

## EQUATIONS AND MULTIDEGREES FOR INVERSE SYMMETRIC MATRIX PAIRS

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We compute the equations and multidegrees of the biprojective variety that parametrizes pairs of symmetric matrices that are inverse to each other. As a consequence of our work, we provide an alternative proof for a result of Manivel, Michałek, Monin, Seynnaeve and Vodička that settles a previous conjecture of Sturmfels and Uhler regarding the polynomiality of maximum likelihood degree.

### 1. Introduction

The purpose of this paper is to study the biprojective variety that parametrizes pairs of symmetric matrices that are inverse to each other. Let  $\mathbb{S}^n$  be the space of symmetric  $n \times n$  matrices over the complex numbers  $\mathbb{C}$ . Let  $\mathbb{P}^{m-1}$  be the projectivization  $\mathbb{P}^{m-1} = \mathbb{P}(\mathbb{S}^n)$  of  $\mathbb{S}^n$ , where  $m = \binom{n+1}{2}$ . We are interested in the biprojective variety  $\Gamma \subset \mathbb{P}^{m-1} \times \mathbb{P}^{m-1}$  given as follows

$$\Gamma := \overline{\{(M, M^{-1}) \mid M \in \mathbb{P}(\mathbb{S}^n) \text{ and } \det(M) \neq 0\}} \subset \mathbb{P}^{m-1} \times \mathbb{P}^{m-1};$$

i.e., the closure of all possible pairs of an invertible symmetric matrix and its inverse.

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Our main results are determining the equations and multidegrees of the biprojective variety  $\Gamma$ . Before presenting them, we establish some notation. Let  $X = (X_{i,j})_{1 \leq i,j \leq n}$  and  $Y = (Y_{i,j})_{1 \leq i,j \leq n}$  be generic symmetric matrices; i.e.,  $X_{i,j}$  and  $Y_{i,j}$  are new variables over  $\mathbb{C}$ . Let  $R$  be the standard graded polynomial ring  $R = \mathbb{C}[X_{i,j}]$ , and  $S$  be the standard bigraded polynomial ring  $S = \mathbb{C}[X_{i,j}, Y_{i,j}]$  where  $\text{bideg}(X_{i,j}) = (1, 0)$  and  $\text{bideg}(Y_{i,j}) = (0, 1)$ .

Let  $\mathfrak{J} \subset S$  be ideal of the defining equations of  $\Gamma$ .

As  $\dim(\Gamma) = m - 1$ , for each  $i, j \in \mathbb{N}$  with  $i + j = m - 1$ , one considers the *multidegree*  $\text{deg}^{i,j}(\Gamma)$  of  $\Gamma$  of type  $(i, j)$ . Geometrically,  $\text{deg}^{i,j}(\Gamma)$  equals the number of points in the intersection of  $\Gamma$  with the product  $L \times M \subset \mathbb{P}^{m-1} \times \mathbb{P}^{m-1}$ , where  $L \subset \mathbb{P}^{m-1}$  and  $M \subset \mathbb{P}^{m-1}$  are general linear subspaces of dimension  $m - 1 - i$  and  $m - 1 - j$ , respectively. Following the notation of [18, §8.5], we say that the *multidegree polynomial* of  $\Gamma$  is given by

$$\mathcal{C}(\Gamma; t_1, t_2) := \sum_{i+j=m-1} \text{deg}^{i,j}(\Gamma) t_1^{m-1-i} t_2^{m-1-j} \in \mathbb{N}[t_1, t_2]$$

(also, see [2, Theorem A], [1, Remark 2.9]).

