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LINEAR SPACES OF SYMMETRIC MATRICES WITH NON-MAXIMAL MAXIMUM LIKELIHOOD DEGREE

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We study the maximum likelihood degree of linear concentration models in algebraic statistics. We relate the geometry of the reciprocal variety to that of semidefinite programming. We show that the Zariski closure in the Grassmannian of the set of linear spaces that do not attain their maximal possible maximum likelihood degree coincides with the Zariski closure of the set of linear spaces defining a projection with non-closed image of the positive semidefinite cone. In particular, this shows that this closure is a union of coisotropic hypersurfaces.

1. Introduction

Maximum likelihood estimation is a widespread optimization approach to fit empirical data to a statistical model. The *maximum likelihood degree* (or short, *ML-degree*) of a model is the number of complex critical points of this optimization problem for generic empirical data [2]. The aim of this paper is to study models whose actual maximum likelihood degree differs from the expected one.

The models we consider are sets of multivariate Gaussian distributions with mean zero that are linear in the space of concentration matrices. The concentration matrices of such a model form a spectrahedron, that is the intersection of a

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linear subspace $\mathcal{L}$ of the space $\mathbb{S}^n$ of symmetric $n \times n$ matrices with the cone of positive definite matrices.

Since the ML-degree counts the generic number of *complex* critical points, it is not just an invariant of the model itself, but also of its complex Zariski closure. In fact, the ML-degree is well-defined for any linear subspace $\mathcal{L}$ of the space $\mathbb{S}^n$ of *complex* symmetric $n \times n$ matrices.

The ML-degree of a generic subspace $\mathcal{L} \subset \mathbb{S}^n$ is known to be the degree of its *reciprocal variety* that is parametrized by the inverses of all matrices in $\mathcal{L}$ (see [17, Theorem 2.3]).

In fact, the ML-degree of any linear subspace $\mathcal{L} \subset \mathbb{S}^n$ is upper bounded by the degree of its reciprocal variety (see Theorem 1.1). We say that a linear subspace $\mathcal{L} \subset \mathbb{S}^n$ whose ML-degree equals the degree of its reciprocal variety is *ML-maximal*.

1.1. Main results

We fix the bilinear pairing $(X,Y) \mapsto \mathrm{trace}(XY)$ on the vector space $\mathbb{S}^n$ of complex symmetric $n \times n$ matrices. This is an inner product when restricting to the real symmetric matrices. For a linear subspace $\mathcal{L} \subset \mathbb{S}^n$, we write

$$\mathcal{L}^{\perp} := \{Y \in \mathbb{S}^n \mid \mathrm{trace}(XY) = 0 \text{ for all } X \in \mathcal{L}\}$$

for its *annihilator* or *polar linear space* with respect to the trace pairing. Moreover, we consider the Zariski closure

$$\mathcal{L}^{-1} := \overline{\{X^{-1} \mid X \in \mathcal{L},\ \mathrm{rank}(X) = n\}} \subset \mathbb{S}^n$$

and call its projectivization $\mathbb{P}\mathcal{L}^{-1} \subset \mathbb{P}\mathbb{S}^n$ the reciprocal variety of $\mathcal{L}$. The definition of $\mathcal{L}^{-1}$ makes sense if $\mathcal{L}$ contains at least one full-rank matrix. We call such a linear space $\mathcal{L}$ *regular*. We provide the following exact characterization of ML-maximal linear spaces in terms of the intersection of their reciprocal varieties and their polar spaces. A formula for the ML-degree of $\mathcal{L}$ in terms of Segre classes of this intersection is given in [1].

Theorem 1.1. *The ML-degree of a linear subspace $\mathcal{L} \subset \mathbb{S}^n$ is at most the degree of its reciprocal variety. Moreover, $\mathcal{L}$ is ML-maximal (i.e., its ML-degree equals $\deg(\mathbb{P}\mathcal{L}^{-1})$) if and only if $\mathcal{L}^{-1} \cap \mathcal{L}^{\perp} = \{0\}$.*

In addition, we give alternative sufficient and necessary conditions for a linear space to be ML-maximal; see Remark 4.10.

Since generic linear spaces of symmetric matrices are ML-maximal, we want to study the fine structure of the complementary property. For integers k and n, we are interested in the set of all k-dimensional linear subspaces of $\mathbb{S}^n$

that are *not* ML-maximal. This subset of the Grassmannian $\mathrm{Gr}(k,\mathbb{S}^n)$ is neither Zariski closed nor open (see Remark 4.1). Therefore, we study its Zariski closure $\mathrm{NM}_{k,n}$ in $\mathrm{Gr}(k,\mathbb{S}^n)$.

We show that this variety equals the *bad locus* $\mathrm{Bad}_{k,n}$ studied in [8, Section 3]. The authors consider k-dimensional linear subspaces $\mathcal{L}$ of the real symmetric $n\times n$ matrices $\mathbb{S}^n_{\mathbb{R}}$ and corresponding projections $\mathbb{S}^n_{\mathbb{R}} \to \mathrm{Hom}(\mathcal{L},\mathbb{R})$ dual to the inclusion $\mathcal{L}\subset \mathbb{S}^n_{\mathbb{R}}$. Such a linear space $\mathcal{L}$ is called *bad* if the image of the cone of positive semidefinite matrices under the projection $\mathbb{S}^n_{\mathbb{R}} \to \mathrm{Hom}(\mathcal{L},\mathbb{R})$ is not closed. Bad subspaces of $\mathbb{S}^n_{\mathbb{R}}$ are those for which strong duality in semidefinite programming fails, which has been thoroughly studied by Pataki [11, 14–16]. The bad locus $\mathrm{Bad}_{k,n}$ is the Zariski closure in $\mathrm{Gr}(k,\mathbb{S}^n)$ of the set of k-dimensional bad subspaces of $\mathbb{S}^n_{\mathbb{R}}$.

Theorem 1.2. *The Zariski closure* $\mathrm{NM}_{k,n}$ *in* $\mathrm{Gr}(k,\mathbb{S}^n)$ *of the set of non-ML-maximal k-dimensional linear subspaces of* $\mathbb{S}^n$ *equals the bad locus* $\mathrm{Bad}_{k,n}$.

We describe the irreducible components of $\mathrm{NM}_{k,n}$ in terms of the determinantal varieties D_s of matrices of rank at most s. The *coisotropic variety* in $\mathrm{Gr}(k,\mathbb{S}^n)$ associated to D_s is the Zariski closure of the set of all k-dimensional linear subspaces of $\mathbb{S}^n$ that intersect D_s at some smooth point non-transversely. For all s as in the following corollary, the coisotropic variety associated to D_s has codimension one in $\mathrm{Gr}(k,\mathbb{S}^n)$ [9, Corollary 6].

Corollary 1.3. *The non-ML-maximal locus* $\mathrm{NM}_{k,n}$ *is the union of the coisotropic hypersurfaces in* $\mathrm{Gr}(k,\mathbb{S}^n)$ *associated to the determinantal varieties* D_s*, where s ranges over the integers such that* $\binom{n-s+1}{2} < k \le \binom{n+1}{2} - \binom{s+1}{2}$.

In particular, a generic linear space $\mathcal{L}\in \mathrm{Gr}(k,\mathbb{S}^n)$ *is ML-maximal.*

Proof. By [8, Theorem 2], the bad locus $\mathrm{Bad}_{k,n}$ is the union of the coisotropic hypersurfaces described above. Hence, Theorem 1.2 implies the assertion. □

Remark 1.4. Theorem 1.2 and Corollary 1.3 provide a geometric proof for the ML-maximality of generic linear spaces of symmetric matrices. An alternative argument, using different techniques from commutative algebra, is given in [17, Theorem 2.3]. ML-maximality for generic linear spaces has also been conjectured in a more general setting in [13, Conjecture 5.8], and a positive answer has since been known to follow from a result of Teissier. However, this has not been written down in the current literature, and therefore we include this argument in Section 3. We also note that Theorem 1.1 is in fact a special case of [13, Theorem 5.5], but we provide a more detailed argument.

We prove Theorem 1.1 in Section 2 and Theorem 1.2 in Sections 5 and 6.

2. Maximum likelihood estimation

In this section we prove Theorem 1.1. Let $\mathbb{S}^n_{\mathbb{R}}$ be the space of real symmetric $n \times n$ matrices. The log-likelihood function of a linear subspace $\mathcal{L}_{\mathbb{R}} \subset \mathbb{S}^n_{\mathbb{R}}$ and a sample covariance matrix $S \in \mathbb{S}^n_{\mathbb{R}}$ is

$$\begin{aligned} \ell_S : \mathcal{L}_{\mathbb{R}} &\longrightarrow \mathbb{R}, \\ X &\longmapsto \log\det(X) - \operatorname{trace}(SX). \end{aligned}$$

The gradient of the log-likelihood at a matrix $X \in \mathcal{L}_{\mathbb{R}}$ is $\nabla_X \ell_S = X^{-1} - S$, so X is a critical point of the log-likelihood if and only if $X^{-1} - S \in \mathcal{L}^{\perp}_{\mathbb{R}}$. Although the log-likelihood itself is only well-defined on the real space $\mathcal{L}_{\mathbb{R}}$, its critical equations are well-defined on the complex Zariski closure $\mathcal{L} \subset \mathbb{S}^n$ of $\mathcal{L}_{\mathbb{R}}$. The *maximum likelihood degree (ML-degree)* of $\mathcal{L}_{\mathbb{R}}$ resp. $\mathcal{L}$ is the number of complex critical points of ℓ_S (i.e., the number of invertible matrices X in $\mathcal{L}$ satisfying $X^{-1} - S \in \mathcal{L}^{\perp}$) for a generic matrix $S \in \mathcal{L}$. Note that this definition of ML-degree extends to any linear subspace of $\mathbb{S}^n$.

