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REGULARITY FOR THE SOLUTIONS OF DOUBLE OBSTACLE PROBLEMS INVOLVING NONLINEAR ELLIPTIC OPERATORS ON THE HEISENBERG GROUP

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We study the Hölder continuity of the homogeneous gradient of the weak solutions $u \in W^{1,p}$ of double obstacle problems involving nonlinear elliptic operators on the Heisenberg group.

1. Introduction.

Let $\mathbb{H}^n$, $n \geq 1$ be the Heisenberg group and let X_i, $i = 1, \dots, 2n$, be the generators of the corresponding Lie algebra with their commutators up to the first order. Let $Xu = (X_1u, \dots, X_{2n}u)$ and let Ω be an open bounded subset of $\mathbb{H}^n$. The purpose of this paper is to investigate the local Hölder continuity of the gradient Xu of the weak solutions $u \in W^{1,p}(\Omega, X)$, $p > 1$, of the double obstacle problem for operators of the form

$$\mathrm{div}_{\mathbb{H}} A(x, Xu) - B(x, u, Xu)$$

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where A and B denote, respectively, vector and scalar valued functions

$$A : \mathbb{H}^n \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$$

$$B : \mathbb{H}^n \times \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}$$

satisfying the following structure conditions for fixed $p > 1$:

(a) $A(x, h) = g(x)|h|^{p-2}h$, for each $(x, h) \in \mathbb{H}^n \times \mathbb{R}^{2n}$, where $g : \mathbb{H}^n \to \mathbb{R}$ satisfies $\|g\|_{L^\infty(\Omega)} \leq \Gamma_1$ and
(b) $|g(x) - g(x')| \leq |x - x'|^{\alpha_1}$, for all $x, x' \in \mathbb{H}^n$.
(c) $|B(x, u, h)| \leq \Gamma_2(|h|^{p(1-\frac{1}{p^*})} + |u|^{p^*-1} + a(x))$ for all $(x, u, h) \in \mathbb{H}^n \times \mathbb{R} \times \mathbb{R}^{2n}$, where $a \in L^t_{loc}(\Omega)$, $t > \frac{p^*}{p^*-1}$.

The obstacles ψ_1, ψ_2 satisfient

(d) $\psi_1, \psi_2 \in C^{1,\alpha_2}(\Omega)$
(e) $\psi_1 \geq \psi_0 \geq \psi_2$ a.e. in Ω

where $\psi_0 \in W^{1,p}(\Omega, X)$. Here $\alpha_1, \Gamma_1, \Gamma_2$ are positive constants and $\alpha_2 \in (0, 1)$. Moreover, whereas $|g(x) - g(x')|$ and $|h|$ denote resp. scalar and vectorial Euclidean norms, $|x - x'|$ denotes the Heisenberg norm in $\mathbb{H}^n$. Finally if Q is the homogeneous dimension of $\mathbb{H}^n$ (we will remind these definitions in section 2), then $p^* := \frac{Qp}{Q-p}$ if $p < Q$ and p^* is an arbitrary number greater than p if $p \geq Q$.

We denote by $W^{1,p}(\Omega, X)$ the space of functions $f \in L^p(\Omega)$ such that $X_k f \in L^p(\Omega)$ for $k = 1, \ldots, 2n$, with norm $\|f\|_{1,p} = \|f\|_{L^p(\Omega)} + \sum_{k=1}^{2n} \|X_k f\|_{L^p(\Omega)}$ and by $W_0^{1,p}(\Omega, X)$ the closure of $C_0^\infty(\Omega)$ with respect to the norm of $W^{1,p}(\Omega, X)$.

A function $u \in W^{1,p}(\Omega, X)$ is said to be a weak solution of the double obstacle problem related to the operator $\operatorname{div}_{\mathbb{H}} A + B$ if $\psi_1 \geq u \geq \psi_2$ and

$$\int_\Omega \left[A(x, Xu)X\varphi + B(x, u, Xu)\varphi\right] dx \geq 0 \tag{1}$$

holds for all $\varphi \in W_0^{1,p}(\Omega, X)$ such that $\psi_1 \geq u + \varphi \geq \psi_2$.

We let C denote a positive constant which may depend only on the structural constants $Q, n, p, \alpha_1, \alpha_2, \Gamma_1, \Gamma_2, t$, not necessarily the same at each occurrance.

The result we mean establish about the solutions of the problem (1) is linked to the anologous one relative to the solutions of the equation

$$\int_\Omega |Xv|^{p-2} Xv\, X\varphi\, dx = 0 \tag{2}$$

$v \in W^{1,p}_{loc}(\Omega, X)$, for all $\varphi \in W^{1,p}_0(\Omega, X)$. We will require that any solution v of (2) has Hölder continuous horizontal derivatives. Moreover we will suppose that there are positive constants C and $\alpha \in (0,1)$ such that any solution v of (2) satisfies

$$\sup_{B(x_0,R-\sigma R)} |Xv| \le \gamma(\sigma)\Big(\frac{1}{|B(x_0,R)|}\int_{B(x_0,R)} |Xv|^p\,dx\Big)^{1/p} \tag{3}$$

$$\max_{i=1,\dots,2n} \operatorname{osc}_{B(x_0,\rho)} X_i v \le C\,(\frac{\rho}{R})^{\alpha} \sup_{B(x_0,R/2)} |Xv| \tag{4}$$

for any homogeneous ball (see section 2) $B(x_0,R) \subset\subset \Omega$ and for all $\rho < R/2$.