A fundamental idea in our approach is to reduce the study of  $\Gamma$  to instead considering the biprojective variety of pairs of symmetric matrices with product zero. Let  $\Sigma \subset \mathbb{P}^{m-1} \times \mathbb{P}^{m-1}$  be the biprojective variety parametrized by pairs of symmetric matrices with product zero; i.e., by pairs of symmetric matrices  $(M, N) \in \mathbb{P}^{m-1} \times \mathbb{P}^{m-1}$  such that  $MN = 0$ . The ideal of defining equations of  $\Sigma$  is clearly given by

$$I_1(XY),$$

where  $I_1(XY)$  denotes the ideal generated by the  $1 \times 1$ -minors (i.e, the entries) of the matrix  $XY$ . Similarly, since  $\dim(\Sigma) = m - 2$ , we define the multidegree polynomial

$$\mathcal{C}(\Sigma; t_1, t_2) := \sum_{i+j=m-2} \text{deg}^{i,j}(\Sigma) t_1^{m-1-i} t_2^{m-1-j} \in \mathbb{N}[t_1, t_2]$$

of  $\Sigma$ .

The theorem below provides the defining equations of  $\Gamma$ . It also shows that the study of  $\mathcal{C}(\Gamma; t_1, t_2)$  can be substituted to considering  $\mathcal{C}(\Sigma; t_1, t_2)$  instead. Our proof depends on translating our questions in terms of Rees algebras and on using the results of Kotsev [12] (also, see [4, 19, 21]).

**Theorem A.** Under the above notations, the following statements hold:

(i)  $\mathfrak{J}$  is a prime ideal given by

$$\mathfrak{J} = I_1(XY - b\text{Id}_n) = \left( \begin{array}{l} \sum_{k=1}^n X_{i,k} Y_{k,j}, \quad 1 \leq i \neq j \leq n \\ \sum_{k=1}^n X_{i,k} Y_{k,i} - \sum_{k=1}^n X_{j,k} Y_{k,j}, \quad 1 \leq i, j \leq n \end{array} \right)$$

where  $b = (XY)_{1,1} = \sum_{l=1}^n X_{1,k} Y_{k,1} \in S$  and  $\text{Id}_n$  denotes the  $n \times n$  identity matrix.

(ii) We have the following equality relating multidegree polynomials

$$t_1^m + t_2^m + \mathcal{C}(\Sigma; t_1, t_2) = (t_1 + t_2) \cdot \mathcal{C}(\Gamma; t_1, t_2).$$

Our second main result is obtaining general formulas for the multidegrees of  $\Gamma$  and  $\Sigma$ . Here our approach depends on previous computations that were made by Nie, Ranestad and Sturmfels [20], and by von Bothmer and Ranestad [7]. The formula we obtained is expressed in terms of a function on subsequences of  $\{1, \dots, n\}$ . Let

$$\psi_i = 2^{i-1}, \quad \psi_{i,j} = \sum_{k=i}^{j-1} \binom{i+j-2}{k} \quad \text{when } i < j,$$

and for any  $\alpha = (\alpha_1, \dots, \alpha_r) \subset \{1, \dots, n\}$  let

$$\psi_\alpha = \begin{cases} \text{Pf}(\psi_{\alpha_k, \alpha_l})_{1 \leq k < l \leq n} & \text{if } r \text{ is even,} \\ \text{Pf}(\psi_{\alpha_k, \alpha_l})_{0 \leq k < l \leq n} & \text{if } r \text{ is odd,} \end{cases}$$

where  $\psi_{\alpha_0, \alpha_k} = \psi_{\alpha_k}$  and Pf denotes the Pfaffian. For any  $\alpha \subset \{1, \dots, n\}$ , the complement  $\{1, \dots, n\} \setminus \alpha$  is denoted by  $\alpha^c$ . By an abuse of notation we set  $\psi_\emptyset = 1$ .

**Theorem B.** Under the above notations, the following statements hold:

(i) The multidegree polynomial of  $\Sigma$  is determined by the equation

$$t_1^m + t_2^m + \mathcal{C}(\Sigma; t_1, t_2) = \sum_{d=0}^m \beta(n, d) t_1^{m-d} t_2^d,$$

where

$$\beta(n, d) := \sum_{\substack{\alpha \subset \{1, \dots, n\} \\ \|\alpha\| = d}} \psi_\alpha \psi_{\alpha^c};$$

in the last sum  $\alpha$  runs over all strictly increasing subsequences of  $\{1, \dots, n\}$ , including the case  $\alpha = \emptyset$ , and  $\|\alpha\|$  denotes the sum of the entries of  $\alpha$ .