Our main tool to prove Theorem 1.1 is the projection away from $\mathcal{L}^{\perp}$:

$$\begin{aligned} \pi_{\mathcal{L}^{\perp}} : \mathbb{P}\mathbb{S}^n &\dashrightarrow \left\{ \mathcal{K} \in \operatorname{Gr}(\dim \mathcal{L}^{\perp} + 1, \mathbb{S}^n) \mid \mathcal{L}^{\perp} \subset \mathcal{K} \right\} \cong \mathbb{P}^{\dim \mathbb{P}\mathcal{L}}, \\ S &\longmapsto L_S := \operatorname{span}\{\mathcal{L}^{\perp}, S\}. \end{aligned}$$

In [1] it is shown that the ML-degree of $\mathcal{L}$ is the degree of this projection restricted to the reciprocal variety $\mathbb{P}\mathcal{L}^{-1}$. In other words, the ML-degree of $\mathcal{L}$ is the cardinality of the generic fiber of the restricted projection $\pi_{\mathcal{L}^{\perp}}|_{\mathbb{P}\mathcal{L}^{-1}}$:

$$\mathbb{P}\left(L_S \cap \mathcal{L}^{-1} \setminus \mathcal{L}^{\perp} \right) \text{ for generic } S \in \mathbb{P}\mathbb{S}^n. \tag{1}$$

Proof of Theorem 1.1. Let us first assume that we have $\mathbb{P}\mathcal{L}^{-1} \cap \mathbb{P}\mathcal{L}^{\perp} = \emptyset$. Then it follows from the above that the ML-degree of $\mathcal{L}$ is the cardinality of the intersection $\mathbb{P}L_S \cap \mathbb{P}\mathcal{L}^{-1}$ for generic $S \in \mathbb{P}\mathbb{S}^n$. Since the dimension of the projective space $\mathbb{P}L_S$ is the codimension of $\mathbb{P}\mathcal{L}^{-1}$, they intersect in either $\deg(\mathbb{P}\mathcal{L}^{-1})$ many points (counted with multiplicity) or in infinitely many points. The latter cannot happen for generic S, since domain and codomain of the map $\pi_{\mathcal{L}^{\perp}}|_{\mathbb{P}\mathcal{L}^{-1}}$ have the same dimension, so its generic fiber (1) must be finite. Thus, the intersection $\mathbb{P}L_S \cap \mathbb{P}\mathcal{L}^{-1}$ consists of $\deg(\mathbb{P}\mathcal{L}^{-1})$ many points, counted with multiplicity. In [1] it is shown that the generic fiber (1) is reduced, so $\mathbb{P}L_S \cap \mathbb{P}\mathcal{L}^{-1}$ consists of $\deg(\mathbb{P}\mathcal{L}^{-1})$ distinct points for generic S, and we conclude that we have ML-degree$(\mathcal{L}) = \deg(\mathbb{P}\mathcal{L}^{-1})$.

Conversely, we assume that the intersection $\mathbb{P}\mathcal{L}^{-1} \cap \mathbb{P}\mathcal{L}^{\perp}$ is non-empty. If it is finite, then, by the same reasoning as before, the intersection $\mathbb{P}L_S \cap \mathbb{P}\mathcal{L}^{-1}$

is again finite for generic S, and thus must consist of $\deg(\mathbb{P}\mathcal{L}^{-1})$ many points (counted with multiplicity). Since $\mathbb{P}\mathcal{L}^{-1}\cap\mathbb{P}\mathcal{L}^{\perp}\neq\emptyset$, we see that $\mathbb{P}L_S\cap\mathbb{P}\mathcal{L}^{-1}$ consists of strictly more points than (1). All in all, we have for generic S that

$$\text{ML-degree}(\mathcal{L}) = |\mathbb{P}\left(L_S\cap\left(\mathcal{L}^{-1}\setminus\mathcal{L}^{\perp}\right)\right)| < |\mathbb{P}L_S\cap\mathbb{P}\mathcal{L}^{-1}| \le \deg(\mathbb{P}\mathcal{L}^{-1}).$$

Hence, we are left to consider the case when the intersection $\mathbb{P}\mathcal{L}^{-1}\cap\mathbb{P}\mathcal{L}^{\perp}$ is infinite. Since the generic fiber (1) is finite, the intersection $\mathbb{P}L_S\cap\mathbb{P}\mathcal{L}^{-1}$ consists of positive-dimensional components inside $\mathbb{P}\mathcal{L}^{\perp}$ as well as k points, among which those outside of $\mathbb{P}\mathcal{L}^{\perp}$ contribute to the ML-degree of $\mathcal{L}$. From the following standard fact from projective geometry it follows that we have $\text{ML-degree}(\mathcal{L})\le k<\deg(\mathbb{P}\mathcal{L}^{-1})$. □

Proposition 2.1. *Let $X\subset\mathbb{P}^m$ be an irreducible projective variety of degree d, and let $L\subset\mathbb{P}^m$ be a projective subspace of complementary dimension (i.e., $\dim L = m-\dim X$). If the intersection of X and L consists of positive dimensional components and k points, then we have $k<d$.*

Proof. The following argument is due to Kristian Ranestad.

If $k=0$, there is nothing to show. Hence, we assume from now on that $k>0$. Let Y_1 be the union of all maximal-dimensional irreducible components of the intersection $X\cap L$. We denote the remaining lower-dimensional components by Z_1 (i.e. $X\cap L = Y_1\cup Z_1$), so in particular Z_1 contains the k points. The codimension c_1 of Y_1 in X satisfies $0<c_1<\dim X$. We consider a general projective space $L_1\subset\mathbb{P}^m$ of codimension c_1 that contains L. All components in the intersection $X\cap L_1$ have codimension c_1 in X. The latter can be seen by iteratively intersecting X with general hyperplanes $H_1,\dots,H_{c_1}$ containing L: at each step, the irreducible components of $X\cap H_1\cap\ldots\cap H_{i-1}$ are not contained in H_i such that intersecting with H_i reduces the dimension by 1.

Some of the irreducible components in $X\cap L_1$ form Y_1. We denote the remaining components by X_1 (i.e., $X\cap L_1 = Y_1\cup X_1$). Recall that d is the degree of X; from the equality $d=\deg(X\cap L_1)=\deg Y_1+\deg X_1$ it follows that $\deg X_1<d$. Moreover, from $Y_1\cup Z_1 = X\cap L = (Y_1\cup X_1)\cap L = Y_1\cup(X_1\cap L)$, it follows that we have $X_1\cap L = (Y_1\cap X_1)\cup Z_1$. In particular, the intersection $X_1\cap L$ contains the k points. We also note that X_1 and L have complementary dimension inside L_1: $\operatorname{codim}_{L_1}(X_1) = \dim L_1-\dim X_1 = (m-c_1)-(\dim X-c_1) = m-\dim X = \dim L$. Hence, if the intersection $X_1\cap L$ is finite, then we have shown that $k\le\deg X_1<d$, so we are done.

Otherwise, if the intersection $X_1\cap L$ is not finite, we consider the union Y_2 of the maximal-dimensional irreducible components of $X_1\cap L$. We observe that $\dim X_1<\dim X$ and $\dim Y_2<\dim Y_1$. Now we repeat our construction above: We let Z_2 be the remaining irreducible components of $X_1\cap L = Y_2\cup Z_2$, choose a general projective space $L_2 \subset \mathbb{P}^m$ of codimension $c_2 := \operatorname{codim}_{X_1}(Y_2)$ that contains L, and denote by X_2 the irreducible components away from Y_2 in $X_1 \cap L_2 = Y_2 \cup X_2$. If $X_2 \cap L$ is finite, the same arguments as above show that $k \leq \deg X_2 < d$. Otherwise, since $\dim X_2 < \dim X_1$ and $\dim(X_2 \cap L) < \dim Y_2$, we can repeat the above process several times until eventually $X_i \cap L$ will be finite for some $i \in \mathbb{N}$. At that point we can conclude the proof as $k \leq \deg X_i < d$. □

3. Generic ML-maximality

In this section we show that a general linear space of symmetric matrices is ML-maximal. This result is not new, see Remark 1.4. What we show here is the equivalent statement (by Theorem 1.1) that for a generic linear space $\mathcal{L}$, we have $\mathcal{L}^{-1} \cap \mathcal{L}^{\perp} = \{0\}$. This was conjectured in more generality in [13, Conjecture 5.8], and shown in even more generality to follow from a statement by Teissier in [12, Corollary 2.6]. However, the authors do not write down how this follows exactly, which is why we include it here. It was explained to us by Mateusz Michałek.