As a matter of fact in [14] we proved the estimates (3) and (4) for any $2 \le p < 1+\sqrt{5}$ and we are stating them also for $1+\frac{1}{\sqrt{5}} < p < 2$ [16]. The strong limitation on the range of admissibility of p required in [14] is linked to the method used which relies on the local L^p regularity of the derivative in the degenerate direction of the solutions of the approximating equations. Notwithstanding we couldn't extend this result to other values of p, we true it possible to improve [14], perhaps by a different method. Here we wish to prove the following result.

Theorem 1. *Let $p > 1$ such that the solutions v of* (2) *satisfy* (3) *and* (4) *(this happens at least for $1+\frac{1}{\sqrt{5}} < p < 1+\sqrt{5}$). Then the solutions u of* (1) *belong to $C^{1,\beta}_{loc}(\Omega)$, for some $\beta \in (0,1)$ depending only on the structural constants and the constants in* (3) *and* (4).

The conditions (3) and (4) concerning the solutions v of (2) are consistent with the method we adopt to prove Theorem 1. In fact we link u and v with the solutions u_0 and u_1 of resp. double and single obstacle problems relative to the operator A freezed in $x_0 \in \Omega$, that is $A_0(h) = g(x_0)|h|^{p-2}h \sim |h|^{p-2}h$. Precisely the estimates of

$$\int_{B(x_0,R)} |Xu - Xu_0|^p\,dx, \quad \int_{B(x_0,R)} |Xu_0 - Xu_1|^p\,dx, \quad \int_{B(x_0,R)} |Xu_1 - Xv|^p\,dx$$

we will establish in section 3 for sufficiently small $R < 1$, fournish an estimate for $\int_{B(x_0,R)} |Xu - Xv|^p\,dx$ which together with the conditions (3) and (4) enables us to prove that, in $B(x_0,R)$, u belongs to some Morrey's space and Xu to some Campanato's space, finally u belongs to $C^{1,\beta}(B(x_0,R))$ for some $\beta \in (0,1)$ independent on x_0, R.

The proof of Theorem 1 is largely inspired to [16]. However, whereas in the euclidean setting of [16] the Hölder continuity of the solutions of the inequality (1) could exploited, as stated by many authors as [6], [5], [19], [3], [13], we don't have this information and we have to work with lack of boundedness of the solutions.

For this reason, as in [20], we adopt the direct approach described in [8] to estimate $\int_{B(x_0,R)}(|Xu|^p + |u|^{p^*})\,dx$ instead of $\int_{B(x_0,R)}|Xu|^p\,dx$ and $\int_{B(x_0,R)}|u|^{p^*}dx$ separately.

This technique agrees with the condition (**c**) we have imposed to B in place of the usual one

$$|B(x,u,Xu)| \le \Gamma(|h|^{p-1} + |u|^{p-1} + a(x))$$

of [13], [16] (but also of [3], [18], [5], [4],...). With respect to [16] we give sometimes a simpler proof of some estimates, specially in the case $p < 1$.

2. Basic knowledge.

The Heisenberg group $\mathbb{H}^n$ is the Lie group whose underlying manifold is $\mathbb{R}^{2n+1}$ with the following group law : for all $x = (x', t) = (x_1, \dots, x_{2n}, t)$, $y = (y', s) = (y_1, \dots, y_{2n}, s)$,

$$x \circ y = (x' + y', t + s + 2[x', y'])$$

where $[x', y'] := \sum_{i=1}^{n}(y_i x_{i+n} - x_i y_{i+n})$.

$\mathbb{H}^n$ is a homogeneous group, that is a group with dilations, defined as

$$\delta_\lambda(x', t) = (\lambda x', \lambda^2 t)$$

where the direction t plays a particular role (the space is non-isotropic) corresponding to the definition of the group action.

A norm for $\mathbb{H}^n$ which is homogeneous of degree 1 with respect to the dilations is the Heisenberg norm

$$|x|^4 = |(x', t)|^4 = |x'|^4 + t^2, \text{ for any } x = (x', t) \in \mathbb{H}^n.$$

If for a moment we denote by $|x|_H$ and $|x|_E := (|x'|^2 + t^2)^{1/2}$ the Heisenberg norm and resp. the Euclidean norm of a point $x = (x', t) \in \mathbb{H}^n$, the following obvious inequalities hold

$$|x|_E \le |x|_H \le |x|_E^{1/2} \quad \text{when } |x|_H \le 1$$

The distance associated to the Heisenberg norm is

$$d(x, y) := |y^{-1} \circ x|, x, y \in \mathbb{H}^n, \text{ where } y^{-1} = -y.$$

$B(x, r)$ will denote the homogeneous ball with center in $x \in \mathbb{H}^n$ and radius $r > 0$.

For every function w defined on $\mathbb{H}^n$, both left and right translations are defined on $\mathbb{H}^n$ as

$$L_y w(x) = w(y \circ x)$$

$$R_y w(x) = w(x \circ y)$$

The Lebesgue measure is invariant with respect to the translations of the group, though the shape of the ball changes if one shifts its center, and it is proportional to the Q-th power of the radius, where $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$, that is $|B(x, r)| \simeq r^Q |B(0, 1)|$.

An operator N on H^n is left-invariant if $L_y(Nw) = N(L_y w)$, and similarly for right-invariance.