(ii) For each  $0 \leq d \leq m - 1$ , we have the equality

$$\text{deg}^{m-1-d,d}(\Gamma) = \sum_{j=0}^d (-1)^j \beta(n, d - j).$$

Our last interest is on the maximum likelihood degree (ML-degree) of the general linear concentration model (see [22], [17] for more details). Let  $\mathcal{L}$  be a general linear subspace of dimension  $d$  in  $\mathbb{S}^n$ , and denote by  $\mathcal{L}^{-1}$  the  $(d - 1)$ -dimensional projective subvariety of  $\mathbb{P}^{m-1} = \mathbb{P}(\mathbb{S}^n)$  obtained by inverting the matrices in  $\mathcal{L}$ . From [22, Theorem 1], the ML-degree of the general linear concentration model, denoted as  $\phi(n, d)$ , is equal to the degree of the projective variety  $\mathcal{L}^{-1}$ . From the way  $\Gamma$  is defined, it then follows that

$$\phi(n, d) = \text{deg}^{m-d, d-1}(\Gamma). \tag{1}$$

So, the computation of the invariants  $\phi(n, d)$  can be reduced to determining the multidegrees of  $\Gamma$  (which we did in Theorem B).

Finally, by using Theorem B and a result of Manivel, Michałek, Monin, Seynnaeve and Vodička regarding the polynomiality in  $n$  of the function  $\psi_{\{1, \dots, n\} \setminus \alpha}$  (see Theorem 4.1), we obtain an alternative proof to a previous conjecture of Sturmfels and Uhler (see [22, p. 611]).

**Corollary C** (Manivel-Michałek-Monin-Seynnaeve-Vodička; [14, Theorem 1.3]). For each  $d \geq 1$ , the function  $\phi(n, d)$  coincides with a polynomial of degree  $d - 1$  in  $n$ .

The basic outline of this paper is as follows. In Section 2, we compute the defining equations of  $\Gamma$ . In Section 3, we determine the multidegrees of  $\Gamma$  and  $\Sigma$ . In Section 4, we show the polynomiality of  $\phi(n, d)$ .

## 2. The defining equations of $\Gamma$

During this section, we compute the defining equations of the variety  $\Gamma$ . The following setup is used throughout the rest of this paper.

**Setup 2.1.** Let  $X = (X_{i,j})_{1 \leq i, j \leq n}$  and  $Y = (Y_{i,j})_{1 \leq i, j \leq n}$  be generic symmetric matrices over  $\mathbb{C}$ . Let  $R$  be the standard graded polynomial ring  $R = \mathbb{C}[X_{i,j}]$ , and  $S$  be the standard bigraded polynomial ring  $S = \mathbb{C}[X_{i,j}, Y_{i,j}]$  where  $\text{bideg}(X_{i,j}) = (1, 0)$  and  $\text{bideg}(Y_{i,j}) = (0, 1)$ . Let  $I = I_{n-1}(X)$  be the ideal of  $(n - 1) \times (n - 1)$ -minors of  $X$ . Let  $t$  be a new indeterminate. The Rees algebra  $\mathcal{R}(I) := \bigoplus_{n=0}^{\infty} I^n t^n \subset R[t]$  of  $I$  can be presented as a quotient of  $S$  by using the map

$$\begin{aligned} \Psi : S &\longrightarrow \mathcal{R}(I) \subset R[t] \\ Y_{i,j} &\mapsto Z_{i,j}t, \end{aligned}$$

where  $Z_{i,j} \in R$  is the signed minor obtained by deleting  $i$ -th row and the  $j$ -th column. We set  $\text{bideg}(t) = (-n + 1, 1)$ , which implies that  $\Psi$  is bihomogeneous of degree zero, and so  $\mathcal{R}(I)$  has a natural structure of bigraded  $S$ -algebra.

