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CONDUCTOR DEGREE AND SOCLE DEGREE

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1. Introduction.

Let K be an infinite field and let $R = K[x_0, \dots, x_n]$ be the usual coordinate ring of $\mathbb{P}^n$. Let $\mathbb{X} = \{P_1, \dots, P_s\}$ be s distinct points in $\mathbb{P}^n$ and let $I_{\mathbb{X}} = \wp_1 \cap \dots \cap \wp_s$ be the ideal of $\mathbb{X}$. We call $A = R/I_{\mathbb{X}}$ the *homogeneous coordinate ring* of $\mathbb{X} \subseteq \mathbb{P}^n$.

There exists a canonical embedding of A into its integral closure, i.e.,

$$A = R/(\wp_1 \cap \dots \cap \wp_s) \hookrightarrow \bigoplus_{i=1}^{s} R/\wp_i = \bigoplus_{i=1}^{s} K[v_i] = T$$

Each ideal $I \subseteq A$ can be extended to T, associating to I the ideal generated by I in T (cfr [8], Ch. VIII, sec. 5).

We call the *conductor* of A in T the biggest ideal J in A that coincides with its extension to T. The conductor, considered as an ideal in T, is generated by homogeneous forms, i.e.

$$J = < v_1^{\alpha_1}, \dots, v_s^{\alpha_s} >$$

The sequence $\{\alpha_1, \dots, \alpha_s\}$ is called the *conductor sequence of A* and the numbers $\alpha_1, \dots, \alpha_s$ are called the *conductor degrees in* $\mathbb{X}$.

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From a theorem by Orecchia ([10], Th. 4.3) we can characterize the conductor sequence in terms of the points of $\mathbb{X}$; namely, we have that α_i is the minimal degree of a homogeneous polynomial F in R such that $F \in \wp_1 \cap \ldots \cap \wp_{i-1} \cap \wp_{i+1} \ldots \cap \wp_s$ and $F \notin \wp_i$.

In view of this Theorem, we give the following

Definition 1.1. Let $\mathbb{X} = \{P_1, \ldots, P_s\}$ be a set of s distinct points in $\mathbb{P}^n$. We say that $F \in R$ is a *separator* for P_i $(1 \leq i \leq s)$ if $F(P_j) = 0$ for all $j \neq i$, $1 \leq j \leq s$, and $F(P_i) \neq 0$. We call the minimal degree of a separator for P_i the *degree of P_i in $\mathbb{X}$*.

Thus, Orecchia's Theorem can be reformulated to say that the *degree of P_i in $\mathbb{X}$* is a *conductor degree in $\mathbb{X}$*. For this reason, we often denote by α_i the *degree of P_i in $\mathbb{X}$* or simply by α when no confusion arises.

In Section 2, following [6], we see that the conductor sequence of A is contained in a set of *permissible values*, which depend on the Hilbert function of A.

It is well known that there are different set of points having the same Hilbert function but with different conductor sequences (Example 1). In other words, the conductor sequence is not, in general, uniquely determined by the Hilbert function.

We restrict our attention to sets $\mathbb{X}$ of distincts points in $\mathbb{P}^2$. In the first part of Section 3 we show, by adapting a general result by Geramita, Kreuzer and Robbiano ([4]), that the maximum of the *permissible values* always appears in the conductor sequence of any set of points with Hilbert function H.

So, two questions naturally arise. Given a Hilbert function, H, for s distinct points,

Question 1. *Is there any* permissible value *(besides the maximum one) guaranteed to appear in the conductor sequence of every set of points with Hilbert function H?*

Question 2. *Are there other more restrictive conditions on the conductor sequence?*

In the second part of Section 3 we give a positive answer to Question 2. In fact the main result of this paper is to get different restrictions on the possible values of conductor degrees. Given a set of distinct points $\mathbb{X}$ in $\mathbb{P}^2$, we can consider the *socle* of any artinian reduction of the homogeneous coordinate ring of $\mathbb{X}$, $A = R/I_{\mathbb{X}}$. It is a known result of homological algebra (see [9], Th. 1.3.6), that the *graded Betti numbers* of A are the same for any artinian reduction of A and that they are connected to the degrees of the minimal generators of the socle of any artinian reduction of A. So these degrees are invariants too. We call them the *socle-permissible values* for A and we prove, in Theorem 3.9, that the possible values for the conductor degrees in $\mathbb{X}$ are *socle-permissible values* for A. So it is possible to associate to any set of points $\mathbb{X}$ in $\mathbb{P}^2$ a set of *socle-permissible values* for A, $\mathcal{S}_{\mathbb{X}}$. We show (see Example 4) that not all *socle-permissible values* for A are realized by some point $P \in \mathbb{X}$. So a new question arises.

Question 3. *Given a set of distinct points $\mathbb{X} \subseteq \mathbb{P}^2$, does there exist a configuration of points $\mathbb{X}'$ such that $\mathcal{S}_{\mathbb{X}} = \mathcal{S}_{\mathbb{X}'}$ and such that any socle-permissible value for $R/I_{\mathbb{X}'}$ is realized by at least one point $P \in \mathbb{X}'$?*

In Section 4, we show that it is possible to consider particular configurations of points $\mathbb{X}$, that we call *lex-configurations*, such that all *socle-permissible values* for $R/I_{\mathbb{X}}$ are realized by at least one point $P \in \mathbb{X}$ (Proposition 4.6).

In Section 5, we answer Question 3 in another particular case, that is when $\mathbb{X}$ is a set of generic points. In this case, we succeed in constructing a set of point $\mathbb{X}'$ with the required property.

2. The conductor and the Hilbert function.

It is well known that the Hilbert function of a graded R-module $M = \bigoplus_{t\geq 0} M_t$ is

$$H_M(t) := \dim_K(M_t)$$

If $I_{\mathbb{Y}} \subseteq R$ is the (saturated) ideal associated to a projective scheme $\mathbb{Y} \subseteq \mathbb{P}^n$, we'll use the notation $H_{\mathbb{Y}}$ to indicate $H_{R/I_{\mathbb{Y}}}$.