For a positive integer n, we denote by I_n the identity matrix of rank n. For an $n \times n$-matrix X we denote by $\operatorname{adj}(X)$ its adjugate; we have $X \operatorname{adj}(X) = \det(X) \cdot I_n$, so if X is invertible we have

$$X^{-1} = (\det(X))^{-1} \cdot \operatorname{adj}(X). \tag{2}$$

Lemma 3.1. *Let V be a complex vector space of dimension n with dual space V^*, and $L \subset V$ a linear subspace. Let $L^{\perp} \subset V^*$ be the space of all linear forms that vanish on L. Moreover, let f be a homogeneous polynomial on V, and ∇f its gradient map. If L is generic, we have $\overline{\mathbb{P}(\nabla f)(L)} \cap \mathbb{P}L^{\perp} = \emptyset$, where $\overline{\mathbb{P}(\nabla f)(L)}$ is the Zariski closure of $\mathbb{P}(\nabla f)(L)$ in $\mathbb{P}V^*$.*

Proof. The following is due to Mateusz Michałek.

Let k be the dimension of L. We may choose coordinates $(x_1, \ldots, x_n)$ on V and an inner product to identify the dual space V^* with V such that L is defined by the equations $x_{k+1} = \cdots = x_n = 0$, and $L^{\perp}$ is given by $x_1 = \cdots = x_k = 0$. Assume that L is generic, and assume by contradiction that there is a sequence of elements $(X_j)_{j \geq 1}$ in $\mathbb{P}(\nabla f)(L)$ with limit contained in $\mathbb{P}L^{\perp}$. Let $(Y_j)_{j \geq 1}$ be a sequence in $\mathbb{P}L$ such that we have

$$(X_j)_{j \geq 1} = \left(\left(\tfrac{\partial f}{\partial x_1}(Y_j) : \ldots : \tfrac{\partial f}{\partial x_n}(Y_j) \right) \right)_{j \geq 1}.$$

Then $\lim_{j \to \infty} X_j \in \mathbb{P}L^{\perp}$ implies that there exists an $l \in \{k+1, \ldots, n\}$ such that for all $i \in \{1,\dots,k\}$ we have

$$\lim_{j\to\infty} \frac{\frac{\partial f}{\partial x_i}(Y_j)}{\frac{\partial f}{\partial x_l}(Y_j)} = 0.$$

Fix such an l. Since L is generic, it follows from [20, II.2.1.3] that $\frac{\partial f}{\partial x_l}$ is integral over the ideal $I = \left(\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_k}\right)$ in the ring $\mathbb{C}[x_1,\dots,x_n]$. By definition, the latter means that there is an integer p, and elements $a_i \in I^{p-i}$ for $i \in \{1,\dots,p-1\}$, such that

$$\frac{\left(\frac{\partial f}{\partial x_l}\right)^p + \sum_{i=0}^{p-1} a_i \left(\frac{\partial f}{\partial x_l}\right)^i}{\left(\frac{\partial f}{\partial x_l}\right)^p} = 0.$$

Plugging in Y_j and taking the limit, we find the contradiction

$$0 = \lim_{j\to\infty} \frac{\left(\frac{\partial f}{\partial x_l}(Y_j)\right)^p}{\left(\frac{\partial f}{\partial x_l}(Y_j)\right)^p} + \lim_{j\to\infty} \frac{\sum_{i=0}^{p-1} a_i(Y_j)\left(\frac{\partial f}{\partial x_l}(Y_j)\right)^i}{\left(\frac{\partial f}{\partial x_l}(Y_j)\right)^p} = \lim_{j\to\infty} \frac{\left(\frac{\partial f}{\partial x_l}(Y_j)\right)^p}{\left(\frac{\partial f}{\partial x_l}(Y_j)\right)^p} = 1,$$

where the second equality follows from the fact that a_i is contained in I^{p-i}. Hence, we conclude that there is no sequence of elements in $\mathbb{P}(\nabla f)(L)$ with limit contained in $\mathbb{P}L^\perp$ if L is generic. This finishes the proof. □

Corollary 3.2. *For generic linear subspaces $\mathcal{L} \subset \mathbb{S}^n$, we have $\mathbb{P}\mathcal{L}^{-1} \cap \mathbb{P}\mathcal{L}^\perp = \emptyset$.*

Proof. Let f be the map $f\colon \mathbb{S}^n \longrightarrow \mathbb{C}$ given by $X \longmapsto \det(X)$. By the Jacobi formula and the fact that we work with symmetric matrices, for $X \in \mathbb{S}^n$ we have $(\nabla f)(X) = \operatorname{adj}(X) \in \mathbb{S}^n$. From this and (2) it follows that for a linear subspace in $\mathbb{S}^n$ we have

$$\mathbb{P}\mathcal{L}^{-1} \cap \mathbb{P}\mathcal{L}^\perp = \overline{\mathbb{P}(\nabla f)(\mathcal{L})} \cap \mathbb{P}\mathcal{L}^\perp,$$

where $\overline{\mathbb{P}(\nabla f)(\mathcal{L})}$ is the Zariski closure of $\mathbb{P}(\nabla f)(\mathcal{L})$ in $\mathbb{P}\mathbb{S}^n$. The statement now follows from Lemma 3.1. □

Remark 3.3. As we will show in Lemma 4.3, for the determinant map f as in Corollary 3.2 the image $\mathbb{P}(\nabla f)(\mathcal{L})$ is disjoint from $\mathbb{P}\mathcal{L}^\perp$ if and only if $\mathcal{L}$ is not contained in the hyperplane tangent to the zero locus $Z(f)$ of f at a smooth point belonging $\mathcal{L}$ (this also follows from the more general statement for any *hyperbolic* polynomial in [13, Proposition 5.9]). But this is true for generic $\mathcal{L}$ by Bertini's Theorem [7, Theorem 17.16]. The added value of Lemma 3.1 is thus to show that the *closure* of $\mathbb{P}(\nabla f)(\mathcal{L})$ is disjoint from $\mathbb{P}\mathcal{L}^\perp$ for generic $\mathcal{L}$. As we saw in the proof of the lemma, where Bertini shows that the radical of the ideal of the singular locus $\mathrm{Sing}(Z(f)\cap\mathcal{L})$ of $Z(f)\cap\mathcal{L}$ equals the radical of the ideal of $\mathrm{Sing}(Z(f))\cap\mathcal{L}$, Teissier shows that these two ideals are in fact *integral* over each other. There are several versions of Teissier's 'Théorème de Bertini idéaliste'; see [18, Proposition 2.7], [19, p.42], [6, Theorem 2.6].

4. Sufficient and necessary conditions for ML-maximality

Let k and n be two integers. By $\mathrm{NM}^{\circ}_{k,n}$ we denote the set in $\mathrm{Gr}(k,\mathbb{S}^n)$ of k-dimensional linear subspaces in $\mathbb{S}^n$ that are not ML-maximal. Note that $\mathrm{NM}_{k,n}$ is its Zariski closure by definition, and by Theorem 1.1, we have

$$\mathrm{NM}^{\circ}_{k,n} = \{\mathcal{L}\in\mathrm{Gr}(k,\mathbb{S}^n) \mid \mathbb{P}\mathcal{L}^{-1}\cap\mathbb{P}\mathcal{L}^{\perp}\neq\emptyset\}.$$

In this section we show that $\mathrm{NM}^{\circ}_{k,n}$ is neither Zariski open nor closed (Remark 4.1), and we provide sufficient and necessary conditions for a linear space to be ML-maximal (Remark 4.10). More specifically, we describe a subset of $\mathrm{NM}^{\circ}_{k,n}$ in terms of tangency to the determinantal hypersurface (Lemma 4.3), and we show that $\mathrm{NM}^{\circ}_{k,n}$ is contained in the closed set

$$C_{k,n} := \{\mathcal{L}\in\mathrm{Gr}(k,\mathbb{S}^n) \mid \exists\,(X,Y)\in\mathbb{P}\mathcal{L}\times\mathbb{P}\mathcal{L}^{\perp}\colon XY=0\} \tag{3}$$

(Corollary 4.7). The latter is one of the main ingredients in the proof of Theorem 1.2. The set $C_{k,n}$ is the union of the coisotropic hypersurfaces in $\mathrm{Gr}(k,\mathbb{S}^n)$ associated to the determinantal variety D_s, where s ranges over the integers such that $\binom{n-s+1}{2}\leq k\leq\binom{n+1}{2}-\binom{s+1}{2}$ [8, Theorems 2 and 3].

Remark 4.1. The set $\mathrm{NM}^{\circ}_{k,n}$ is in general neither Zariski open nor closed. To illustrate this, we consider the stratification of the Grassmannian $\mathrm{Gr}(2,\mathbb{S}^n)$ in terms of Segre symbols described in [5, Section 5]. In [5, Example 1.3], we see that the ML-maximal elements of $\mathrm{Gr}(2,\mathbb{S}^3)$ lie in the strata of $\mathrm{Gr}(2,\mathbb{S}^3)$ defined by Segre symbols with only 1's. In other words, the complement of $\mathrm{NM}^{\circ}_{2,3}$ is the union of the two strata $\mathrm{Gr}_{[1,1,1]}$ and $\mathrm{Gr}_{[(1,1),1]}$. However, in Figure 1 of the same paper, we find the following inclusions of Zariski closures of strata of codimensions 2, 1 and 0 in $\mathrm{Gr}(2,\mathbb{S}^3)$:

$$\overline{\mathrm{Gr}_{[(1,1),1]}}\subset\overline{\mathrm{Gr}_{[2,1]}}\subset\overline{\mathrm{Gr}_{[1,1,1]}}=\mathrm{Gr}(2,\mathbb{S}^3).$$

We conclude that the complement of $\mathrm{NM}^{\circ}_{2,3}$ in $\mathrm{Gr}(2,\mathbb{S}^3)$ is neither Zariski open nor closed, hence neither is $\mathrm{NM}^{\circ}_{2,3}$. By [5, Example 3.1], the ML-maximal elements of $\mathrm{Gr}(2,\mathbb{S}^4)$ lie again in the strata with Segre symbols containing only 1's. The same argument shows that $\mathrm{NM}^{\circ}_{2,4}$ is neither Zariski open nor closed.