Our point of departure comes from the following simple remarks.

**Remark 2.2.** For any matrix  $M \in \mathbb{S}^n$ , we denote its adjoint matrix as  $M^+$ . For any  $M \in \mathbb{S}^n$  with  $\det(M) \neq 0$ , since  $M^{-1} = \frac{1}{\det(M)}M^+$ , it follows that  $M^{-1}$  and  $M^+$  represent the same point in  $\mathbb{P}^{m-1} = \mathbb{P}(\mathbb{S}^n)$ . Thus, we have that  $\Gamma$  can be equivalently described as

$$\Gamma = \overline{\{(M, M^+) \mid M \in \mathbb{P}(\mathbb{S}^n) \text{ and } \det(M) \neq 0\}} \subset \mathbb{P}^{m-1} \times \mathbb{P}^{m-1}.$$

Denote by  $\mathcal{F} : \mathbb{P}^{m-1} \dashrightarrow \mathbb{P}^{m-1}$  the rational map determined by signed minors  $Z_{i,j}$ , that is,

$$\mathcal{F} : \mathbb{P}^{m-1} \dashrightarrow \mathbb{P}^{m-1}, \quad (X_{1,1} : X_{1,2} : \cdots : X_{n,n}) \mapsto (Z_{1,1} : Z_{1,2} : \cdots : Z_{n,n}).$$

Therefore, we obtain that  $\Gamma$  coincides with

$$\Gamma = \overline{\text{graph}(\mathcal{F})} \subset \mathbb{P}^{m-1} \times \mathbb{P}^{m-1},$$

the closure of the graph of the rational map  $\mathcal{F}$ .

**Remark 2.3.** Notice that  $I = I_{n-1}(X)$  by construction is the base ideal of the rational map  $\mathcal{F}$  – the ideal generated by a linear system defining the rational map. So, it is a basic result that the Rees algebra  $\mathcal{R}(I)$  coincides with the bihomogeneous coordinate ring of the closure of the graph of  $\mathcal{F}$ . By Remark 2.2, the bihomogeneous coordinate ring of  $\Gamma$  is given by the Rees algebra  $\mathcal{R}(I)$ . Hence, in geometrical terms, we have the identification

$$\Gamma = \text{BiProj}(\mathcal{R}(I)) \subset \mathbb{P}^{m-1} \times \mathbb{P}^{m-1}.$$

In more algebraic terms: the ideal  $\mathfrak{J} \subset S$  considered in the Introduction coincides with the defining equations of the Rees algebra, that is,  $\mathfrak{J} = \text{Ker}(\Psi)$ . For the relations between rational maps and Rees algebras, see, e.g., [3, Section 3].

In general the Rees algebra is a very difficult object to study, but, under the present conditions we shall see that it coincides with the symmetric algebra  $\text{Sym}(I)$  of  $I$  (i.e., the ideal  $I$  is of linear type). So, the main idea is to bypass the Rees algebra and consider the symmetric algebra instead.

From a graded presentation of  $I$

$$F_1 \xrightarrow{\varphi} F_0 \xrightarrow{(Z_{1,1}, Z_{1,2}, \dots, Z_{n,n})} I \rightarrow 0,$$

the symmetric algebra  $\text{Sym}(I)$  automatically gets the presentation

$$\text{Sym}(I) \cong S/I_1([Y_{i,j}] \cdot \varphi) \tag{2}$$

and obtains a natural structure of bigraded  $S$ -algebra (for more details on the symmetric algebra, see, e.g., [5, §A2.3]). In general, we have a canonical exact sequence of bigraded  $S$ -modules relating both algebras

$$0 \rightarrow \mathcal{K} \rightarrow \text{Sym}(I) \rightarrow \mathcal{R}(I) \rightarrow 0,$$

where  $\mathcal{K}$  equals the  $R$ -torsion of  $\text{Sym}(I)$  (see [16]). However, in the present case, we shall see that  $\text{Sym}(I) = \mathcal{R}(I)$ .