We recall now some results from ([6]). Let $\mathbb{X}$ be a set of distinct points in $\mathbb{P}^n$ and let α be the degree of a point $P \in \mathbb{X}$. By definition, there exists a homogeneous polynomial F, of degree α, that is a separator of minimal degree for P.

Let $\mathbb{X}' = \mathbb{X} \setminus \{P\}$. Then

$$F \in (I_{\mathbb{X}'})_\alpha \setminus (I_{\mathbb{X}})_\alpha$$

So we get ([6], Lemma 2.3)

$$\dim(I_{\mathbb{X}'})_t = \begin{cases} \dim(I_{\mathbb{X}})_t & t < \alpha \\ \dim(I_{\mathbb{X}})_t + 1 & t \geq \alpha \end{cases}$$

and the Hilbert function of $\mathbb{X}'$ is the following

$$H_{\mathbb{X}'}(t) = \begin{cases} H_{\mathbb{X}}(t) & t < \alpha \\ H_{\mathbb{X}}(t) - 1 & t \geq \alpha \end{cases}$$

i.e.

$$\Delta H_{\mathbb{X}'}(t) = \begin{cases} \Delta H_{\mathbb{X}}(t) & t \neq \alpha \\ \Delta H_{\mathbb{X}}(t) - 1 & t = \alpha \end{cases}$$

This motivates the following

Definition 2.1. ([6], Def. 4.1). Let $S = \{b_i\}$, $i \geq 0$, be a zero-dimensional differentiable 0-sequence. We say that l is a *permissible value for* S if the sequence $T = \{c_i\}$

$$c_i = \begin{cases} b_i & i < l \\ b_i - 1 & i \geq l \end{cases}$$

is again a zero-dimensional differentiable 0-sequence.

Then, by the previous remark, we can conclude

Remark 2.2. The degree of a point $P \in \mathbb{X}$ is a permissible value for the Hilbert function of $A = R/I_{\mathbb{X}}$.

In other words: given a set of points $\mathbb{X}$, then simply by looking at the graph of $\Delta H_{\mathbb{X}}$, the possible values for the conductor degrees in $\mathbb{X}$ are given by the integers t for which the $\Delta H_{\mathbb{X}}(t)$ can decrease by 1 and still remain a Hilbert function.It is trivial to see that if $\sigma(\mathbb{X}) = \min\ \{t | \Delta H_{\mathbb{X}}(t) = 0\}$ then $\sigma(\mathbb{X}) - 1$ is the maximum of the permissible values for the Hilbert function of $\mathbb{X}$.

Example 1. Consider the following zero-dimensional differentiable 0-sequence (which thus can be the Hilbert function of 11 points in $\mathbb{P}^2$):

$$\begin{array}{lcccccccccc} H_{\mathbb{X}} & = & 1 & 3 & 6 & 8 & 10 & 11 & 11 & \dots \\ \Delta H_{\mathbb{X}} & = & 1 & 2 & 3 & 2 & 2 & 1 & 0 & \dots \end{array}$$

We draw the graph of $\Delta H_{\mathbb{X}}$ (see Figure 1) and we see from the graph that the permissible values are exactly 2, 4, 5. In Figures 2 and 3 we give examples of two different configurations of points in $\mathbb{P}^2$ with the Hilbert function given above but with different conductor sequences.

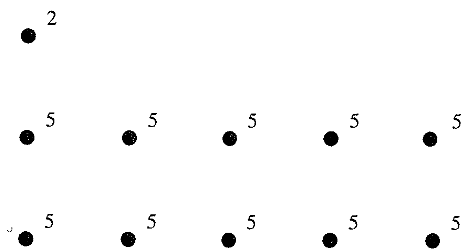


Figure 3: Configuration with conductor sequence $\{2, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5\}$

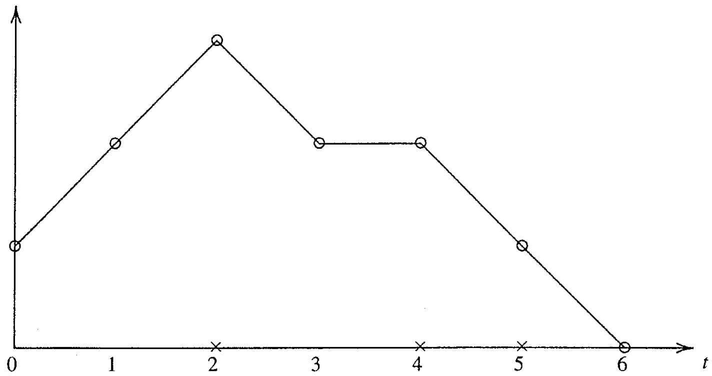


Figure 1: Graph of the first difference function $\Delta H_{\mathbb{X}}$

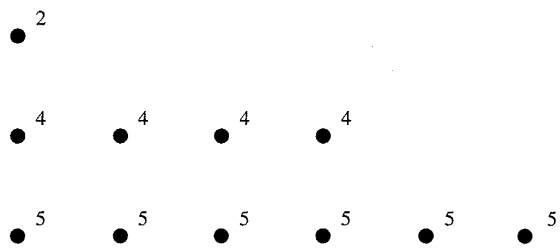


Figure 2: Configuration with conductor sequence $\{2, 4, 4, 4, 4, 5, 5, 5, 5, 5, 5\}$

3. Some results on the degrees of the conductor.

3.1 A necessary value.

We know that the possible values of the conductor degrees of a set of points $\mathbb{X}$, with Hilbert function H, can be read from the first difference function $\Delta H_{\mathbb{X}}$. Thanks to a result of Geramita, Kreuzer and Robbiano ([4], Prop. 1.14), we show that there is always a point $P \in \mathbb{X}$ such that the degree of P in $\mathbb{X}$ is the largest permissible value.