The Lie algebra $\mathcal{L}(X)$ of left-invariant vector fields corresponding to $\mathbb{H}^n$ is generated by

$$\begin{aligned} X_i &= \partial_{x_i} + 2x_{i+n}\partial_t \\ X_{i+n} &= \partial_{x_{i+n}} - 2x_i\partial_t \\ T &= -4\partial_t \end{aligned}$$

for $i = 1, \ldots, n$. Since $[X_i, X_{i+n}] = -[X_{i+n}, X_i] = T$, $i = 1, \ldots, n$, and $[X_i, X_j] = 0$ in any other case, the vector fields X_i, $i = 1, \ldots, 2n$ satisfy the Hörmander condition of order 1 [10], that is together with their first order commutators they span the whole Lie algebra.

The vector fields X_i don't commute with right translations.

We introduce the definitions (slightly different from the usual ones) of the Morrey's and Campanato's spaces.

Let $0 < \lambda < 1$, $1 \le p < \infty$ and $R_0 > 0$.

Definition 1. The function $w \in L^p_{loc}(\Omega)$ belongs to the *Morrey space* $M^{p,\lambda}(\Omega)$ if and only if

$$\frac{1}{|B(x_0, \rho) \cap \Omega|} \int_{B(x_0,\rho)\cap\Omega} |w|^p \, dx \le C\rho^{p(\lambda-1)}$$

for every $x_0 \in \Omega$ and $0 < \rho < \min\{R_0, \text{ diam } \Omega\}$.

Definition 2. The function $w \in L^p_{\text{loc}}(\Omega)$ belongs to the *Campanato's space* $L^{p,\lambda}(\Omega)$ if and only if

$$\frac{1}{|B(x_0,\rho)\cap\Omega|}\int_{B(x_0,\rho)\cap\Omega} |w(x) - w_{x_0,\rho}|^p\,dx \le C\rho^{p\lambda}$$

for every $x_0 \in \Omega$ and $0 < \rho < \ min\ \{R_0,\ diam\ \Omega\}$, where $w_{x_0,\rho} = \frac{1}{|B(x_0,\rho)\cap\Omega|}\int_{B(x_0,\rho)\cap\Omega} w(x)\,dx$.

Definition 3. The function w belongs to the *Folland-Stein space* $\Gamma^\lambda(\Omega)$ if and only if

$$\sup_{x,y\in\Omega}\frac{|w(x)-w(y)|}{|x-y|^\lambda} < \infty$$

and $\Gamma^\lambda_{loc}(\Omega) \equiv \{u \mid \eta w \in \Gamma^\lambda(\Omega) \text{ for every } \eta \in C_0^\infty(\Omega)\}$.

As an easy consequence of the Poincaré inequality [12], Theorem C, we have

$$Xu \in M^{p,\lambda}(B(x_0,2\rho)) \ \Rightarrow\ u \in L^{p,\lambda}(B(x_0,\rho)) \tag{5}$$

Moreover, since the balls $B(x_0,\rho)$ have the exterior corkscrew property, we have

$$L^{p,\lambda}(B(x_0,\rho) \subset \Gamma^\lambda_{loc}(B(x_0,\rho)) \tag{6}$$

Finally from the confront between the $|\cdot|_H$ and $|\cdot|_E$ norms we deduce

$$C^\lambda_{loc}(B(x_0,\rho)) \subset \Gamma^\lambda_{loc}(B(x_0,\rho)) \subset C^{\lambda/2}_{loc}(B(x_0,\rho)) \tag{7}$$

3. Preliminaries results.

Lemma 1. [18], Lemma 1. *There is a positive constant γ_0 depending only on Q, p, α_1, Γ_1, such that*

$$[A(x,h) - A(x,h')][h-h'] \ge \begin{cases} (|h|+|h'|)^{p-2}\ |h-h'|^2 & \text{if } 1<p<2 \\ |h-h'|^p & \text{if } p \ge 2 \end{cases}$$

for all $h, h' \in \mathbb{R}^{2n}$.

Let $x_0 \in \Omega$ and let $0 < R < 1$ such that $B(x_0, R) \subset \Omega$. Let $A_0(h) \equiv A(x_0, h)$ and let u be a solution of the double obstacle problem related to $\mathrm{div}_{\mathbb{H}} A + B$.

We say that $u_0 \in W^{1,p}(B(x_0, R), X)$ is a weak solution of the double obstacle problem related to $\mathrm{div}_{\mathbb{H}} A_0$ if $u_0 - u \in W_0^{1,p}(B(x_0, R), X)$, $\psi_1 \geq u_0 \geq \psi_2$ and

$$\int_{B(x_0,R)} A_0(Xu_0)\, X\varphi \, dx \geq 0 \tag{8}$$

holds for any $\varphi \in W_0^{1,p}(B(x_0, R), X)$ such that $\psi_1 \geq u_0 + \varphi \geq \psi_2$.

Lemma 2. *Let the hypothesis of Theorem 1 hold. If u_0 is a weak solution of (8), then there exist positive structural constants C, δ and $\sigma(u, u_0)$ such that*

$$\int_{B(x_0,R)} |Xu - Xu_0|^p \, dx \leq C\Theta(R) \int_{B(x_0,R)} (|Xu|^p + |u|^{p^*}) dx + C\, R^{Q+\sigma(u,u_0)} \tag{9}$$

where $\Theta(R) = O(R^\delta)$ as $R \to 0$.