We are now ready to compute the defining equations of  $\Gamma$ .

*Proof of Theorem A (i).* Due to Remark 2.3, it suffices to compute the defining equations of the Rees algebra  $\mathcal{R}(I)$ . From [12, Theorem A] we have that  $I$  is of linear type, i.e., the canonical map

$$\text{Sym}(I) \twoheadrightarrow \mathcal{R}(I)$$

is an isomorphism. So,  $\mathfrak{J}$  coincides with the ideal of defining equations of  $\text{Sym}(I)$ . By using [11] or [6] we obtain an explicit  $R$ -free resolution for the ideal  $I$  which is of the form  $0 \rightarrow J_3 \rightarrow J_2 \xrightarrow{\varphi} J_1 \rightarrow R \rightarrow R/I \rightarrow 0$ . From the presentation  $\varphi$  of  $I$ , we obtain the ideal

$$\mathfrak{J} = I_1([Y_{i,j}] \cdot \varphi)$$

of defining equations of the symmetric algebra (see (2) above). Therefore,  $\mathcal{R}(I) = \text{Sym}(I)$  is a bigraded  $S$ -algebra presented by the quotient

$$\mathcal{R}(I) = \text{Sym}(I) \cong S/\mathfrak{J},$$

and from the description of  $\varphi$  (the syzygies of  $I$ ) given in [11] or [6] we obtain

$$\mathfrak{J} = \left( \begin{array}{l} \sum_{k=1}^n X_{i,k} Y_{k,j}, \quad 1 \leq i \neq j \leq n \\ \sum_{k=1}^n X_{i,k} Y_{k,i} - \sum_{k=1}^n X_{j,k} Y_{k,j}, \quad 1 \leq i, j \leq n \end{array} \right).$$

Finally, it is clear that  $\mathfrak{J} = I_1(XY - b\text{Id}_n)$ . □

### 3. Computation of the multidegrees of $\Gamma$

In this section, we concentrate on computing the multidegrees of  $\Gamma$ . The idea is to reduce this computation to instead compute the multidegrees of  $\Sigma$  and then to use previous results obtained in [20] and [7].

For each  $r_1, r_2 \in \mathbb{N}$ , we define the following ideal

$$J(r_1, r_2) := I_1(XY) + I_{r_1+1}(X) + I_{r_2+1}(Y) \subset S.$$

The following proposition yields a primary decomposition of the ideal  $I_1(XY)$  in terms of the ideals  $J(r_1, r_2)$ . Its proof is easily obtained by using results from [12]. Similarly, the properties of  $I_1(XY)$  described below are known in a more geometric language (see [10, Proposition 16] and the references given therein).

**Proposition 3.1.** The following statements hold:

- (i) If  $r_1 + r_2 \leq n$ , then  $J(r_1, r_2)$  is a prime ideal.
- (ii) The ideal  $I_1(XY)$  is equidimensional of dimension  $m = \binom{n+1}{2}$  and radical with primary decomposition

$$I_1(XY) = \bigcap_{r=0}^n J(r, n-r).$$

*Proof.* (i) From [12, Proposition 4.5], we have that  $B(r_1, r_2) = S/J(r_1, r_2)$  is a domain, so the result is clear.