Proposition 3.1. *Let $\mathbb{X}$ be a set of s distincts points in $\mathbb{P}^n$ and let $H_{\mathbb{X}}$ be the Hilbert function of $\mathbb{X}$. Then there exists $P \in \mathbb{X}$ such that the degree of P in $\mathbb{X}$ is $\alpha = \sigma(\mathbb{X}) - 1$.*

Proof. By Proposition 1.14 of [4], there is a point P such that $\mathbb{X}' = \mathbb{X} \setminus \{P\}$ has the following Hilbert function

$$H_{\mathbb{X}'}(t) = \begin{cases} H_{\mathbb{X}}(t) & t < \sigma(\mathbb{X}) - 1 \\ s - 1 & t \geq \sigma(\mathbb{X}) \end{cases}$$

This means exactly that the degree of this point P in $\mathbb{X}$ is $\alpha = \sigma(\mathbb{X}) - 1$. □

3.2 Socle-permissible values.

We have already said (see Remark 2.2) that the possible values for the conductor degrees in a set of points $\mathbb{X} \in \mathbb{P}^n$ with Hilbert function H can be deduced from H.

We will now show that, if $\mathbb{X} \subseteq \mathbb{P}^2$, they can be also read from the last term of a minimal free resolution of the homogeneous coordinate ring of $\mathbb{X}$, $R/I_{\mathbb{X}}$. First we recall some definitions and some general results of homological algebra (see [9], sec. 1.1).

Let I be an homogeneous ideal of $R = K[x_0, \ldots, x_n]$ and let $A = R/I$. If we denote by h the homological dimension of A, then a minimal free resolution of A is:

$$0 \to F_h \to \ldots \to F_1 \to R \to A \to 0$$

where the F_j's are free R-modules, i.e. $F_j = \bigoplus R(-k)^{\beta_{jk}}$. The exponents β_{jk} are invariants of A which are called the *graded Betti numbers of* A. Starting from the graded Betti numbers of A, the *Betti numbers*, β_j, are defined as follows:

$$\beta_j = \sum_k \beta_{jk}.$$

Definition 3.2. Let $A = R/I$ be a Cohen-Macaulay ring and $L_1, \ldots, L_g$ be a maximal R-sequence in R_1 on A. The artinian ring $B = A/(\bar{L}_1, \ldots, \bar{L}_g)$ is called an *artinian reduction* of A.

Remark 3.3. By an important result of homological algebra ([9], Th. 1.3.6), we know that the graded Betti numbers in a minimal free resolution of a Cohen-Macaulay ring A (as R-module) do not change when we consider any artinian reduction of A.

We now suppose that J is an artinian homogeneous ideal of $S = K[x_0, \ldots, x_t]$ and $B = S/J$. Then

$$B = K \oplus B_1 \oplus \ldots \oplus B_p \quad \text{where } B_p \neq 0$$

Let $\mathfrak{M} = B_1 \oplus \ldots \oplus B_p$ denote the maximal homogeneous ideal of B.

Definition 3.4. The *socle of B*, denoted $soc(B)$, is:

$$soc(B) := Ann_B(\mathfrak{M}).$$

It is well known that, if J is an artinian homogeneous ideal in S and $B = S/J$, then the rank (as a S-module) of the last free module in a minimimal free resolution of B is the dimension of the socle of B as K vector space, i.e.

$$\beta_{t+1} = dim_K(soc(B)).$$

Moreover, under the same hypothesis,

Remark 3.5. The socle of B has $\beta_{t+1,k}$ minimal generators of degree $k-(t+1)$.

We now give the following

Definition 3.6. Let I be a homogeneous Cohen-Macaulay ideal in $R = K[x_0, \ldots, x_n]$ and $A = R/I$. We call the *socle-permissible values for A* the distinct degrees of the minimal generators of the socle of any artinian reduction of A.

Notice that the Definition 3.6 works thanks to Remark 3.3. Moreover, by Remark 3.5, the Definition 3.6 can be restated to say that $l \in \mathbb{N}$ is a socle-permissible value for A if and only if $R(-(l+n))$ appears in the last term of a minimal free resolution of A.

Remark 3.7. Let $\mathbb{X}$ be a set of s distincts points in $\mathbb{P}^n$ and let $H_{\mathbb{X}}$ be the Hilbert function of $\mathbb{X}$. We know that the function $\delta H_{\mathbb{X}}$ is the Hilbert function of any artinian reduction of $A = R/I_{\mathbb{X}}$. Let B denote an artinian reduction of A. If we write:

$$B = K \oplus B_1 \oplus \ldots \oplus B_p \quad \text{where } B_p \neq 0$$

then, by the obvious inclusion $B_p \subseteq soc(B)$, we obtain that p is a socle-permissible value for A. It is immediate to see that $p = \sigma(\mathbb{X}) - 1$, so p is the maximum socle-permissible value for A and it coincides with the maximum permissible value for $H_{\mathbb{X}}$.

Let's apply these observations to an example.

Example 2. Let $\mathbb{X}$ be a set of points defined by the saturated ideal

$$I_{\mathbb{X}} = (x(x-z), xy, y(y-z)(y-2z)) \subseteq K[x, y, z].$$

Let A be the homogeneous coordinate ring of $\mathbb{X}$ and let $B = A/(z)$ be an artinian reduction of A.

So, if we denote by $S = K[x, y]$, we have $B = S/(x^2, xy, y^3)$ and the minimal free resolution of B as a S-module is

$$0 \to S(-4) \oplus S(-3) \to S(-3) \oplus S(-2)^2 \to S \to B \to 0$$

From this resolution we see that $dim_K(soc(B)) = 2$ with minimal generators in degrees $4-2 = 2$ and $3-2 = 1$. So the socle-permissible values are exactly 2 and 1.

In the following, we specialize our discussion to sets of points in $\mathbb{P}^2$ and denote again by $R = K[x, y, z]$ the usual homogeneous coordinate ring.

Lemma 3.8. *Let $\mathbb{X}$ be a finite set of distinct points in $\mathbb{P}^2$. Let F be a separator of minimal degree for the point $P \in \mathbb{X}$. Then the ideal $(I_{\mathbb{X}}, F) \subseteq R$ is saturated.*

Proof. Let deg $(F) = \alpha$.