Proof. Let's select the test function $\varphi = u - u_0$ in (8). This is admissible because $\varphi \in W_0^{1,p}(B(x_0, R), X)$ and $\psi_1 \geq \varphi + u_0 \geq \psi_2$. Thus we obtain

$$\int_{B(x_0,R)} [A_0(Xu) - A_0(Xu_0)] \cdot [Xu - Xu_0] \, dx \leq$$

$$\leq \int_{B(x_0,R)} [A_0(Xu) - A(x, Xu)] \cdot [Xu - Xu_0] \, dx + \int_{B(x_0,R)} A(x, Xu) \cdot [Xu - Xu_0] \, dx$$

$$\leq C R^{\alpha_1} \int_{B(x_0,R)} |Xu|^{p-1} |Xu - Xu_0| \, dx - \int_{B(x_0,R)} B(x, u, Xu)\, (u - u_0) \, dx$$

$$\leq C R^{\alpha_1} \Big(\int_{B(x_0,R)} |Xu|^p \, dx \Big)^{(p-1)/p} \Big(\int_{B(x_0,R)} |Xu - Xu_0|^p \, dx \Big)^{1/p}$$

$$+ C \int_{B(x_0,R)} (|Xu|^{p(p^*-1)/p^*} + |u|^{p^*-1} + |a|)\, |u - u_0| \, dx$$

(applying Sobolev's inequality [12], Theorem C)

$$(10)\qquad \le CR^{\alpha_1}\Big(\int_{B(x_0,R)} |Xu|^p\,dx\Big)^{(p-1)/p}\Big(\int_{B(x_0,R)} |Xu - Xu_0|^p\,dx\Big)^{1/p}$$

$$+ C\Big\{\Big(\int_{B(x_0,R)} (|Xu|^p + |u|^{p^*})\,dx\Big)^{1-1/p^*}$$

$$+ \|a\|_t R^{Q(1-\frac{1}{p^*}-\frac{1}{t})}\Big\}\Big(\int_{B(x_0,R)} |Xu - Xu_0|^p\,dx\Big)^{1/p}.$$

If $p \ge 2$ then (9) follows from (10) and Lemma 1 with

$$\Theta(R) = \Big(\int_{B(x_0,R)} (|Xu|^p + |u|^{p^*})\,dx\Big)^{\frac{p}{(p-1)Q}} + R^{\alpha_1}$$

(Let's observe that $\int_{B(x_0,R)} |u|^{p^*}\,dx < +\infty$ for small $R < 1$, as proved in [1], [12]), and $\sigma(u,v) = \frac{Qp}{p-1}(\frac{1}{p} - \frac{1}{p^*} - \frac{1}{t})$. Let now $1 < p < 2$. If we set $\varphi = u - u_0$ in (1) we easily obtain

$$(11)\qquad \int_{B(x_0,R)} |Xu_0|^p\,dx \le \int_{B(x_0,R)} |Xu|^p\,dx$$

Moreover

$$(12)\quad |Xu - Xu_0|^p = (|Xu| + |Xu_0|)^{p(2-p)/2}\{(|Xu| + |Xu_0|)^{p-2}|Xu - Xu_0|^2\}^{p/2}$$

Applying Hölder inequality gives

$$\int_{B(x_0,R)} |Xu - Xu_0|^p\,dx \le \Big(\int_{B(x_0,R)} |Xu|^p\,dx\Big)^{(2-p)/2}\cdot$$

$$\cdot\Big[\int_{B(x_0,R)} (|Xu| + |Xu_0|)^{p-2}|Xu - Xu_0|^2\,dx\Big]^{p/2}$$

(by (10) and Lemma 1)

$$\le C\Big(\int_{B(x_0,R)} |Xu|^p\,dx\Big)^{(2-p)/2}\Big(\int_{B(x_0,R)} |Xu - Xu_0|^p\,dx\Big)^{1/2}\cdot$$

$$\cdot C\Big\{R^{\alpha_1}\Big(\int_{B(x_0,R)} |Xu|^p\,dx\Big)^{(p-1)/p}$$
$$+\Big[\int_{B(x_0,R)} (\,|Xu|^p + |u|^{p^*}\,)\,dx\Big]^{(1-1/p^*)} + R^{\gamma}\Big\}^{p/2}$$

where $\gamma = Q\,(1 - \frac{1}{p^*} - \frac{1}{t}\,)$. Then

$$(13)\qquad \Big(\int_{B(x_0,R)} |Xu - Xu_0|^p\,dx\Big)^{1/2} \le C\Big(\int_{B(x_0,R)} |Xu|^p\,dx\Big)^{(2-p)/2}\cdot$$
$$\cdot\Big\{R^{\alpha_1 p/2}\Big(\int_{B(x_0,R)} |Xu|^p\,dx\Big)^{(p-1)/2}$$
$$+\Big[\int_{B(x_0,R)} (\,|Xu|^p + |u|^{p^*}\,)\,dx\Big]^{(1-1/p^*)p/2} + R^{\gamma p/2}\Big\}$$
$$\le C\Big\{R^{\alpha_1 p/2}\Big(\int_{B(x_0,R)} |Xu|^p\,dx\Big)^{1/2} + \Big[\int_{B(x_0,R)} (\,|Xu|^p + |u|^{p^*}\,)\,dx\Big]^{1-p/2p^*}$$
$$+\Big(\int_{B(x_0,R)} |Xu|^p\,dx\Big)^{(2-p)/2} R^{\gamma p/2}\Big\} \le C\Big\{R^{\alpha_1 p/2}\Big(\int_{B(x_0,R)} |Xu|^p\,dx\Big)^{1/2}$$
$$+\Big[\int_{B(x_0,R)} (\,|Xu|^p + |u|^{p^*}\,)\,dx\Big]^{1-p/2p^*} + \Big(\int_{B(x_0,R)} |Xu|^p\,dx\Big)^{(2-p)q/2} + R^{\gamma p q'/2}\Big\}$$