(ii) By [12, Lemma 4.6], we know that the canonical map

$$\frac{S}{I_1(XY)} \rightarrow \prod_{r=0}^n \frac{S}{J(r, n-r)}$$

is injective. So, it is clear that  $I_1(XY) = \bigcap_{r=0}^n J(r, n-r)$ . The dimension of the Rees algebra is equal to  $\dim(\mathcal{R}(I)) = \dim(R) + 1 = m + 1$  (see, e.g., [8, Theorem 5.1.4]). By (5),  $S/I_1(XY) \cong \mathcal{R}(I)/b\mathcal{R}(I)$ , and so Krull’s Principal Ideal Theorem (see, e.g., [15, Theorem 13.5]) yields that

$$\dim(S/J(r, n-r)) = \dim(\mathcal{R}(I)) - 1 = m$$

for each  $0 \leq r \leq n$ . Therefore, the result follows. □

We now recall how to define the multidegree polynomial  $\mathcal{C}(\mathcal{R}(I); t_1, t_2)$  of  $\mathcal{R}(I)$  by using the Hilbert series of  $\mathcal{R}(I)$  (see [18, §8.5]). We can write the Hilbert series

$$\text{Hilb}_{\mathcal{R}(I)}(t_1, t_2) := \sum_{v_1, v_2 \in \mathbb{N}} \dim_{\mathbb{C}}([\mathcal{R}(I)]_{v_1, v_2}) t_1^{v_1} t_2^{v_2} \in \mathbb{N}[[t_1, t_2]]$$

in the following way

$$\text{Hilb}_{\mathcal{R}(I)}(t_1, t_2) = \frac{K(\mathcal{R}(I); t_1, t_2)}{(1-t_1)^m (1-t_2)^m},$$

where  $K(\mathcal{R}(I); t_1, t_2)$  is called the  $K$ -polynomial of  $\mathcal{R}(I)$  (for instance, by just computing a bigraded free  $S$ -resolution of  $\mathcal{R}(I)$ ). Then, we define

$$\mathcal{C}(\mathcal{R}(I); t_1, t_2) := \text{sum of the terms of } K(\mathcal{R}(I); 1-t_1, 1-t_2) \text{ of degree } = m-1.$$

Additionally, we remark that  $m - 1$  is the minimal degree of the terms of

$$K(\mathcal{R}(I); 1 - t_1, 1 - t_2).$$

In a similar way, we define the multidegree polynomials

$$\mathcal{C}(S/I_1(XY); t_1, t_2) \quad \text{and} \quad \mathcal{C}(S/J(r, n - r); t_1, t_2)$$

for each  $0 \leq r \leq n$ .

The multidegrees of the particular cases  $S/J(0, n)$  and  $S/J(n, 0)$  are easily handled by the following remark.

**Remark 3.2.** Since  $J(0, n) = I_1(X) = (X_{i,j})$  and  $J(n, 0) = I_1(Y) = (Y_{i,j})$ , it follows from the definition of multidegrees that

$$\mathcal{C}(S/J(0, n); t_1, t_2) = t_1^m \quad \text{and} \quad \mathcal{C}(S/J(n, 0); t_1, t_2) = t_2^m.$$

For notational purposes, we denote by  $\mathfrak{N} := (X_{i,j}) \cap (Y_{i,j}) \subset S$  the irrelevant ideal in the current biprojective setting. We have the following equivalent descriptions of  $\Gamma$  and  $\Sigma$  in terms of the BiProj construction

$$\Gamma = \text{BiProj}(\mathcal{R}(I)) = \{P \in \text{Spec}(\mathcal{R}(I)) \mid P \text{ is bihomogeneous and } P \not\supseteq \mathfrak{N}\mathcal{R}(I)\} \tag{3}$$

and

$$\Sigma = \text{BiProj}(T) = \{P \in \text{Spec}(T) \mid P \text{ is bihomogeneous and } P \not\supseteq \mathfrak{N}T\}, \tag{4}$$

where  $T = S/I_1(XY)$ . For more details on the BiProj construction, the reader is referred to [9, §1].

Next, we have a remark showing that the multidegree polynomial of  $\Gamma$  as introduced before coincides with the multidegree polynomial of the Rees algebra  $\mathcal{R}(I)$ .