We can construct the following exact sequence:

$$(1) \qquad 0 \to R/(I_{\mathbb{X}} : F)(-\alpha) \xrightarrow{\cdot F} R/I_{\mathbb{X}} \to R/(I_{\mathbb{X}}, F) \to 0$$

where the first map is multiplication by F and the second is the canonical map onto a quotient. We notice that $(I_{\mathbb{X}} : F) = \wp$ is the ideal of the separated point P and the saturation of the ideal $(I_{\mathbb{X}}, F)$ is the ideal of the remaining points $\mathbb{X}' = \mathbb{X} \setminus \{P\}$. We know that $(I_{\mathbb{X}}, F) \subseteq \mathrm{sat}(I_{\mathbb{X}}, F)$. In order to prove the other inclusion, it is enough to show both ideals have the same Hilbert function.

By Lemma 2.3 of [6], we have

$$H_{R/\mathrm{sat}(I_{\mathbb{X}},F)}(t) = H_{\mathbb{X}'}(t) = \begin{cases} H_{\mathbb{X}}(t) & t < \alpha \\ H_{\mathbb{X}}(t) - 1 & t \geq \alpha \end{cases}$$

On the other hand, from the exact sequence (1) we get that

$$H_{R/(I_{\mathbb{X}},F)}(t) = H_{\mathbb{X}(t)} - H_P(t-\alpha) = \begin{cases} H_{\mathbb{X}}(t) & t < \alpha \\ H_{\mathbb{X}}(t) - 1 & t \geq \alpha \end{cases}$$

We see that the Hilbert functions are the same, so the two ideals have to coincide. □

Theorem 3.9. *Let $\mathbb{X}$ a finite set of distinct points in $\mathbb{P}^2$. Let*

$$0 \to \bigoplus_{j \in B_2} R(-j)^{\beta_{2j}} \to \bigoplus_{k \in B_1} R(-k)^{\beta_{1k}} \to R \to R/I_{\mathbb{X}} \to 0$$

be a graded minimal free resolution of $R/I_{\mathbb{X}}$. Then for any point $P \in \mathbb{X}$ the degree of P in $\mathbb{X}$, α, is a socle-permissible value for $R/I_{\mathbb{X}}$ i.e.

$$\alpha + 2 \in B_2$$

where B_2 is the set of syzygies of the module $R/I_{\mathbb{X}}$.

Proof. Let F be a separator of deg $(F) = \alpha$ for the point $P \in \mathbb{X}$.

We recall that $(I_{\mathbb{X}} : F) = \wp$ is the ideal of the separated point P and $(I_{\mathbb{X}}, F)$ is the ideal of the remaining points $\mathbb{X}' = \mathbb{X} \setminus \{P\}$ by Lemma 3.8. Applying the "mapping cone" construction (cfr [9], sec. 1.1) to the resolutions of $R/I_{\mathbb{X}}$ and $R/(I_{\mathbb{X}} : F) = R/\wp$, and from the sequence (1), we get a resolution for the ideal $R/(I_{\mathbb{X}}, F)$, namely:

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & 0 & & 0 & & & R(-\alpha-2) & & \\
 & \downarrow & & \downarrow & & & \downarrow & & \\
 & R(-\alpha-2) & & \bigoplus_j R(-j)^{\beta_{2j}} & & & R(-\alpha-1)^2 \oplus \bigoplus_j R(-j)^{\beta_{2j}} & & \\
 & \downarrow & & \downarrow & & & \downarrow & & \\
 & R(-\alpha-1)^2 & & \bigoplus_k R(-k)^{\beta_{1k}} & & & R(-\alpha) \oplus \bigoplus_k R(-k)^{\beta_{1k}} & & \\
 & \downarrow & & \downarrow & & & \downarrow & & \\
 & R(-\alpha) & & R & & & R & & \\
 & \downarrow & & \downarrow & & & \downarrow & & \\
0 \longrightarrow & R/(-\alpha) & \xrightarrow{\cdot F} & R/I_{\mathbb{X}} & & \longrightarrow & R/(I_{\mathbb{X}}, F) & \longrightarrow & 0
\end{array}
$$

This construction gives a resolution that, generally, is not minimal, but since $(I_{\mathbb{X}}, F)$ is saturated, the resolution of $(I_{\mathbb{X}}, F)$ must be of length 2. This means that the term $R(-\alpha - 2)$ must cancel with something in $R(-\alpha - 1)^2 \oplus \bigoplus_j R(-j)^{\beta_{2j}}$, that is $R(-\alpha - 2)$ must be one of the $R(-j)$'s, i.e. $\alpha + 2 \in B_2$ and we are done. □

Example 3. We first determine the two minimal free resolutions associated to the sets of points $\mathbb{X}_1$ and $\mathbb{X}_2$ represented in Figure 2 and in Figure 3.
We obtain, for the first configuration of points, $\mathbb{X}_1$ (Figure 2)

$$0 \to R(-7) \oplus R(-6) \oplus R(-4) \to R(-6) \oplus R(-5) \oplus R(-3)^2 \to R \to R/I_{\mathbb{X}_1} \to 0$$

and so the socle-permissible values for $R/I_{\mathbb{X}_1}$ are exactly $7 - 2 = 5$, $6 - 2 = 4$ and $4 - 2 = 2$. For the second configuration of points, $\mathbb{X}_2$ (Figure 3), we have

$$0 \to R(-7) \oplus R(-4) \to R(-5) \oplus R(-3)^2 \to R \to R/I_{\mathbb{X}_2} \to 0$$

and so the socle-permissible values for $R/I_{\mathbb{X}_2}$ are $7 - 2 = 5$ and $4 - 2 = 2$.

We observe that in both cases illustrated in the above example, for any socle-permissible value, α, for $R/I_{\mathbb{X}_i}$, $(i = 1, 2)$, there exists at least one point $P \in \mathbb{X}_i$ such that the degree of P in $\mathbb{X}_i$ is exactly α. This is not true in general, as the following example shows.