where $q, q' > 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Let's observe that $2(1 - \frac{p}{2p^*}) > 1$. Moreover, as $t > \frac{p^*}{p^*-1}$, we can choose q in such a way that $\gamma p q' > Q$ and $(2-p)q > 1$. Then (9) follows from (13) with $\Theta(R) = (\,\int_{B(x_0,R)}(\,|Xu|^p + |u|^{p^*}\,)\,dx\,)^{\nu}$, where $\nu = (1 - \frac{p}{p^*}) \wedge [(2-p)q - 1]$, and $\sigma(u, v) = \gamma p q' - Q$.

We say that $u_1 \in W^{1,p}(B(x_0, R), X)$ is a weak solution of the single obstacle problem related to the operator $\mathrm{div}_{\mathbb{H}} A_0$ if $u_1 \ge \psi_2$, $u_0 - u_1 \in W_0^{1,p}(B(x_0, R), X)$ and

$$(14)\qquad \int_{B(x_0,R)} A_0(Xu_1)\cdot X\varphi\,dx \ge 0$$

for all $\varphi \in W_0^{1,p}(B(x_0, R), X)$ such that $\varphi + u_1 \ge \psi_2$.

Lemma 3. *Let the hypothesis of Theorem 1 hold. If* u_1 *is a weak solution of* (14)*, then there exist positive structural constants* C *and* $\sigma(u_0, u_1)$ *such that*

$$\int_{B(x_0,R)} |Xu_0 - Xu_1|^p\,dx \le C\,R^{\sigma(u_0,u_1)}\left(R^Q + \int_{B(x_0,R)} |Xu_0|^p\,dx\right) \tag{15}$$

Proof. The proof is largely due to [16].

The preliminar part of the proof insists on the estimate of the following term:

$$I \equiv \int_{B(x_0,R)} [A_0(Xu_0) - A_0(Xu_1)]\cdot[Xu_0 - Xu_1]\,dx.$$

Let's select the test function $\varphi = u_0 - u_1$ in (14). This is admissible because $\varphi \in W_0^{1,p}(B(x_0, R), X)$ and $\varphi + u_1 = u_0 \ge \psi_2$. We have

$$\int_{B(x_0,R)} A_0(Xu_1)\cdot[Xu_0 - Xu_1]\,dx \ge 0$$

Then

$$I = \int_{B(x_0,R)} A_0(Xu_0)\cdot[Xu_0 - Xu_1]\,dx - \int_{B(x_0,R)} A_0(Xu_1)\cdot[Xu_0 - Xu_1]\,dx$$

$$\le \int_{B(x_0,R)} A_0(Xu_0)\cdot(Xu_0 - Xu_1)\,dx$$

$$= \int_{B(x_0,R)} A_0(Xu_0)\cdot[Xu_0 - X(u_1\wedge\psi_1)]\,dx$$

$$+ \int_{B(x_0,R)} A_0(Xu_0)\cdot[X(u_1\wedge\psi_1) - Xu_1]\,dx = I_1 + I_2.$$

Now $I_1 \le 0$. In fact the function $\varphi = u_1\wedge\psi_1 - u_0$ satisfies $\psi_1 \ge \varphi + u_0 = u_1\wedge\psi_1 \ge \psi_2$. Moreover it belongs to $W_0^{1,p}(B(x_0, R), X)$ because if $u_1 \ge \psi_1$, then $0 \le \varphi = \psi_1 - u_0 \le u_1 - u_0 = 0$ on $\partial B(x_0, R)$, and if $u_1 \le \psi_1$, then $\varphi = u_1 - u_0 = 0$ on $\partial B(x_0, R)$.
Thus

$$I \le \int_{B(x_0,R)} A_0(Xu_0)\cdot[X(u_1\wedge\psi_1) - Xu_1]\,dx \tag{16}$$

$$\leq \Big(\int_{B(x_0,R)} |A_0(Xu_0)|^{p'}\, dx \Big)^{1/p'} \Big(\int_{B(x_0,R)} |X(u_1-\psi_1)^+|^p\, dx \Big)^{1/p}$$

since $u_1 - (u_1 \wedge \psi_1) = (u_1-\psi_1)^+$. To carry on with the estimate of I we need now the following estimate

$$\int_{B(x_0,R)} |X(u_1-\psi_1)^+|^p\, dx \leq \begin{cases} C\, R^{Q+\alpha_2 p'} & \text{if } p \geq 2 \\ C\, R^{Q+\alpha_2 (p-1) p} & \text{if } 1 < p < 2. \end{cases} \tag{17}$$