**Remark 3.3.** Due to (3), the fact that  $(0 :_{\mathcal{R}(I)} \mathfrak{N}^\infty) = 0$  and [1, Remark 2.9], it follows that

$$\mathcal{C}(\Gamma; t_1, t_2) = \mathcal{C}(\mathcal{R}(I); t_1, t_2).$$

On the other hand, the following remark shows that the multidegree polynomials of  $\Sigma$  and  $S/I_1(XY)$  do not agree. Indeed, the minimal primes  $J(0, n)$  and  $J(n, 0)$  of  $I_1(XY)$  are irrelevant from a geometric point of view, and so they are taken into account in the multidegree polynomial of  $I_1(XY)$  but not in the one of  $\Sigma$ .



**Remark 3.4.** For ease of notation, set  $T = S/I_1(XY)$ . Directly from (4), we get that

$$\Sigma = \text{BiProj}(T) = \text{BiProj}\left(\frac{T}{(0 :_T \mathfrak{N}^\infty)}\right) = \text{BiProj}\left(\frac{S}{\bigcap_{r=1}^{n-1} J(r, n-r)}\right).$$

Let  $T' = S/\bigcap_{r=1}^{n-1} J(r, n-r)$ . Thus, since  $(0 :_{T'} \mathfrak{N}^\infty) = 0$ , [1, Remark 2.9] yields the equality  $\mathcal{C}(\Sigma; t_1, t_2) = \mathcal{C}(T'; t_1, t_2)$ . Therefore, from Proposition 3.1, Remark 3.2 and the additivity of multidegrees (see [18, Theorem 8.53]) we obtain the equality

$$\mathcal{C}(S/I_1(XY); t_1, t_2) = t_1^m + t_2^m + \mathcal{C}(\Sigma; t_1, t_2).$$

The next result provides an important relation between the multidegrees of  $\Gamma$  and  $\Sigma$ .

*Proof of Theorem A (ii).* First, we note the following trivial equality

$$\mathfrak{J} + bS = I_1(XY - b\text{Id}_n) + bS = I_1(XY).$$

As  $\mathcal{R}(I) \cong S/\mathfrak{J}$  is clearly a domain and  $\text{bideg}(b) = (1, 1)$ , we obtain the short exact sequence

$$0 \rightarrow \mathcal{R}(I)(-1, -1) \xrightarrow{b} \mathcal{R}(I) \rightarrow S/I_1(XY) \rightarrow 0 \tag{5}$$

and that  $\dim(S/I_1(XY)) = \dim(\mathcal{R}(I)) - 1$ . Consequently, we get the following equality relating Hilbert series

$$\text{Hilb}_{S/I_1(XY)}(t_1, t_2) = (1 - t_1 t_2) \cdot \text{Hilb}_{\mathcal{R}(I)}(t_1, t_2).$$

It then follows that  $K(S/I_1(XY); t_1, t_2) = (1 - t_1 t_2) \cdot K(\mathcal{R}(I); t_1, t_2)$ , and the substitutions  $t_1 \mapsto 1 - t_1, t_2 \mapsto 1 - t_2$  yield the equation

$$K(S/I_1(XY); 1 - t_1, 1 - t_2) = (t_1 + t_2 - t_1 t_2) \cdot K(\mathcal{R}(I); 1 - t_1, 1 - t_2).$$

By choosing the terms of minimal degree in both sides of the last equation, we obtain

$$\mathcal{C}(S/I_1(XY); t_1, t_2) = (t_1 + t_2) \cdot \mathcal{C}(\mathcal{R}(I); t_1, t_2),$$

and so the result follows Remark 3.3 and Remark 3.4. □

In [20] it was introduced the notion of algebraic degree of semidefinite programming. By using [20, Theorem 10], these invariants can be seen as the multidegrees of  $S/J(r, n-r)$  for  $0 < r < n$ .

**Theorem 3.5** (Nie - Ranestad - Sturmfels; [20, Theorem 10]). For  $0 < r < n$ , we have that

$$\mathcal{C}(S/J(n-r, r); t_1, t_2) = \sum_{d=0}^m \delta(d, n, r) t_1^{m-d} t_2^d,$$

where  $\delta(d, n, r)$  denotes the algebraic degree of semidefinite programming.