Example 4. We consider the set $\mathbb{X}$ of 7 distinct points in $\mathbb{P}^2$ chosen such that no three of them lie on a line, nor six of them lie on a conic. Using the CoCoA system (see [1]), we compute the graded minimal free resolution of $R/I_{\mathbb{X}}$:

$$0 \to R(-5) \oplus R(-4) \to R(-3)^3 \to R \to R/I_{\mathbb{X}} \to 0$$

By Definition 3.6, we know that the socle-permissible values for $R/I_{\mathbb{X}}$ are $5-2=3$ and $4-2=2$.

We know that the maximum value, 3, always appears, so we have only to show that the degree of P_i cannot be 2 for $i = 1, \dots, 7$. But it is true since no six points are on a conic, so the only degree that appears is 3.

4. A particular case: the lex-configuration.

In this section we prove that given a set of points $\mathbb{X}$ in a particular configuration, that we'll call the *lex-configuration*, all socle-permissible values for $R/I_{\mathbb{X}}$ do appear.

This case specializes the more general one of permissible values discussed in [6], where the authors give (in Theorem 4.3) a positive answer to the question of finding a configuration realizing all permissible values.

We now give some basic definitions.

Definition 4.1. Let $\prec$ denote the lexicographic order on the monomials in n indeterminates $\{x_1, \dots, x_n\}$. We say that a set of monomials $M \subseteq K[x_1, \dots, x_n]$ of degree d is a *lex-segment* if

$$t \in M \text{ and } t' \succ t \Longrightarrow t' \in M.$$

Definition 4.2. Let $J \in K[x_1, \dots, x_n]$ be a monomial ideal. We say that J is a *lex-segment ideal* if the monomial basis of J_d is a lex-segment for all $d \in \mathbb{N}$.

As a matter of notation, we denote by $\bar{J}$ the ideal obtained by applying the Hartshorne lifting procedure (see [3]) to a monomial ideal J.

Definition 4.3. Let I be an artinian lex-segment ideal in $K[x_1, \dots, x_n]$. We call a *lex-configuration* the zero-dimensional projective scheme $\subseteq {}^n$ associated to the ideal $\bar{I}$.

Example 5. We consider the artinian lex-segment ideal

$$I = (x^3, x^2y, xy^4, y^6) \subseteq K[x, y].$$

The lifting ideal we want is:

$$\bar{I} = (x(x-z)(x-2z), xy(x-z), xy(y-z)(y-2z)(y-3z), \\ y(y-z)(y-2z)(y-3z)(y-4z)(y-5z)) \subseteq K[x, y, z].$$

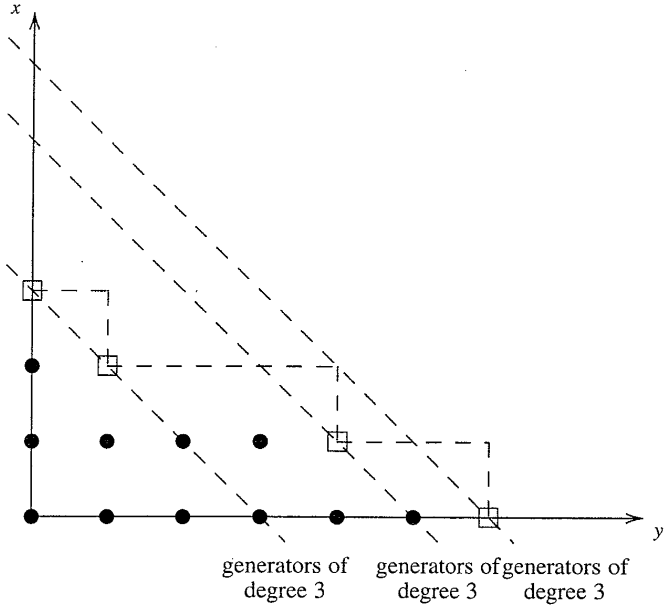


Figure 4: Costruction of a lex-configuration

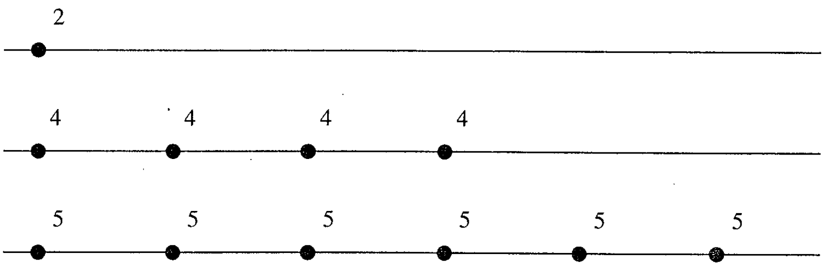


Figure 5: Values of conductor degrees and notations for a lex-configuration

Let $\mathbb{X}$ be the set of points associated to $\bar{I}$. In Figure 4 we represent the artinian ring $K[x, y]/I$ (the squares indicate the minimal generators of I). The lex-configuration is given by Figure 5. If we compute the conductor sequence of $A = R/\bar{I}$, we can notice that the degree in $\mathbb{X}$ of all the points lying on the same horizontal line is the same (see Figure 5). This is a general property of lex-configurations as Proposition 4.4 shows.

Proposition 4.4. *Let $\mathbb{X}$ be a lex-configuration in $\mathbb{P}^2$. We suppose that the points of $\mathbb{X}$ are placed on m horizontal lines l_k, $k = 1, \dots, m$ (as in Figure 5) and we denote by n_k the number of points on the k-th line, $n_1 > n_2 > \dots > n_m \neq 0$. For any point $P \in l_k$, let α_k denote the degree of P in $\mathbb{X}$. Then*

$$\alpha_k = n_k + k - 2 \qquad k = 1, \dots, m$$

Proof. This is a recursive proof based on iterated applications of Bézout's Theorem.