On this aim let

$$J \equiv \int_{B(x_0,R)} [A_0(Xu_1) - A_0(X\psi_1)\,] \cdot [\,X(u_1-\psi_1)^+]\, dx$$

$$= \int_{B(x_0,R)} A_0(Xu_1) \cdot [\,X(u_1-\psi_1)^+]\, dx$$

$$- \int_{B(x_0,R)} A_0(X\psi_1) \cdot [\,X(u_1-\psi_1)^+]\, dx \equiv J_1 - J_2.$$

Defining $\varphi = (u_1-\psi_1)^+$, then $-\varphi + u_1 \geq \psi_2$; moreover $\psi_1 \geq u_0 = u_1$ on $\partial B(x_0,R)$ and then $\varphi \in W_0^{1,p}(B(x_0,R),X)$. Inserting $-\varphi$ in (14) gives $J_1 \leq 0$. Hence

$$J \leq -J_2 = - \int_{B(x_0,R)} [\,A_0(X\psi_1) - A_0(X(\psi_1(x_0))\,] \cdot [\,X(u_1-\psi_1)^+]\, dx \tag{18}$$

Let now $p \geq 2$.

Since A_0 is Lipschitz in this case, then from (18) and (d) we have

$$J \leq C(\epsilon) \int_{B(x_0,R)} |X\psi_1 - X\psi_1(x_0)|^{p'} dx + \epsilon \int_{B(x_0,R)} |X(u_1-\psi_1)^+|^p dx \tag{19}$$

$$\leq C(\epsilon) \|\psi_1\|_{1,\alpha_2} R^{Q+\alpha_2 p'} + \epsilon \int_{B(x_0,R)} |X(u_1-\psi_1)^+|^p\, dx$$

Moreover, from Lemma 1, we have

$$J = \int_{B(x_0,R)} \chi_{u_1 \geq \psi_1} [A_0(Xu_1) - A_0(X\psi_1)\,] \cdot [\,Xu_1 - X\psi_1]\, dx \tag{20}$$

$$\geq \gamma_0 \int\limits_{B(x_0,R)} |X(u_1-\psi_1)^+|^p\,dx$$

where χ_E denotes the characteristic function of a set E. Now (17) easily follows from (19) and (20). Let now $1 < p < 2$.

Since A_0 is Hölder continuous of constant $p-1$, then, from (18) and (**d**) we have

$$J \leq C \int_{B(x_0,R)} |X\psi_1 - X\psi_1(x_0)|^{p-1}|X(u_1-\psi_1)^+|\,dx \tag{21}$$

$$\leq C\,R^{(Q+\alpha_2 p)/p'}\Big(\int\limits_{B(x_0,R)} |X(u_1-\psi_1)^+|^p\,dx\Big)^{1/p}$$

Moreover, as in Lemma 2 we have

$$\int_{B(x_0,R)} |X(u_1-\psi_1)^+|^p\,dx$$

$$\leq \Big(\int\limits_{B(x_0,R)} |X\psi_1|^p dx\Big)^{(2-p)/2}\Big[\int\limits_{B(x_0,R)} (|Xu_1|+|X\psi_1|)^{p-2}|Xu_1-X\psi_1|^2\,dx\Big]^{p/2}$$

(by Lemma 1)

$$\leq C\Big(\int\limits_{B(x_0,R)} |X\psi_1|^p\,dx\Big)^{(2-p)/2}\Big\{\int\limits_{B(x_0,R)} [A_0(Xu_1)-A_0(X\psi_1)]\cdot \tag{22}$$

$$\cdot[X(u_1-\psi_1)^+dx]\Big\}^{p/2}$$

$$\leq C\Big(\int_{B(x_0,R)} |X\psi_1|^p\,dx\Big)^{(2-p)/2} J^{p/2}$$

By (21), (22) and (d) then we have

$$\int\limits_{B(x_0,R)} |X(u_1-\psi_1)^+|^p\,dx \leq C\,R^{Q(2-p)/2}\Big[R^{(Q+\alpha_2 p)/p'}\cdot$$

$$\cdot\Big(\int\limits_{B(x_0,R)} |X(u_1-\psi_1)^+|^p\,dx\Big)^{1/p}\Big]^{p/2}$$

and (17) easily follows.

We are now in a position to establish (15).

Let $p \geq 2$. From (16), (17) and (a) we have

$$I \leq C\, R^{(Q+\alpha_2 p')/p}\bigg(\int_{B(x_0,R)} |A_0(Xu_0)|^{p'}\,dx\bigg)^{1/p'}$$

$$\leq C\, R^{\alpha_2 p'/(2p)}\bigg(\int_{B(x_0,R)} |Xu_0|^{p}\,dx\bigg)^{1/p'} R^{(Q+\alpha_2 p'/2)/p}$$

$$\leq C\bigg\{R^{\alpha_2 (p')^2/(2p)}\int_{B(x_0,R)} |Xu_0|^{p}\,dx + R^{Q+\alpha_2 p'/2}\bigg\}$$

(as $R < 1$)

$$\leq C\bigg\{R^{\alpha_2 (p')^2/(2p)}\int_{B(x_0,R)} |Xu_0|^{p}\,dx + R^{Q+\alpha_2 (p')^2/(2p)}\bigg\}$$

Taking into account Lemma 1, this establishes (15) with $\sigma(u_0, u_1) = \frac{\alpha_2 (p')^2}{2p}$.

Let now $1 < p < 2$.