Lemma 4.2. *Let $\mathcal{L} \subset \mathbb{S}^n$ be a regular linear space. The intersection $\mathcal{L}^{-1} \cap \mathcal{L}^{\perp}$ does not contain matrices of full rank.*

Proof. If the intersection would contain a full-rank matrix Y, then $Y^{-1} \in \mathcal{L}$ and $Y \in \mathcal{L}^{\perp}$, which yields $0 = \operatorname{trace}(Y^{-1}Y) = \operatorname{trace}(I_n) = n > 0$; a contradiction. □

For a linear space $\mathcal{L}$ in $\mathbb{S}^n$, we denote by $\mathbb{P}\mathring{\mathcal{L}}^{-1}$ the open subset of $\mathbb{P}\mathcal{L}^{-1}$ given by the adjugates of all matrices in $\mathbb{P}\mathcal{L}$ of rank at least $n-1$. We now describe the following subset of $\mathrm{NM}^{\circ}_{k,n}$:

$$\{\mathcal{L} \in \mathrm{Gr}(k, \mathbb{S}^n) \mid \mathbb{P}\mathring{\mathcal{L}}^{-1} \cap \mathbb{P}\mathcal{L}^{\perp} \neq \emptyset\}. \tag{4}$$

The next lemma also follows from the more general [13, Proposition 5.9].

Lemma 4.3. *The subset of* $\mathrm{NM}^{\circ}_{k,n}$ *given by (4) is equal to*

$$\{\mathcal{L} \in \mathrm{Gr}(k, \mathbb{S}^n) \mid \exists\, X \in \mathrm{Reg}(D_{n-1})\colon X \in \mathcal{L} \subset T_X(D_{n-1})\}.$$

Proof. If $\mathcal{L}$ is contained in (4), then there is a matrix X in $\mathbb{P}\mathcal{L}$ of rank at least $n-1$, whose adjugate is contained in $\mathbb{P}\mathcal{L}^{\perp}$. By Lemma 4.2, X has rank $n-1$, hence it is a regular point of D_{n-1}. Since $\operatorname{adj}(X)$ is in the annihilator of the tangent hyperplane $T_X(D_{n-1})$, and $\operatorname{adj}(X)$ is contained in $\mathbb{P}\mathcal{L}^{\perp}$, it follows that $\mathcal{L}$ is a subset of $T_X(D_{n-1})$. Conversely, let X be a regular point of D_{n-1} such that $\mathcal{L}$ is tangent to D_{n-1} at X. Then X has rank $n-1$, so $\operatorname{adj}(X)$ lies in $\mathbb{P}\mathring{\mathcal{L}}^{-1}$. Moreover, since $\operatorname{adj}(X)$ is in the annihilator of $T_X(D_{n-1})$, it is also in $\mathbb{P}\mathcal{L}^{\perp}$. □

Remark 4.4. Lemma 4.3 says that (4) is exactly the set of k-dimensional tangent spaces at smooth points of the determinantal hypersurface D_{n-1}. Thus, its Zariski closure is the irreducible coisotropic variety $\mathrm{Ch}_{k-1}(D_{n-1})$. [9]

Example 4.5. The following linear space is an element in $\mathrm{NM}^{\circ}_{3,3}$ that is not contained in its subset (4). Let $\mathcal{L}$ be spanned by $\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}$. The intersection $\mathbb{P}\mathcal{L}^{-1} \cap \mathbb{P}\mathcal{L}^{\perp}$ consists of the single element $M = \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$, so $\mathcal{L}$ is contained in $\mathrm{NM}^{\circ}_{3,3}$. However, $\mathbb{P}\mathring{\mathcal{L}}^{-1} \cap \mathbb{P}\mathcal{L}^{\perp}$ is empty, since M is not the adjugate of any matrix in $\mathcal{L}$.

The linear space $\mathcal{L}$ is a net of conics of Type C according to Wall's classification [21]. This means that it is a generic point in the intersection of the Chow hypersurface $\mathrm{Ch}_0(D_1)$ of the rank-one locus D_1 with the Zariski closure $\mathrm{Ch}_2(D_2)$ of (4) described in Remark 4.4. By duality, $\mathcal{L} \in \mathrm{Ch}_2(D_2)$ if and only if $\mathcal{L}^{\perp} \in \mathrm{Ch}_0(D_1)$. Hence $\mathcal{L}$ is generic among all linear spaces with the property that both $\mathcal{L}$ and $\mathcal{L}^{\perp}$ contain a rank-one matrix. We also see from [4, Table 1] that $\text{ML-degree}(\mathcal{L}) = 2 < 3 = \deg(\mathbb{P}\mathcal{L}^{-1})$, so $\mathcal{L}$ is not ML-maximal.

We end this section by showing that $\mathrm{NM}^{\circ}_{k,n}$ is contained in the set $C_{k,n}$ defined in (3). For any $\mathcal{L} \in G(k,\mathbb{S}^n)$, consider the Zariski closed set

$$C_{\mathcal{L}} = \{(X,Y) \in \mathbb{P}\mathcal{L} \times \mathbb{P}\mathbb{S}^n \mid XY = t \cdot I_n \text{ for some } t \in \mathbb{C}\}$$

with the projection $\pi_{\mathcal{L}} : C_{\mathcal{L}} \to \mathbb{S}^n$ to the second coordinate.

Lemma 4.6. *Let $\mathcal{L} \subset \mathbb{S}^n$ be a regular linear space. For every $Y \in \mathcal{L}^{-1}$ with $\operatorname{rank}(Y) < n$, there is an $X \in \mathcal{L}$, $X \neq 0$, such that $XY = 0$.*

Proof. For any $Z \in \mathbb{P}\mathcal{L}$ of full rank, we have $(Z, Z^{-1}) \in C_{\mathcal{L}}$, so $Z^{-1} \in \pi_{\mathcal{L}} C_{\mathcal{L}}$. Since $C_{\mathcal{L}}$ is a projective variety, $\pi_{\mathcal{L}} C_{\mathcal{L}}$ is closed, so $\mathcal{L}^{-1}$ is contained in $\pi_{\mathcal{L}} C_{\mathcal{L}}$. Therefore, if Y is in $\mathcal{L}^{-1}$, then Y is in the image of $\pi_{\mathcal{L}}$. So there is an $X \in \mathcal{L}$ with $XY = t \cdot I_n$ for some constant t. If Y is not of full rank, this implies $XY = 0$. □

Corollary 4.7. *The set $\mathrm{NM}^{\circ}_{k,n}$ is contained in $C_{k,n}$.*

Proof. Let $\mathcal{L} \in \mathrm{NM}^{\circ}_{k,n}$ be a regular subspace of $\mathbb{S}^n$. By definition, there is a non-zero matrix Y in the intersection $\mathcal{L}^{-1} \cap \mathcal{L}^{\perp}$ which has rank $< n$ by Lemma 4.2. Lemma 4.6 now guarantees the existence of a non-zero matrix $X \in \mathcal{L}$ such that $XY = 0$, which shows that $\mathcal{L}$ is contained in $C_{k,n}$. □

Remark 4.8. The Zariski closure $\mathrm{NM}_{k,n}$ of $\mathrm{NM}^{\circ}_{k,n}$ is in general not equal to $C_{k,n}$, as can be seen by the following argument: under the involution $\mathcal{L} \mapsto \mathcal{L}^{\perp}$, the set $C_{k,n}$ gets mapped to the set $C_{\binom{n+1}{2}-k,n}$, but $\mathrm{NM}_{k,n}$ is in general not mapped to $\mathrm{NM}_{\binom{n+1}{2}-k,n}$ as Example 4.9 illustrates.

In fact, by Theorem 1.2 and [8, Theorem 3], we have the following: if there is an integer s such that $k = \binom{n-s+1}{2}$, then $C_{k,n}$ is the union of $\mathrm{NM}_{k,n}$ and the Chow hypersurface $\mathrm{Ch}_0(D_s)$ of D_s; otherwise, $C_{k,n} = \mathrm{NM}_{k,n}$.

Example 4.9. The integers $k = 3, n = 3, s = 1$ satisfy $k = \binom{n-s+1}{2}$. The linear space $\mathcal{L}$ in $\mathbb{S}^3$ spanned by diagonal matrices is contained in $C_{3,3}$ but not in $\mathrm{NM}_{3,3}$. According to Wall's classification [21], $\mathcal{L}$ is a net of conics of Type E. Its projectivization is a trisecant plane of the rank-one locus D_1, so in particular $\mathcal{L}$ is in the Chow hypersurface $\mathrm{Ch}_0(D_1)$. However, $\mathcal{L}$ is not contained in the coisotropic hypersurface $\mathrm{Ch}_2(D_2)$, which is by Corollary 1.3 equal to $\mathrm{NM}_{3,3}$.