1. Let F be a separator of degree d for the point P on the k-th horizontal line. We prove that $d \geq n_k + k - 2$. By hypothesis $n_m < n_{m-1} < \dots < n_1$, so we have

   $$(2) \qquad n_k \leq n_{k-1} - 1 \leq n_{k-2} - 2 \leq \dots \leq n_2 - (k-2) \leq n_1 - (k-1)$$

   For a contradiction we suppose $d = \deg(F) < n_k + k - 2$. By (2) we obtain $n_k + (k-2) \leq n_2 < n_1$, so $d < n_1$. Since F intersects the line l_1 (of equation $L_1 = 0$) in $n_1 > d$ points, by Bézout Theorem it factors as $F = G_1 L_1$, where $\deg(G_1) = d - 1$. Now, for (2) $n_k + (k-3) \leq n_3 < n_2$, so $d - 1 < n_2$ and by Bézout Theorem G_1 factors as $G_1 = L_2 G_2$. By the same reasoning, we get that $F = L_1 L_2 \dots L_{k-1} G_{k-1}$, where $\deg(G_{k-1}) = d - (k-1)$ and G_{k-1} vanishes on all points of $\mathbb{X} \setminus \{P\}$ except those lying on l_i, for $i < k$. Again, since $d < n_k + k - 2 = n_k + (k-1) - 1$, then $\deg(G_{k-1}) < n_k - 1$, so G_{k-1} has to vanish on the line l_k and therefore on P, too. Contradiction. We conclude that $d \geq n_k + k - 2$.

2. There exists a separator of degree exactly $n_k + k - 2$. We first consider the curve of equation $G = 0$ obtained as the union of the horizontal lines $l_1, \dots, l_{k-1}$, $G = L_1 L_2 \cdots L_{k-1}$. Then we will prove that there exists a form F of degree $n_k - 1$ which vanishes at all points of $\mathbb{X} \setminus \{P\}$ except those lying on the horizontal lines l_i's ($i = 1, \dots, k-1$). We observe that FG is a separator for P and that the degree of FG is exactly $n_k + k - 2$, so we are done. We only need to prove the existence of the form F of degree $n_k - 1$. To do this, we notice that the number of points in the set $\mathbb{X} \setminus \bigcup_{i=1}^{k-1} l_i$

is $\sum_{j=1}^{n_k} j = \frac{n_k(n_k+1)}{2}$. So F has to vanish on $\frac{n_k(n_k+1)}{2} - 1$ points. But we know that $dim_K R_{n_k-1} = \binom{n_k+1}{2}$, thus we conclude that there always exists a form F of degree $n_k - 1$ such that F vanishes on all points of $\mathbb{X}$ except those lying on the horizontal lines l_i's $(i = 1, \ldots, k-1)$. It is trivial to prove that F cannot vanish on the point P. So FG is a separator for P of degree $n_k + k - 2$.

As the proof is independent of the choice of the point P on l_k, we conclude that all points lying on the k-th line have the same degree in $\mathbb{X}$. □

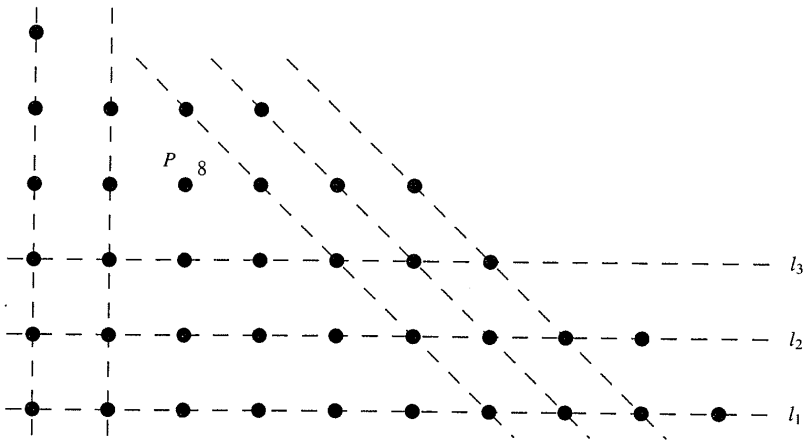


Figure 6: Example of separator

Remark 4.5. Let $\mathbb{X} \subseteq \mathbb{P}^2$ be a lex-configuration. So $\mathbb{X}$ is the zero-dimensional projective scheme associated to the ideal $\bar{I}$ obtained by applying the Hartshorne lifting procedure to an artinian lex-segment ideal $I \subseteq K[x, y]$. We suppose that the points of $\mathbb{X}$ are placed on m lines l_k, $k = 1, \ldots, m$ (as in Figure 6) and we denote by n_k the number of points on the k-th line. Recall that $n_1 > n_2 > \ldots > n_m$. We observe, (see Figure 4), that the degrees of generators of the lex-segment ideal I determine the number of collinear points in the associated lex-configuration: in fact, if $x^p y^q$ is a minimal generator of I, then on the $(p+1)$-th horizontal line are placed exactly q points. By Proposition 4.4, it follows that the degrees of the points P's belonging to the $(p+1)$-th horizontal line is:

$$\alpha_{p+1} = q + (p+1) - 2 = p + q - 1$$

i.e. for any minimal generator of I of degree d there exists at least one point $P \in \mathbb{X}$ such that the degree of P in $\mathbb{X}$ is exactly $d-1$.

Proposition 4.6. *Let $\mathbb{X} \subseteq \mathbb{P}^2$ be a lex configuration. Then all the socle-permissible values for $R/I_{\mathbb{X}}$ occur in the conductor sequence of $R/I_{\mathbb{X}}$.*

Proof. From a result of Eliahou-Kervaire ([2]), the Betti numbers for a lex-configuration are known and come from the degrees of the generators of the associated lex-segment ideal. Namely, let G be the set of the generators of the lex-segment ideal J and let $G^* = G \setminus \{x^d\}$ ($x^d \in G$ by definition of lex-segment). Then [2] proved:

- $\beta_{1i} = |G_i|$
- $\beta_{2,i+1} = |G_i^*|$

where by G_i we mean $G \cap R_i$.
So a graded minimal free resolution of $R/I_{\mathbb{X}}$ is the following:

$$0 \to \bigoplus_i R(-i-1)^{\beta_{2,i+1}} \to \bigoplus_i R(-i)^{\beta_{1i}} \to R \to R/I_{\mathbb{X}} \to 0 \tag{3}$$

We observe that if α is a socle-permissible value, then $R(-\alpha-2)$ appears in the last free module of the resolution (3), i.e. $\beta_{2,\alpha+2} \neq 0$. By [2] $\beta_{1,\alpha+1} \neq 0$ too, so there exists a minimal generator of J of degree $\alpha+1$. By Remark 4.5 we have that there exists at least one point P such that the degree of P in $\mathbb{X}$ is exactly α.