From (16), (17) and (a) we have in this case

$$I \leq C\, R^{[Q+\alpha_2(p-1)p]/p}\bigg(\int_{B(x_0,R)} |Xu_0|^{p}\,dx\bigg)^{1/p'} \tag{23}$$

Moreover, proceeding as in Lemma 2

$$\int_{B(x_0,R)} |Xu_0 - Xu_1|^p\,dx \leq C\bigg(\int_{B(x_0,R)} |Xu_0|^{p}\,dx\bigg)^{(2-p)/2}\cdot$$

$$\cdot\bigg[\int_{B(x_0,R)} (|Xu_0| + |Xu_1|)^{p-2}|Xu_0 - Xu_1|^2\,dx\bigg]^{p/2}$$

(by Lemma 1)

$$\leq C\bigg(\int_{B(x_0,R)} |Xu_0|^{p}\,dx\bigg)^{(2-p)/2} I^{p/2}$$

(by (23))

$$\leq C\bigg(\int_{B(x_0,R)} |Xu_0|^{p}\,dx\bigg)^{1/2} R^{[Q+\alpha_2(p-1)p]/2}$$

(by Young's inequality)

$$\leq C\left\{R^{\eta}\int_{B(x_0,R)}|Xu_0|^p dx + R^{Q+\alpha_2(p-1)p-\eta}\right\}$$

for any $\eta > 0$, and (15) follows with $\sigma(u_0, u_1) = \eta = \frac{\alpha_2(p-1)p}{2}$.

We say that $v \in W^{1,p}(B(x_0, R), X)$ is a weak solution of the equation $\mathrm{div}_{\mathbb{H}} A_0 = 0$ if $v - u \in W_0^{1,p}(B(x_0, R), X)$ and

$$\int_{B(x_0,R)} A_0(Xv)X\varphi\, dx = 0 \tag{24}$$

holds for any $\varphi \in W_0^{1,p}(B(x_0, R), X)$.

Lemma 4. *Let the hypothesis of Theorem 1 hold. If u_1 is a weak solution of* (14) *and v is a weak solution of* (24), *then there exist positive structural constants C and $\sigma(u_1, v)$ such that*

$$\int_{B(x_0,R)} |Xu_1 - Xv|^p\, dx \leq C\, R^{\sigma(u_1,v)}\left(R^Q + \int_{B(x_0,R)}|Xu_0|^p\, dx\right) \tag{25}$$

Proof. It follows from Lemma 3 taking $\psi_2 \equiv -\infty$.

Lemmas 2,3 and 4 form a chain between u and v which allows us to conclude that u and v are linked. This is formally stated by the following

Proposition 1. *If v is a solution of* (24), *then there exist positive structural constants C, δ' and $\sigma(u, v)$ such that*

$$\int_{B(x_0,R)} |Xu - Xv|^p\, dx \leq C\Theta(R)\int_{B(x_0,R)}(|Xu|^p + |u|^{p^*})dx + C\, R^{Q+\sigma(u,v)} \tag{26}$$

where $\Theta(R) = O(R^{\delta'})$ as $R \to 0$.

Proof. Let u_0 and u_1 be weak solutions of resp. (8) and (14). First of all we wish to show that u and u_1 are linked through u_0. From Lemmas 2,3 and (11) we have

$$\int_{B(x_0,R)} |Xu - Xu_1|^p\, dx \leq 2^{p-1}\left[\int_{B(x_0,R)}|Xu - Xu_0|^p dx + \int_{B(x_0,R)}|Xu_0 - Xu_1|^p\, dx\right]$$

$$\leq C\Big\{\Theta(R)\int\limits_{B(x_0,R)}(|Xu|^p+|u|^{p^*})dx+R^{Q+\sigma(u,u_0)}+R^{\sigma(u_0,u_1)}\int\limits_{B(x_0,R)}|Xu|^p\,dx$$

$$+R^{Q+\sigma(u_0,u_1)}\Big\}\leq C\Big\{\Theta(R)\int\limits_{B(x_0,R)}(|Xu|^p+|u|^{p^*})dx+R^{\sigma(u,u_1)}\Big\}.$$

This proves that u and u_1 are linked. We know that u_1 and v are linked by (25) and thus, by a similar argument, we can show that u and v are linked.

Lemma 5. *Let $v \in W^{1,p}(B(x_0,R),X)$ be a weak solution of the equation* (24). *Then*

$$\int_{B(x_0,\rho)}|Xv|^p\,dx \leq C(\frac{\rho}{R})^Q\int_{B(x_0,R)}|Xv|^p\,dx \tag{27}$$

$$\int_{B(x_0,\rho)}|Xv-(Xv)_{x_0,\rho}|^p\,dx \leq C(\frac{\rho}{R})^{Q+\alpha p}\int_{B(x_0,R)}|Xv|^p\,dx \tag{28}$$

for all $0 \leq \rho \leq R$. Here C and $\alpha \in (0,1)$ are positive structural constants.

Proof. It easily follows from (3) and (4) (see [16]).

Let

$$\varphi(\rho)\equiv\int_{B(x_0,\rho)}(|Xu|^p+|u|^{p^*})\,dx$$

for all $\rho > 0$.

Lemma 6. *For any $\tau > 0$ there are constants $C = C(\tau)$ and $\bar{R} = \bar{R}(\tau) < 1$ depending only on τ and on the data such that*

$$\varphi(\rho)\leq C\Big[(\frac{\rho}{R})^{Q-\tau}\varphi(R)+R^{Q-\tau}\Big]$$

for all $0 \leq \rho \leq R \leq \bar{R}$.