On the other hand, its polar net $\mathcal{L}^{\perp}$ is of type E^*, so by [4, Table 1] we have $\text{ML-degree}(\mathcal{L}^{\perp}) = 1 < 4 = \deg(\mathbb{P}(\mathcal{L}^{\perp})^{-1})$. Hence the polar net $\mathcal{L}^{\perp}$ is not ML-maximal, i.e. $\mathcal{L}^{\perp} \in \mathrm{NM}^{\circ}_{3,3}$. Corollary 4.7 and Remark 4.8 imply that both $\mathcal{L}^{\perp}$ and $\mathcal{L}$ are contained in $C_{3,3}$.

Remark 4.10. Lemma 4.3 gives a sufficient condition, and Corollary 4.7 gives a necessary condition for a linear subspace not to be ML-maximal. On the one hand, every linear space that is tangent to the manifold of corank-one matrices is not ML-maximal. On the other hand, for every linear space $\mathcal{L}$ that is not ML-maximal, there are non-zero matrices $X \in \mathcal{L}$ and $Y \in \mathcal{L}^{\perp}$ such that $XY = 0$.

5. Generic bad subspaces are not ML-maximal

Recall that $\mathrm{Bad}_{k,n}$ is the Zariski closure in $\mathrm{Gr}(k, \mathbb{S}^n)$ of the set of k-dimensional bad subspaces of $\mathbb{S}^n_{\mathbb{R}}$ as defined in the introduction. In this and the next section, we prove that $\mathrm{NM}_{k,n}$ equals $\mathrm{Bad}_{k,n}$. We start with one inclusion.

Proposition 5.1. $\mathrm{Bad}_{k,n}$ *is contained in* $\mathrm{NM}_{k,n}$.

To show the proposition, we outsource all the hard work to the following lemma.

Lemma 5.2. *Let s be an integer with $0 < s < n$ and $k > \binom{n-s+1}{2}$. For fixed $X, Y \in \mathbb{S}^n$ with $\mathrm{rank}(X) = s$, $\mathrm{rank}(Y) = n - s$ and $XY = 0$, we consider the variety*

$$\mathcal{G}_{X,Y} := \left\{ \mathcal{L} \in \mathrm{Gr}(k, \mathbb{S}^n) \mid X \in \mathcal{L}, Y \in \mathcal{L}^{\perp} \right\}.$$

A general $\mathcal{L}$ in $\mathcal{G}_{X,Y}$ satisfies

$$\{Z \in \mathbb{S}^n \mid XZ = 0\} \subseteq \mathcal{L}^{-1};$$

in particular, we have that Y is contained in $\mathcal{L}^{-1}$.

We first prove Proposition 5.1 to see how we can apply the lemma. Afterwards, we give the proof of Lemma 5.2.

Proof of Proposition 5.1. The bad locus $\mathrm{Bad}_{k,n}$ is the union of the irreducible coisotropic hypersurfaces in $\mathrm{Gr}(k, \mathbb{S}^n)$ associated to the bounded-rank loci D_s where s is in the range $\binom{n-s+1}{2} < k \leq \binom{n+1}{2} - \binom{s+1}{2}$ [8, Theorem 2]. A general point $\mathcal{L}$ in one of these hypersurfaces satisfies the following property:

$$\exists\, X \in \mathcal{L} \cap \mathrm{Reg}(D_s) : \mathcal{L} + T_X D_s \neq \mathbb{S}^n.$$

This implies that $\mathcal{L}$ and the tangent space $T_X D_s$ have a common non-zero element Y in their annihilators. In other words, there is a non-zero matrix $Y \in \mathcal{L}^{\perp}$ satisfying $XY = 0$. The rank of that matrix Y is at most $n - s$. Since $\mathcal{L}$ is general, we may assume that $\mathrm{rank}(Y) = n - s$. In fact, we may choose $\mathcal{L}$ by first fixing any $X \in \mathcal{L}$ of rank s, then fixing any $Y \in \mathcal{L}^{\perp}$ of rank $n - s$ with $XY = 0$, and finally choosing the remaining basis vectors of $\mathcal{L}$ arbitrarily. In other words, $\mathcal{L}$ is a general point of $\mathcal{G}_{X,Y}$, so by Lemma 5.2 we see that Y is contained in $\mathcal{L}^{-1}$. It follows that $\mathcal{L}$ is contained in $\mathrm{NM}_{k,n}$. $\square$

In the proof of Lemma 5.2, we compute the total transform of a point in the blow-up of a linear space along the indeterminacy locus of the adjugate map. Since this is a technical construction, we first do this in a concrete example. It was shown in [3] that the blow-up of $\mathbb{PS}^n$ along D_{n-2}, i.e. the Zariski closure of the graph of matrix inversion on $\mathbb{PS}^n$, is

$$\Gamma := \{(X,Y) \in \mathbb{PS}^n \times \mathbb{PS}^n \mid XY = t \cdot I_n \text{ for some } t \in \mathbb{C}\}.$$

For a regular linear space $\mathcal{L}$ in $\mathbb{S}^n$, we use the Zariski closure

$$\Gamma_{\mathcal{L}} := \overline{\{(X, X^{-1}) \in \mathbb{PS}^n \times \mathbb{PS}^n \mid X \in \mathbb{P}\mathcal{L}\}}$$

of the graph of matrix inversion restricted to $\mathbb{P}\mathcal{L}$ to understand the reciprocal variety $\mathbb{P}\mathcal{L}^{-1}$, as $\mathbb{P}\mathcal{L}^{-1}$ is the image of the projection of $\Gamma_{\mathcal{L}}$ to the second factor. In particular, for a point $X \in \mathbb{P}\mathcal{L}$, we are interested in its *total transform*

$$\Gamma_{\mathcal{L}}(X) := \{Z \in \mathbb{PS}^n \mid (X,Z) \in \Gamma_{\mathcal{L}}\} \subseteq \{Z \in \mathbb{P}\mathcal{L}^{-1} \mid XZ = t \cdot I_n \text{ for some } t \in \mathbb{C}\}.$$

Example 5.3. $(n = 3,\ k = 5)$ Let $X, Y \in \mathbb{S}^3$ be the matrices given by

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $\mathcal{L}$ be the polar linear space $Y^{\perp}$ of Y in $\mathbb{S}^3$, and note that X is contained in $\mathcal{L}$. Setting $s = 1$, the matrices X,Y satisfy the conditions in Lemma 5.2, and $\mathcal{L}$ is contained in $\mathcal{G}_{X,Y}$. We compute that (the affine cone over) the total transform $\Gamma_{\mathcal{L}}(X)$ is $\{Z \in \mathbb{S}^3 \mid XZ = 0\}$, so the latter is contained in $\mathcal{L}^{-1}$.

A basis for $\mathcal{L}$ is given by $\{X, B_{01}, B_{02}, B_1, B_2\}$, where $(B_{01}, B_{02}, B_1, B_2)$ is

$$\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right).$$

For a matrix $M \in \mathbb{S}^n$ we denote its lower-right 2×2 block by $\overline{M}$. Note that $\overline{B}_{01}$ and $\overline{B}_{02}$ are both 0, and $\overline{B_1}$, $\overline{B_2}$ together span $I_2^{\perp}$ in $\mathbb{S}^2$. To determine the total transform $\Gamma_{\mathcal{L}}(X)$, we perturb X, and then compute its adjugate. Let ε be an indeterminate. The first perturbation we compute is

$$X + \varepsilon(b_{01}B_{01} + b_{02}B_{02} + b_1B_1 + b_2B_2) = \begin{bmatrix} 1 & \varepsilon b_{02} & \varepsilon b_{01} \\ \varepsilon b_{02} & \varepsilon b_1 & \varepsilon b_2 \\ \varepsilon b_{01} & \varepsilon b_2 & -\varepsilon b_1 \end{bmatrix},$$

where $(b_{01}, b_{02}, b_1, b_2)$ is a vector in $\mathbb{C}^4 \setminus \{0\}$. The adjugate of this matrix is

$$\begin{bmatrix} -\varepsilon^2(b_1^2+b_2^2) & \varepsilon^2(b_{02}b_1+b_{01}b_2) & \varepsilon^2(b_{02}b_2-b_{01}b_1) \\ \varepsilon^2(b_{02}b_1+b_{01}b_2) & -\varepsilon(b_1+\varepsilon b_{01}^2) & -\varepsilon(b_2-\varepsilon b_{01}b_{02}) \\ \varepsilon^2(b_{02}b_2-b_{01}b_1) & -\varepsilon(b_2-\varepsilon b_{01}b_{02}) & \varepsilon(b_1-\varepsilon b_{02}^2) \end{bmatrix}.$$

Note that the lowest degree terms are all in the 2×2 lower-right block. Dividing by ε and setting $\varepsilon = 0$, we obtain the matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -b_1 & -b_2 \\ 0 & -b_2 & b_1 \end{bmatrix} = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \operatorname{adj}(b_1\bar{B}_1+b_2\bar{B}_2) \end{array}\right].$$

Since $\overline{B}_1, \overline{B}_2$ span $I_2^\perp$, this implies $\{Z \mid XZ = 0,\ \bar{Z} \in (I_2^\perp)^{-1}\} \subseteq \Gamma_{\mathcal{L}}(X)$.