Remark 4.7. The lex-configuration defined above is essentially the same as the one constructed in Th. 3.3 of [6]. There the authors proved that this configuration realizes all permissible values. Their result, together with the Proposition 4.6, allows us to conclude that in the case of a lex-configuration, the set of socle-permissible values and the set of permissible values coincide.

5. Open problems.

As we showed in Example 4, there exist sets of points $\mathbb{X} \subseteq \mathbb{P}^2$ for which not all socle-permissible values for $R/I_{\mathbb{X}}$ are realized by a point $P \in \mathbb{X}$. So a first problem in this case is to know whether it is possible to find a configuration $\mathbb{X}'$ such that the set of the socle-permissible values for $R/I_{\mathbb{X}'}$ coincides with the set of the socle-permissible values for $R/I_{\mathbb{X}}$, but such that any socle-permissible value for $R/I_{\mathbb{X}'}$ is realized by at least one point $P \in \mathbb{X}'$.

As a matter of notations, we call $\mathcal{S}_{\mathbb{X}}$ the set of the socle-permissible values for $R/I_{\mathbb{X}}$.

More precisely,we consider the following

Problem. *Given a set of distinct points $\mathbb{X}$ in $\mathbb{P}^2$, find a configuration of points $\mathbb{X}'$ in $\mathbb{P}^2$ such that $\mathcal{S}_{\mathbb{X}} = \mathcal{S}_{\mathbb{X}'}$ and such that for any socle-permissible value for $R/I_{\mathbb{X}'}$, α, there exists at least one point $P \in \mathbb{X}'$ such that the degree of P in $\mathbb{X}'$ is exactly α.*

In Section 4, Proposition 4 solves the problem when $\mathbb{X}$ is a lex-configuration. In the following we give a positive answer in another particular case, that is $\mathbb{X} \subseteq \mathbb{P}^2$ a set of generic points.

5.1 The case of generic points.

Let $\mathbb{X}$ be a set of s generic distinct points in $\mathbb{P}^2$. Let d be the minimal integer such that

$$s = \binom{d+1}{2} + k = \binom{d+2}{2} - (d-k+1)$$

where $0 \leq k < d+1$.

It is known (see [5]) that the ideal $I_{\mathbb{X}}$ can be minimally generated by forms of degrees d and $d+1$, or only by forms of degree d, according to whether $d < 2k$ or not. Moreover, by the Hilbert-Burch Theorem, these forms can be described as the maximal minors of a $\rho \times (\rho+1)$ matrix (called the Hilbert-Burch matrix of $I_{\mathbb{X}}$) whose entries are linear forms L_{ij} and forms Q_{lm} of degree 2.

The integer ρ depends on the values of d and k. We have two possibilities:

| case $d < 2k$ |

In this case $\rho = k$ and the matrix has the following structure

$$M = \begin{pmatrix} L_{1,1} & \dots & L_{1,2k-d} & Q_{1,1} & \dots & Q_{1,d-k+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ L_{k,1} & \dots & L_{k,2k-d} & Q_{k,1} & \dots & Q_{k,d-k+1} \end{pmatrix}$$

The corresponding degree matrix is

$$\partial M = \begin{pmatrix} 1 & \dots & 1 & 2 & \dots & 2 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \dots & 1 & 2 & \dots & 2 \end{pmatrix}$$

from which the minimal free resolution is

$$0 \to R(-d-2)^k \xrightarrow{M} R(-d-1)^{2k-d} \oplus R(-d)^{d-k+1} \to R \to R/I_{\mathbb{X}} \to 0$$

$\boxed{\text{case } d \geq 2k}$

In this case, $\rho = d - k$ and the ideal is generated (only in degree d), by the minors of the following Hilbert-Burch matrix:

$$M = \begin{pmatrix} Q_{1,1} & \dots & Q_{1,d-k+1} \\ \vdots & \ddots & \vdots \\ Q_{k,1} & \dots & Q_{k,d-k+1} \\ L_{1,1} & \dots & L_{1,d-k+1} \\ \vdots & \ddots & \vdots \\ L_{d-2k,1} & \dots & L_{d-2k,d-k+1} \end{pmatrix}$$

and the degree matrix is

$$\partial M = \begin{pmatrix} 2 & \dots & 2 \\ \vdots & \ddots & \vdots \\ 2 & \dots & 2 \\ 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix}$$

The minimal free resolution now is

$$0 \to R(-d-2)^k \oplus R(-d-1)^{d-2k} \xrightarrow{M} R(-d)^{d-k+1} \to R \to R/I_{\mathbb{X}} \to 0$$

At this point, we can prove the following

Proposition 5.1. *Let $\mathbb{X} \subseteq \mathbb{P}^2$ be a set of s generic points. Then there exists a set of points $\mathbb{X}'$ such that $S_{\mathbb{X}} = S_{\mathbb{X}'}$ and such that all socle-permissible values for $R/I_{\mathbb{X}'}$ occur in the conductor sequence of $R/I_{\mathbb{X}'}$.*

Proof. Let d be the minimal integer such that

$$s = \binom{d+1}{2} + k = \binom{d+2}{2} - (d - k + 1)$$

where $0 \leq k < d + 1$. We distinguish the two cases.