Proof.

$$\int\limits_{B(x_0,\rho)}|u|^{p^*}\,dx\leq C\Big[\int\limits_{B(x_0,R)}|u-v|^{p^*}\,dx$$

$$+\int\limits_{B(x_0,R)}|v-v_{x_0,R}|^{p^*}\,dx+\int\limits_{B(x_0,R)}|v_{x_0,R}-u_{x_0,R}|^{p^*}\,dx+\int\limits_{B(x_0,\rho)}|u_{x_0,\rho}|^{p^*}\,dx\Big]$$

If μ is the exponent of the Hölder continuity of v with respect to the homogeneous norm, then

$$\int_{B(x_0,R)} |v - v_{x_0,R}|^{p^*}\, dx \le C\, R^{Q+p^*\mu}$$

Hence, by Sobolev's inequality [12], Theorem C,

$$(29)\quad \int\limits_{B(x_0,\rho)} |u|^{p^*} dx \le C\Big[R^{Q+p^*\mu} + \int\limits_{B(x_0,R)} |u - v|^{p^*}\, dx + (\frac{\rho}{R})^Q \int\limits_{B(x_0,R)} |u|^{p^*}\, dx \Big]$$

$$\le C\Big[R^{Q+p\mu} + (\int\limits_{B(x_0,R)} |Xu - Xv|^p\, dx)^{p^*/p} + (\frac{\rho}{R})^Q \varphi(R) \Big]$$

Moreover

$$(30)\quad \int\limits_{B(x_0,\rho)} |Xu|^p\, dx \le 2^{p-1}\Big[\int\limits_{B(x_0,R)} |Xu - Xv|^p\, dx + \int\limits_{B(x_0,R)} |Xv|^p\, dx \Big]$$

Adding (29) to (30) and taking into account Proposition 1 and that

$$\int_{B(x_0,R)} |Xv|^p\, dx \le \int_{B(x_0,R)} |Xu|^p\, dx\,,$$

we obtain (changing if necessary the value of $\sigma(u, v)$)

$$(31)\qquad \varphi(\rho) \le C\Big[(\frac{\rho}{R})^Q + \Theta(R) \Big] \varphi(R) + C R^{Q+\sigma(u,v)}$$

where $\Theta(R) \to 0$ as $R \to O$. The thesis immediately follows from (31) applying [8], Lemma 2.1, p. 86.

4. Proof of Theorem 1.

Proposition 2. $u \in C^{0,\lambda}_{loc}(B(x_0, R))$, *for any* $0 < \lambda < 1$.

Proof. It's a direct consequence of (5), (6), (7) and Lemma 6. In fact, by Lemma 6, $Xu \in M^{p,\lambda}(B(x_0,\rho))$ for any $0<\rho<R$ and for $\lambda = 1-\frac{\tau}{p}$, for any small $\tau > 0$.

We conclude the proof of Theorem 1 by proving that $Xu \in L^{p,\lambda}(B(x_0,R))$ for a suitable $\lambda \in (0,1)$.

Let $0<\tau<1$. By Lemma 6 there exists positive structural constants C and $0 < R_0 < 1$ such that

$$\varphi(R) \leq C\left(\frac{R}{R_0}\right)^{Q-\tau}\left[\varphi(R_0) + R_0^{Q-\tau}\right] \tag{32}$$

for all $0 \leq R \leq R_0$. Then, on account of (26), (28), (32) and that $\int_{B(x_0,R)} |Xv|^p\,dx \leq \int_{B(x_0,R)} |Xu|^p\,dx$, we estimate as follows:

$$\begin{aligned}
&\int_{B(x_0,\rho)} |Xu-(Xu)_\rho|^p\,dx \\
&\leq C\left\{\int_{B(x_0,\rho)} |Xu-Xv|^p\,dx + \int_{B(x_0,\rho)} |Xv-(Xv)_\rho|^p\,dx \right.\\
&\qquad\left. + \int_{B(x_0,\rho)} |(Xu)_\rho-(Xv)_\rho|^p\,dx\right\}\\
&\leq C\left\{\left(\frac{\rho}{R}\right)^{Q+\alpha p}\int_{B(x_0,R)} |Xv|^p\,dx + \int_{B(x_0,\rho)} |Xu-Xv|^p\,dx\right\}\\
&\leq C\left\{\left(\frac{\rho}{R}\right)^{Q+\alpha p}\int_{B(x_0,R)} |Xu|^p\,dx + \Theta(R)\varphi(R) + R^{Q+\sigma(u,v)}\right\}\\
&\leq C\left\{\left[\left(\frac{\rho}{R}\right)^{Q+\alpha p} + \Theta(R)\right]\varphi(R) + R^{Q+\sigma(u,v)}\right\}\\
&\leq C\left\{\left[\left(\frac{\rho}{R}\right)^{Q+\alpha p} + R^{\delta'}\right]\left(\frac{R}{R_0}\right)^{Q-\tau}[\varphi(R_0)+R_0^{Q-\tau}] + R^{Q+\sigma(u,v)}\right\}.
\end{aligned}$$

Let's take now $R=\rho^{\sigma_0}$, where $0<\sigma_0<1$. Then we attein our goal choosing τ small enough and σ_0 close enough to 1 so that $\alpha p + \sigma_0(Q-\tau) > 0$ and $\sigma_0(\delta' + Q - \tau) > Q$.

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