The second perturbation of X that we compute is the matrix

$$X + \varepsilon(c_{01}B_{01} + c_{02}\varepsilon B_{02} + c_1\varepsilon B_1 + c_2\varepsilon B_2) = \begin{bmatrix} 1 & \varepsilon^2 c_{02} & \varepsilon c_{01} \\ \varepsilon^2 c_{02} & \varepsilon^2 c_1 & \varepsilon^2 c_2 \\ \varepsilon c_{01} & \varepsilon^2 c_2 & -\varepsilon^2 c_1 \end{bmatrix},$$

where $c = (c_{01}, c_{02}, c_1, c_2)$ is a vector in $\mathbb{C}^4 \setminus \{0\}$. We find the adjugate

$$\begin{bmatrix} -\varepsilon^4(c_1^2+c_2^2) & \varepsilon^3(\varepsilon c_{02}c_1+c_{01}c_2) & \varepsilon^3(\varepsilon c_{02}c_2-c_{01}c_1) \\ \varepsilon^3(\varepsilon c_{02}c_1+c_{01}c_2) & -\varepsilon^2(c_1+c_{01}^2) & -\varepsilon^2(c_2-\varepsilon c_{01}c_{02}) \\ \varepsilon^3(\varepsilon c_{02}c_2-c_{01}c_1) & -\varepsilon^2(c_2-\varepsilon c_{01}c_{02}) & \varepsilon^2(c_1-\varepsilon^2 c_{02}^2) \end{bmatrix}.$$

Again, the lowest degree terms are in the 2×2 lower-right block. We now divide by ε^2 and set $\varepsilon = 0$, and obtain

$$Z_c = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -(c_1+c_{01}^2) & -c_2 \\ 0 & -c_2 & c_1 \end{bmatrix}.$$

Let $\mathcal{Z}$ be the closure in $\mathbb{S}^2$ of the set $\{\overline{Z}_c \mid c \in \mathbb{C}^4 \setminus \{0\}\}$. All elements $\overline{Z}_c \in \mathcal{Z}$ with $c = (0, c_{02}, c_1, c_2)$ parametrize the hypersurface $(I_2^\perp)^{-1}$ in $\mathbb{S}^2$ as before, so $(I_2^\perp)^{-1}$ is contained in $\mathcal{Z}$. Since $\mathcal{Z}$ is irreducible, it follows that $\mathcal{Z}$ is either equal to $(I_2^\perp)^{-1}$ or to $\mathbb{S}^2$. However, the element $\overline{Z}_{(1,0,1,0)}$ is contained in $\mathcal{Z}$, and

$$\left(\overline{Z}_{(1,0,1,0)}\right)^{-1} = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \notin I_2^\perp,$$

so $\mathcal{Z}$ is not contained in $(I_2^\perp)^{-1}$, hence $\mathcal{Z} = \mathbb{S}^2$. We have shown that every matrix $Z \in \mathbb{S}^3$ with $XZ = 0$ is contained in $\mathcal{Z}$. Therefore, $\Gamma_{\mathcal{L}}(X) = \{Z \in \mathbb{PS}^3 \mid XZ = 0\}$.

We now give the proof of Lemma 5.2. We generalize the construction that was done in the previous example, and we recommend reading this example alongside the proof for illustration.

Proof of Lemma 5.2. After a change of coordinates, we may assume that

$$X = \left[\begin{array}{c|c} I_s & 0 \\ \hline 0 & 0 \end{array}\right] \text{ and } Y = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & I_{n-s} \end{array}\right]. \tag{5}$$

In the following, we will denote by $\bar{M}$ the lower-right $(n-s)\times(n-s)$ block of a symmetric matrix $M \in \mathbb{S}^n$. For instance, $\bar{X} = 0$ and $\bar{Y} = I_{n-s}$.

A dimension count reveals that every $\mathcal{L} \in \mathcal{G}_{X,Y}$ must contain a matrix B_0 with $\bar{B}_0 = 0$ that is linear independent from X. Indeed, the vector space of symmetric matrices M satisfying $\bar{M} = 0$ is contained in the hyperplane $Y^\perp$ and has codimension $\binom{n-s+1}{2} - 1$ in $Y^\perp$. As $\mathcal{L}$ is also contained in $Y^\perp$, its projection away from X yields a $(k-1)$-dimensional vector space inside $Y^\perp$. Since we have $k-1 \geq \binom{n-s+1}{2}$, that vector space contains a non-zero matrix B_0 with $\bar{B}_0 = 0$.

The same dimension count also shows that the image of the projection $M \mapsto \bar{M}$ onto $\mathbb{S}^{n-s}$ restricted to a general $\mathcal{L} \in \mathcal{G}_{X,Y}$ is the whole hyperplane $I_{n-s}^\perp$. In terms of a basis $\{X, B_0, B_1, \ldots, B_{k-2}\}$ (with $\bar{B}_0 = 0$) for a general $\mathcal{L} \in \mathcal{G}_{X,Y}$ this means that $\bar{B}_1, \ldots, \bar{B}_{k-2}$ span the hyperplane $I_{n-s}^\perp$.

Equipped with this knowledge, we will now prove the assertion. Note that, for $Z \in \mathbb{S}^n$ with $(X,Z) \in \Gamma_{\mathcal{L}}$, we have $XZ = 0$. In what follows we show that the converse holds as well; more specifically, we will show that in fact *all* pairs (X,Z) with $Z \in \mathbb{S}^n$, $XZ = 0$, are contained in $\Gamma_{\mathcal{L}}$.

Apply matrix inversion to the blow-up of $\mathcal{L}$ at X. In terms of a basis $\{X, B_0, B_1, \ldots, B_{k-2}\}$ for $\mathcal{L}$, this means to compute the matrix Z_b that appears as the first non-zero coefficient of the following power series in ε:

$$\operatorname{adj}\left(X + \varepsilon(b_0 B_0 + b_1 B_1 + \ldots + b_{k-2} B_{k-2})\right), \tag{6}$$

where $b = (b_0, b_1, \ldots, b_{k-2})$ is a non-zero vector of arbitrary power series b_i in $\mathbb{C}[[\varepsilon]]$. All matrices Z_b obtained in this way satisfy $(X, Z_b) \in \Gamma_{\mathcal{L}}$. In what follows we show that the closure of the set of matrices Z_b where b is a non-zero vector of either constants, or with b_0 constant and the other b_i linear monomials, already contains $\{(X,Z) \mid Z \in \mathbb{S}^n,\ XZ = 0\}$, thus proving the lemma.

Let us first compute Z_b for the case where b is a non-zero vector in $\mathbb{C}^{k-1}$. The lowest degree terms in the matrix (6) are of degree $n-s-1$ and appear exactly in its lower right block. Their coefficients are the minors of size $n-s-1$ of the matrix $b_1\bar{B}_1 + \ldots + b_{k-2}\bar{B}_{k-2}$ (since $\bar{B}_0 = 0$). More precisely, we see that

$$Z_b = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \operatorname{adj}(b_1\bar{B}_1 + \ldots + b_{k-2}\bar{B}_{k-2}) \end{array}\right].$$

Due to the generality of $\mathcal{L}$, the matrices $\bar{B}_1,\ldots,\bar{B}_{k-2}$ span the hyperplane $I_{n-s}^{\perp}$, so the closure of the set $\{\bar{Z}_b \mid b \in \mathbb{C}^{k-1} \setminus \{0\}\}$ in $\mathbb{S}^{n-s}$ equals the reciprocal hypersurface $(I_{n-s}^{\perp})^{-1}$. Hence, we have proven so far that

$$\left\{(X,Z) \mid XZ = 0,\ \bar{Z} \in (I_{n-s}^{\perp})^{-1}\right\} \subseteq \Gamma_{\mathcal{L}}.$$

Finally, we compute Z_b for the case when b is a non-zero vector where b_0 a constant and the power series $b_1,\ldots,b_{k-2}$ have only linear terms (i.e., $b_i = c_i\varepsilon$ for a constant c_i). The lowest degree terms in the matrix (6) are of degree $2(n-s-1)$ and appear again only in its lower right block. Now the coefficients of these terms do not only depend on $b_1,\ldots,b_{k-2}$, but also on b_0. The closure of the set of the resulting $\bar{Z}_b$ forms an irreducible subvariety $\mathcal{Z}$ of $\mathbb{S}^{n-s}$. Setting $b_0 = 0$, we see that $\mathcal{Z}$ contains the reciprocal hypersurface $(I_{n-s}^{\perp})^{-1}$. Hence $\mathcal{Z}$ is either equal to $(I_{n-s}^{\perp})^{-1}$ or it is the whole ambient space $\mathbb{S}^{n-s}$. The condition "$\mathcal{Z} \subseteq (I_{n-s}^{\perp})^{-1}$" is Zariski closed in the entries of the matrices $B_0, B_1,\ldots,B_{k-2}$. Thus, if there is one instance with $\mathcal{Z} \not\subseteq (I_{n-s}^{\perp})^{-1}$, then we know $\mathcal{Z} = \mathbb{S}^{n-s}$ for general choices of $B_0,\ldots,B_{k-2}$ (with $\bar{B}_0 = 0$). For general $\mathcal{L} \in \mathcal{G}_{X,Y}$ we can then conclude that $\{(X,Z) \mid Z \in \mathbb{S}^n,\ XZ = 0\} \subseteq \Gamma_{\mathcal{L}}$, which proves the assertion.