First case: $d < 2k$

From the minimal free resolution

$$0 \to R(-d-2)^k \to R(-d-1)^{2k-d} \oplus R(-d)^{d-k+1} \to R \to R/I_{\mathbb{X}} \to 0$$

we see that there is only one socle-permissible value for $R/I_{\mathbb{X}}$, that is $d = \sigma(\mathbb{X}) - 1$. Since we know from Proposition 3.1 that there exists at least one point $P \in \mathbb{X}$ such that the degree of P in $\mathbb{X}$ is $\sigma(\mathbb{X}) - 1$, we can conclude that $\mathbb{X}$ has the required property.

Second case: $d \geq 2k$

From the minimal free resolution

$$0 \to R(-d-2)^k \oplus R(-d-1)^{d-2k} \to R(-d)^{d-k+1} \to R \to R/I_{\mathbb{X}} \to 0$$

there are two socle-permissible values for $R/I_{\mathbb{X}}$, i.e

$$\mathcal{S}_{\mathbb{X}} = \{d-1, d\}.$$

In order to find a configuration $\mathbb{X}'$ realizing all socle-permissible values, the idea is to construct a monomial ideal, starting from the degree matrix, and to apply to this ideal the Hartshorne lifting procedure.

We see in details how to proceed. Consider the following matrix having $d - 2k$ rows and $d - k + 1$ columns:

$$M' = \begin{pmatrix} x^2 & y^2 & 0 & \dots & & & & 0 \\ 0 & x^2 & y^2 & 0 & & \dots & & 0 \\ \vdots & \ddots & \ddots & \ddots & & & & \vdots \\ 0 & & 0 & x^2 & y^2 & 0 & & 0 \\ 0 & & & 0 & x & y & 0 & \vdots \\ \vdots & & & & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & & & & 0 & x & y \end{pmatrix}$$

obtained by putting on the main diagonals the pure powers of x and y to the degrees given by the degree matrix ∂M and putting 0 on the remaining entries.

Let's consider the monomial ideal J generated by the maximal minors of the matrix M' and show that the configuration $\mathbb{X}'$ obtained lifting this ideal is the requested one. First of all, we observe that $\mathcal{S}_{\mathbb{X}'} = \mathcal{S}_{\mathbb{X}}$. So it is enough to show that there exists at least one point $P \in \mathbb{X}'$ such that the degree of P in $\mathbb{X}'$ is $d - 1$ (Proposition 3.1 tells us that there exists one point $Q \in \mathbb{X}'$ such that the degree of Q in $\mathbb{X}'$ is d).

We observe that $x^{d-1}y$ and x^d belong to J, because they are the minors obtained by deleting, respectively, the $(d-k)$-th and the $(d-k+1)$-th column from the matrix M'. So by lifting the ideal J, we obtain a set of points $\mathbb{X}'$ placed on d horizontal lines (see Figure 7). In particular, there always exists one line

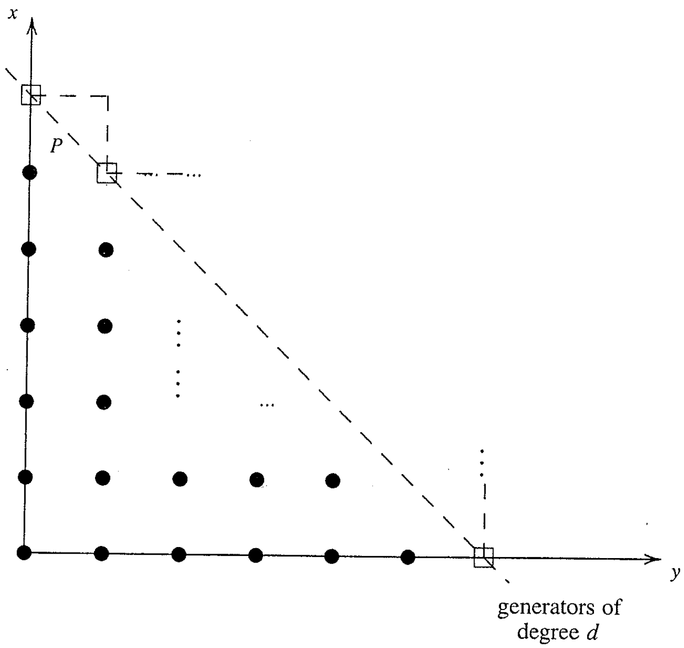


Figure 7

on which there is only one point P. It is trivial to prove that the degree of such P in $\mathbb{X}'$ is exactly $d-1$. $\square$

We make the construction clear by an example.

Example 6. Let's consider a set $\mathbb{X}$ of $7 = \binom{4}{2} + 1$ generic points. In this case, with the previous notations, we have $d = 3$ and $k = 1$ so $d > 2k$. The degree matrix is:

$$\partial M = \begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

So a minimal resolution is

$$0 \to R(-5) \oplus R(-4) \to R(-3)^3 \to R \to R/I_{\mathbb{X}} \to 0$$

and we get that the socle-permissible values for $R/I_{\mathbb{X}}$ are 3 and 2. We have already seen (Example 4) that the degree of P in $\mathbb{X}$ is 3 for any $P \in \mathbb{X}$. As we did in the previous Proposition, we construct the matrix

$$M' = \begin{pmatrix} x^2 & y^2 & 0 \\ 0 & x & y \end{pmatrix}$$

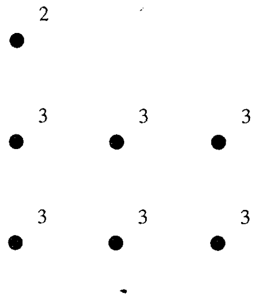
which gives us the monomial ideal:

$$J = (x^3, x^2y, y^3) \subseteq K[x, y]$$

The lifted ideal we want is:

$$\bar{J} = (x(x-z)(x-2z), xy(x-z), y(y-z)(y-2z)) \subseteq K[x, y, z]$$

and the set of points $\mathbb{X}'$ associated to $\bar{J}$ is shown in the following figure:



We notice that the set of points $\mathbb{X}'$ realizes all socle-permissible values for R/J.

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