$$H_1 = (u_1 - 2)^2 + P_1(u_1 + u_2),$$

$$H_2 = (u_2 - 1)^2 + P_2(u_1 - u_2).$$

The costate equations are obtained from the first order open-loop necessary conditions as follows

$$\dot{P_1}(t) = -\frac{\partial H_1}{\partial x_1} = 0, \quad P_1(t_f) = \frac{\partial \varphi(x(t_f))}{\partial x_1} = 1$$

which implies $P_1(t) = 1$

$$\dot{P}_2(t) = -\frac{\partial H_2}{\partial x_2} = 0, \quad P_2(t_f) = \frac{\partial \varphi(x_2(t_f))}{\partial x_2(t_f)} = 2$$

which implies $P_2(t) = 2$.
From the necesary conditions, the Hamiltonian functions is minimized if the first derivative of u_i equal zero i.e.,

$$\frac{\partial H_1}{\partial u_1} = 0, \quad \frac{\partial H_2}{\partial u_2} = 0$$

since $\frac{\partial H_1}{\partial u_1} = 2(u_1 - 2) + P_1$ then $u_1 = \frac{3}{2}$.
Since $\frac{\partial H_1}{\partial u_2} = 2(u_2 - 1) + P_2$ then $u_2 = 2$, and thus

$$J_1^1 = \min J_1 = x_1(t_f) + \int_{t_\circ}^{t_f} (u_1 - 2)^2\, dt = x_1(t_f) + \int_{t_\circ}^{t_f} (\frac{3}{2} - 2)^2\, dt$$

$$= x_1(t_f) + \frac{1}{4}(t_f - t_\circ),$$

$$J_2^1 = \min J_2 = 2x_2(t_f) + \int_{t_\circ}^{t_f} (u_2 - 1)^2\, dt = 2x_2(t_f) + (t_f - t_\circ).$$

In order to elect the membership functions μ_i we assume that the controls $u_1 = \frac{3}{4}$ and $u_2 = 1$ make the cost J_i undersirable (i.e. $\mu_i(J_i) = 0$). Therefore

$$J_1^\circ = x_1(t_f) + \int_{t_\circ}^{t_f} (u_1 - 2)^2 \, dt = x_1(t_f) + \int_{t_\circ}^{t_f} (\frac{3}{4} - 2)^2 \, dt =$$

$$= x_1(t_f) + \frac{25}{16}(t_f - t_\circ),$$

$$J_2^\circ = 2x_2(t_f) + \int_{t_\circ}^{t_f} (u_2 - 1)^2 dt = 2x_2(t_f),$$

Hence

$$\mu(J_1) = \frac{J_1 - J_1^\circ}{J_1^1 - J_1^\circ} = \frac{x_1(t_f) + \int_{t_\circ}^{t_f} (u_1 - 2)^2 dt - x_1(t_f) - \frac{25}{16}(t_f - t_\circ)}{\frac{1}{4}(t_f - t_\circ) - \frac{25}{16}(t_f - t_\circ)}$$

$$= \frac{25}{21} - \frac{16}{21} \int_{t_\circ}^{t_f} \frac{(u_1 - 2)^2}{(t_f - t_\circ)} \, dt$$

$$\mu(J_2) = \frac{J_2 - J_2^\circ}{J_2^1 - J_2^\circ} = \frac{1}{t_f - t_\circ} \int_{t_\circ}^{t_f} (u_2 - 1) \, dt.$$

The flexible formulation of the original problem in the form of a fuzzy continuous differential game problem can be transformed into the following problem

$$\max_{u_1, u_2} \lambda$$

subject to

$$(*) \qquad \lambda \leq \frac{25}{21} - \frac{16}{21} \int_{t_\circ}^{t_f} \frac{(u_1 - 2)^2}{(t_f - t_\circ)} \, dt,$$

$$(**) \qquad \lambda \leq \frac{1}{t_f - t_\circ} \int_{t_\circ}^{t_f} (u_2 - 1)^2 \, dt,$$

$$\dot{x}_1(t) = u_1(t) + u_2(t)$$

$$\dot{x}_2(t) = u_1 - u_2$$

The Hamiltonian functions of membership $\mu_i(J_i)$ for each player i are

$$\hat{H}_1 = -\frac{16(u_1 - 2)^2}{21(t_f - t_\circ)} + P_1(u_1 + u_2)$$

$$\hat{H}_2 = -\frac{(u_2 - 1)^2}{(t_f - t_\circ)} + P_2(u_1 - u_2)$$

from the necessary conditions we have

$$\frac{\partial \hat{H}_1}{\partial u_1} = -\frac{32(u_1 - 2)}{21(t_f - t_\circ)} + P_1 = 0$$

$$\frac{32}{21}(u_1 - 2) = (t_f - t_\circ)$$

$$\hat{u}_1 = \frac{21(t_f - t_\circ)}{32} + 2,$$

and

$$\frac{\partial \hat{H}_2}{\partial u_2} = 0 = \frac{2(u_2 - 1)}{(t_f - t_\circ)} - P_2$$

then

$$\hat{u}_2 = 1 + (t_f - t_\circ).$$

By substituting about u_1, u_2 in $(*)$, $(**)$ we have

$$\lambda \leq \frac{25}{21} - \frac{16}{21}\int_{t_\circ}^{t_f} \frac{(2 + \frac{21}{32}(t_f - t_\circ) - 2)^2}{(t_f - t_\circ)}\, dt$$

$$= \frac{25}{21} - \frac{16}{21}\left[\frac{21(t_f - t_\circ)}{32}\right]^2 (t_f - t_\circ) = \frac{25}{21} - \frac{21}{64}(t_f - t_\circ)^3,$$

and

$$\lambda \leq \frac{1}{(t_f - t_\circ)}\int_{t_\circ}^{t_f} (t_f - t_\circ)^2\, dt \leq (t_f - t_\circ)^2.$$

4. Conclusions.

The necessary conditions for the optimality of a fuzzy differential game are derived and applied on Nash-Equilibrium fuzzy continuous differential game. Deriving these conditions was based on an auxiliary variable λ which can be determined as a solution of deterministic mathematical programming problem.

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