We exhibit such an instance. Since $\bar{B}_1,\ldots,\bar{B}_{k-2}$ span the hyperplane $I_{n-s}^{\perp}$, we may assume that the first $\binom{n-s+1}{2} - 1$ of these matrices are a standard basis of $I_{n-s}^{\perp}$. In particular, we may assume that $\bar{B}_i$, for $1 \le i \le n-s-1$, is the diagonal matrix whose i-th entry is 1, whose $(n-s)$-th entry is -1, and all other entries are 0. We fix B_0 to be the matrix with a 1 as entries at $(1,n)$ and $(n,1)$, and all other entries are 0. When we choose $b_0 = 1$, $b_i = \varepsilon$ for $1 \le i \le n-s-1$, and $b_j = 0$ for $j \ge n-s$, a direct computation reveals that $\bar{Z}_b$ is the diagonal matrix with entries $(s-n,\ldots,s-n,1)$. As $\bar{Z}_b$ is invertible, we can check that $\bar{Z}_b$ is not contained in $(I_{n-s}^{\perp})^{-1}$. Hence, $\mathcal{Z} \not\subseteq (I_{n-s}^{\perp})^{-1}$, which concludes the proof. □

6. Generic non-ML-maximal subspaces are bad

We now prove the other inclusion $\mathrm{NM}_{k,n} \subseteq \mathrm{Bad}_{k,n}$. Let $\mathcal{L}$ be a generic point in $\mathrm{NM}_{k,n}$ for some k,n. Since $\mathcal{L}$ is not ML-maximal, there is a non-zero Y in $\mathcal{L}^{-1} \cap \mathcal{L}^{\perp}$, which means that there is a matrix $X \in \mathcal{L}$ such that Y is contained in $\Gamma_{\mathcal{L}}(X)$, and this implies $XY = 0$. After a change of coordinates we can assume

$$X = \left[\begin{array}{c|c} I_u & 0 \\ \hline 0 & 0 \end{array}\right] \text{ and } Y = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & I_v \end{array}\right], \tag{7}$$

where $u = \operatorname{rank}(X)$, $v = \operatorname{rank}(Y)$. We will show that $\mathcal{L}$ is contained in $\mathrm{Bad}_{k,n}$. We start by treating some special instances of $\mathcal{L}$.

Lemma 6.1. *If $\binom{v+1}{2} < k$, then $\mathcal{L}$ is contained in $\mathrm{Bad}_{k,n}$.*

Proof. Note that, since $XY = 0$ and Y is an element of $\mathcal{L}^\perp$, we have that $\mathcal{L}^\perp$ is contained in the coisotropic variety $\mathrm{Ch}_c(D_v)$, where $c = \dim(\mathcal{L}^\perp) - \mathrm{codim}(D_v)$. By duality, this implies that $\mathcal{L}$ is contained in $\mathrm{Ch}_{k-\mathrm{codim}(D_{n-v})}(D_{n-v})$ [10, Lemma 3]. If we have $\binom{v+1}{2} < k$, then $n-v$ is either contained in the Pataki range described in [8, Theorem 2] or it exceeds the Pataki range. If $n-v$ is in the range, then $\mathcal{L}$ is contained in $\mathrm{Bad}_{k,n}$ by [8, Theorem 2]. If $n-v$ exceeds the range, then the coisotropic variety $\mathrm{Ch}_{k-\mathrm{codim}(D_{n-v})}(D_{n-v})$ is a subvariety of the last coisotropic hypersurface in the Pataki range. So $\mathcal{L}$ is again contained in $\mathrm{Bad}_{k,n}$ by [8, Theorem 2], and we are done in both cases. □

Lemma 6.2. *a) We have* $\mathrm{NM}_{1,n} = \emptyset$.

b) If $(k,u,v) = (3, n-3, 2)$, *then* $\mathcal{L}$ *is contained in* $\mathrm{Bad}_{3,n}$.

Proof. We start with part a). A one-dimensional regular linear subspace of $\mathbb{S}^n$ is a point in $\mathbb{PS}^n$ of full rank. Therefore its ML-degree is 1, and so is the degree of its reciprocal variety. We conclude that $\mathrm{NM}_{1,n} = \emptyset$ for all n.

For part b), let $\{X, B_1, B_2\}$ be a basis for $\mathcal{L}$ and write $\overline{B}_i \in \mathbb{S}^3$ for the lower-right 3×3 block of B_i for $i = 1, 2$. The locus D_1 of rank-one matrices in $\mathbb{S}^3$ is 3-dimensional. The subset of D_1 of matrices that annihilate both $\overline{B}_1$ and $\overline{B}_2$ is cut out by two hyperplanes, and is therefore at least one-dimensional. Hence, there is a non-zero matrix $\overline{N}$ of rank one such that $\mathrm{tr}(\overline{N}\overline{B}_1) = \mathrm{tr}(\overline{N}\overline{B}_2) = 0$. Let $N \in \mathbb{S}^n$ be the matrix with lower-right 3×3 block equal to $\overline{N}$, and zeroes everywhere else: $N = \left[\begin{smallmatrix} 0 & 0 \\ 0 & \overline{N} \end{smallmatrix}\right]$. By construction, the matrix N is in the polar space of X, B_1, and B_2, so we have $N \in \mathcal{L}^\perp$. As the rank of N is one, there is a matrix M in $\mathbb{S}^n$ of rank $n-1$ with $MN = 0$. Since $MN = 0$ poses no conditions on the upper-left $u \times u$ block of M, we can choose M such that this block is the identity matrix I_u. Now consider the matrix $X_\varepsilon = X + \varepsilon(M - X)$. For small enough $\varepsilon \neq 0$, this matrix has rank $n-1$. Since $MN = 0$ and $XN = 0$, we also have $X_\varepsilon N = 0$. Let $\mathcal{L}_\varepsilon$ be the linear space spanned by X_ε, B_1, and B_2. Now $\mathcal{L}_\varepsilon$ is contained in $N^\perp$, which is the tangent hyperplane of D_{n-1} at X_ε. We conclude that $\mathcal{L}_\varepsilon$ is contained in the coisotropic hypersurface $\mathrm{Ch}_2(D_{n-1})$. Thus, $\mathcal{L} = \lim_{\varepsilon \to 0} \mathcal{L}_\varepsilon$ is in $\mathrm{Ch}_2(D_{n-1})$ as well. Since this is within the Pataki range, $\mathcal{L}$ is contained in $\mathrm{Bad}_{3,n}$. □

Lemma 6.3. *If* $u + v < n$, *then* $\mathcal{L}$ *is contained in* $\mathrm{Bad}_{k,n}$.

Proof. By Lemma 6.1, we can assume that we have

$$\binom{v+1}{2} \geq k. \tag{8}$$

Note that the set $\{M \in \mathbb{S}^n \mid XM = 0\}$ is isomorphic to $\mathbb{S}^{n-u}$, and since $\mathcal{L}^\perp$ has codimension k, it follows that the space $A_X := \{M \in \mathcal{L}^\perp \mid XM = 0\}$, considered as a subset of $\mathbb{S}^{n-u}$, has dimension at least $\binom{n-u+1}{2}-k$. Consider the locus D_{v-1} of matrices of rank at most $v-1$ in $\mathbb{S}^{n-u}$, which has codimension $\binom{n-u-v+2}{2}$. We conclude that the dimension of $A_X \cap D_{v-1}$ is at least $\binom{n-u+1}{2}-\binom{n-u-v+2}{2}-k$, which by (8) is at least

$$\binom{n-u+1}{2}-\binom{n-u-v+2}{2}-\binom{v+1}{2}=(v-1)(n-u-v)-1. \tag{9}$$

Set $v'=v-1$, and $\delta=n-u-v$. If $\dim(A_X\cap D_{v-1})=0$, then we have (9) <1, which holds if and only if either $v=1$ or $(v,\delta)=(2,1)$. In the first case, we would have $k=1$ by (8). In the second case, we have $n-u=3$ and $k=3$ by (8), because otherwise, if $v=2\geq k$, then we have a strict inequality in (8), so $\dim(A_X\cap D_{v-1})\geq 1$. Both of the cases are treated in Lemma 6.2.

Now assume $\dim(A_X\cap D_{v-1})\geq 1$, so we have a non-zero matrix in the intersection $A_X\cap D_{v'}$. As in the proof of Lemma 6.1, this implies that $\mathcal{L}$ is contained in $\mathrm{Ch}_{k-\mathrm{codim}(D_{n-v'})}(D_{n-v'})$. If we have $\binom{v'+1}{2}<k$, then $\mathcal{L}$ is in $\mathrm{Bad}_{k,n}$ by (the proof of) Lemma 6.1. If, on the other hand, we have $\binom{v'+1}{2}\geq k$, then we can do the same computation as before using v' instead of v. Therefore, we can continue this until we find an integer $v''<v$ such that $\mathcal{L}$ is contained in $\mathrm{Ch}_{k-\mathrm{codim}(D_{n-v''})}(D_{n-v''})$, and $\binom{v''+1}{2}<k$. ☐

Proposition 6.4. $\mathrm{NM}_{k,n}$ *is contained in* $\mathrm{Bad}_{k,n}$.

Proof. As explained above, it suffices to show that $\mathcal{L}$ is contained in $\mathrm{Bad}_{k,n}$. By Lemma 6.3 we can assume that we have $u+v=n$, and by Lemma 6.1 we can assume $\binom{v+1}{2}\geq k$. If there is no s such that $k=\binom{n-s+1}{2}$, it follows from [8, Theorem 3] that the set $C_{k,n}$ equals $\mathrm{Bad}_{k,n}$. Since $\mathrm{NM}_{k,n}$ is contained in $C_{k,n}$ (Corollary 4.7), we conclude that in that case, $\mathcal{L}$ is contained in $\mathrm{Bad}_{k,n}$. Therefore, from now on we assume that there is an s such that $k=\binom{n-s+1}{2}$. Combining this with (8), we find $v\geq n-s$. We distinguish two cases.

Case 1: Assume that there is a matrix B_1 in $\mathcal{L}$, which is linearly independent of X, and such that its lower-right $v\times v$ block consists of only zeroes. Then we can choose $B_2,\dots,B_{k-1}$ such that $\mathcal{L}$ is spanned by $X,B_1,B_2,\dots,B_{k-1}$. We now show that $\mathcal{L}\in\mathrm{Bad}_{k,n}$ by a similar construction as in the proof of Lemma 6.2. We write $\overline{B}_i$ for the lower-right $v\times v$ block of B_i, for $i\in\{1,\dots,k-1\}$. Recall that the dimension of the determinantal variety D_{n-s-1} of matrices with rank at most $n-s-1$ in $\mathbb{S}^v$ is

$$\binom{v+1}{2}-\binom{v-n+s+2}{2}. \tag{10}$$

We want to show that (10) is strictly larger than $k-2$. This is true if and only if we have $\binom{v+1}{2}+2>\binom{t+1}{2}+\binom{v-t+2}{2}$, where $t=n-s$. After rewriting this, one can see that this is equivalent to $t(v-t)+1 > v-t$. But this holds, since $v \geq t$ and $t \geq 1$ (otherwise k would be 0). We conclude that the dimension of D_{n-s-1} in $\mathbb{S}^v$ is strictly larger than $k-2$.

Thus, the set of matrices in D_{n-s-1} that are in the polar space of $\overline{B}_2, \ldots, \overline{B}_{k-1}$ is at least one-dimensional; so we can pick a matrix $\overline{N}$ in that set. As in the proof of Lemma 6.2, we let N be the matrix in $\mathbb{S}^n$ whose lower-right $v \times v$ block equals $\overline{N}$, and that has zeroes everywhere else. Since $\overline{B}_1 = 0$, we have $N \in \mathcal{L}^{\perp}$. Again, we can choose an $M \in \mathbb{S}^n$ of rank $n - \operatorname{rank}(N)$ such that $MN = 0$, and whose upper-left $u \times u$ block is the identity matrix I_u. Now let $\mathcal{L}_\varepsilon$ be the space spanned by $X_\varepsilon = X + \varepsilon(M - X)$ and $B_1, \ldots, B_{k-1}$. For sufficiently small $\varepsilon \neq 0$, the rank of X_ε is $\operatorname{rank}(M)$. By construction, we obtain $X_\varepsilon N = 0$ and $\mathcal{L}_\varepsilon \subset N^{\perp}$, and therefore $\mathcal{L}_\varepsilon$ is contained in the coisotropic variety in $\operatorname{Gr}(k, \mathbb{S}^n)$ associated to $D_{\operatorname{rank}(M)}$. Since the rank of M is at least $s+1$, it either falls in the Pataki range or exceeds it, and thus the latter coisotropic variety is contained in $\operatorname{Bad}_{k,n}$ (as in the proof of Lemma 6.2). We conclude that $\mathcal{L} = \lim_{\varepsilon \to 0} \mathcal{L}_\varepsilon$ is contained in $\operatorname{Bad}_{k,n}$.

Case 2: Now we assume that $\mathcal{L}$ does not contain a matrix that is both linearly independent of X and has its lower-right $v \times v$ block consisting of only zeroes. We show that this implies that Y is not contained in the total transform $\Gamma_{\mathcal{L}}(X)$, contradicting the fact that Y was chosen in $\Gamma_{\mathcal{L}}(X)$ at the beginning of this section.

As before, the elements in $\Gamma_{\mathcal{L}}(X)$ are the adjugates of all perturbations of X. For a vector $b = (b_1, \ldots, b_{k-1})$ in $\mathbb{C}[[\varepsilon]]^{k-1}$, we write $b_i = b_{i0} + b_{i1}\varepsilon + \cdots$ and

$$X_b := X + \varepsilon \sum_{i=1}^{k-1} b_i B_i = X + \varepsilon X_1 + \varepsilon^2 X_2 + \ldots,$$

where $X_j := \sum_{i=1}^{k-1} b_{i,j-1} B_i \in \mathcal{L}$. By our Case-2 assumption, we have for all j that either X_j is a multiple of X or $\overline{X}_j \neq 0$. If X_b is a multiple of X, the lowest degree term of its adjugate cannot yield Y. Hence, we may assume that X_b is not a multiple of X, which implies $\overline{X}_b \neq 0$ by the Case-2 assumption. Moreover, since $\overline{B}_1, \ldots, \overline{B}_{k-1}$ span a subspace of $I_v^{\perp}$, the lowest degree non-zero term of $\overline{X_b}$ is also in $I_v^{\perp}$ and thus cannot be a multiple of the identity matrix I_v. Now the following lemma concludes the proof as it shows that the lowest degree term of $\operatorname{adj}(X_b)$ cannot be equal to Y. $\square$

As above, for a matrix A, we write $\overline{A}$ for the lower $(v \times v)$-block of A.

Lemma 6.5. *Let X' be a matrix with entries in $\mathbb{C}[[\varepsilon]]$ of the form $X' = X + \sum_{j=1}^{\infty} \varepsilon^j X_j$ such that for each j, either $\overline{X}_j \neq 0$ or X_j is a multiple of X. Assume that $u + v = n$ and that the lowest degree term of $\operatorname{adj}(X')$ is a multiple of Y. Then the lowest degree term of $\overline{X'}$ is a multiple of the identity matrix I_v.*

Proof. Let $a > 0$ be such that $\operatorname{adj}(X') = \varepsilon^a Y_a + \varepsilon^{a+1} Y_{a+1} + \cdots$ and $Y_a \neq 0$, then by assumption we have $Y_a = \lambda Y$ for some constant $\lambda \neq 0$. Let m be the minimal integer for which X_m is not a multiple of X; again by assumption, it follows that $\overline{X}_m \neq 0$. Moreover, for all $j < m$ we have that the matrix X_j is a multiple of X, which implies $\overline{X_j A} = 0$ for any $n \times n$-matrix A. It follows that we have

$$\begin{aligned}\overline{X' \operatorname{adj}(X')} &= \overline{(X + \varepsilon X_1 + \cdots)(\varepsilon^a Y_a + \varepsilon^{a+1} Y_{a+1} + \cdots)}\\ &= \overline{(\varepsilon^m X_m + \varepsilon^{m+1} X_{m+1} + \cdots)(\varepsilon^a Y_a + \varepsilon^{a+1} Y_{a+1} + \cdots)}\\ &= \varepsilon^{m+a}\overline{X_m Y_a} + \text{ h.o.t.}\end{aligned}$$

For any matrix A, we have that $\overline{YAY} = \overline{A}$. Using the fact that $Y_a = \lambda Y$ we obtain

$$\overline{X_m Y_a} = \overline{Y X_m \lambda Y^2} = \overline{\lambda Y X_m Y} = \lambda \overline{X}_m,$$

so the lowest degree non-zero term of $\overline{X' \operatorname{adj}(X')}$ is $\varepsilon^{m+a} \lambda \overline{X}_m$. As $X' \operatorname{adj}(X') = f I_n$ for some $f \in \mathbb{C}[[\varepsilon]]$, we conclude that $\overline{X}_m$ is a multiple of I_v. □

Proof of Theorem 1.2. This follows from Propositions 5.1 and 6.4. □

7. Example: the case $n = 3$

In this section we describe in detail the set $\mathrm{NM}_{k,n}$ for $n = 3$.

The hypersurface $\mathrm{NM}_{5,3} = \mathrm{Ch}_2(D_1)$ has degree three in $\mathrm{Gr}(5, \mathbb{S}^3) \cong \mathbb{P}^5$, consisting of all subspaces $\mathcal{L} = A^\perp$ where $\det A = 0$. This matrix satisfies $\operatorname{trace}(A \cdot \operatorname{adj}(A)) = 0$, so $A \in \mathcal{L}^\perp$. Hence $A \in \mathcal{L}^{-1} \cap \mathcal{L}^\perp$.

The hypersurface $\mathrm{NM}_{4,3} \cong \mathrm{Ch}_1(D_1)$ can be identified with $\mathrm{NM}_{2,3} \cong \mathrm{Ch}_1(D_2)$ under $\mathrm{Gr}(4, \mathbb{S}^3) \cong \mathrm{Gr}(2, \mathbb{S}^3)$. The latter contains those pencils with Segre symbol [2,1] [5], with canonical representation as the span of $A_1 = \begin{bmatrix} 0 & a & 0 \\ a & 1 & 0 \\ 0 & 0 & b \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ for $a \neq b \in \mathbb{R}$. The matrix $A_1 - aA_2$ looks like $\begin{bmatrix} 0 & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}$ with adjoint $\begin{bmatrix} * & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathcal{L}^{-1} \cap \mathcal{L}^\perp$.

The hypersurface $\mathrm{NM}_{3,3} \cong \mathrm{Ch}_2(D_2) \cong \mathrm{Ch}_0(D_1)$. The last term is the Chow form of the Veronese embedding of $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5 = \mathbb{P}(\mathbb{S}^3)$, or equivalently, the resultant of three ternary quadrics.

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