We now present the following explicit formula for the algebraic degree of semidefinite programming that was obtained in [7].

**Theorem 3.6** (von Bothmer - Ranestad; [7, Theorem 1.1]). The algebraic degree of semidefinite programming is equal to

$$\delta(d, n, r) = \sum_{\alpha} \psi_{\alpha} \psi_{\alpha^c},$$

where the sum runs over all strictly increasing subsequences  $\alpha = \{\alpha_1, \dots, \alpha_{n-r}\}$  of  $\{1, \dots, n\}$  of length  $n-r$  and sum  $\alpha_1 + \dots + \alpha_r = d$ , and  $\alpha^c$  is the complement  $\{1, \dots, n\} \setminus \alpha$ .

After the previous discussions, we can now compute the multidegrees of  $\Gamma$ .

*Proof of Theorem B.* (i) First, we concentrate on computing the multidegrees of  $S/I_1(XY)$ . By using the additivity of multidegrees (see [18, Theorem 8.53]) together with Proposition 3.1, we obtain the following equality

$$\mathcal{C}(S/I_1(XY)) = \sum_{r=0}^n \mathcal{C}(S/J(r, n-r); t_1, t_2).$$

Hence, by combining Theorem 3.5, Remark 3.2 and Theorem 3.6 it follows that

$$\begin{aligned} \mathcal{C}(S/I_1(XY); t_1, t_2) &= t_1^m + t_2^m + \sum_{r=1}^{n-1} \sum_{d=0}^m \delta(d, n, r) t_1^{m-d} t_2^d \\ &= \sum_{d=0}^m \left( \sum_{\substack{\alpha \subset \{1, \dots, n\} \\ \|\alpha\|=d}} \psi_{\alpha} \psi_{\alpha^c} \right) t_1^{m-d} t_2^d, \end{aligned}$$

where in the last equation  $\alpha$  runs over all strictly increasing subsequences of  $\{1, \dots, n\}$ , including the case  $\alpha = \emptyset$ , and  $\|\alpha\|$  denotes the sum of the entries of  $\alpha$ . Notice that  $\psi_{\{1, \dots, n\}} = 1$  (see [13, Proposition A.15]) and that by an abuse of notation we are setting  $\psi_{\emptyset} = 1$ . Finally, by setting

$$\beta(n, d) = \sum_{\substack{\alpha \subset \{1, \dots, n\} \\ \|\alpha\|=d}} \psi_{\alpha} \psi_{\alpha^c},$$

the result of this part follows from [Remark 3.4](#).

(ii) Notice that [Theorem A](#) (ii) yields the equation

$$\beta(n, d) = \text{deg}^{d, m-1-d}(\Gamma) + \text{deg}^{d-1, m-d}(\Gamma).$$

Since the ideal  $\mathfrak{J}$  of defining equations of  $\mathcal{R}(I)$  is symmetric under swapping the variables  $X_{i,j}$  and  $Y_{i,j}$ , it follows that  $\text{deg}^{d, m-1-d}(\Gamma) = \text{deg}^{m-1-d, d}(\Gamma)$  for all  $0 \leq d \leq m-1$ . Accordingly, we have the equality

$$\beta(n, d) = \text{deg}^{m-1-d, d}(\Gamma) + \text{deg}^{m-d, d-1}(\Gamma).$$

Therefore, the equation  $\text{deg}^{m-1-d, d}(\Gamma) = \sum_{j=0}^d (-1)^j \beta(n, d-j)$  is obtained iteratively. □

#### 4. Polynomiality of ML-degree

During this short section, we show [Corollary C](#). Our proof is an easy consequence of [Theorem B](#) and the following result.

**Theorem 4.1** (Manivel-Michałek-Monin-Seynaeve-Vodička; [[14](#), Theorem 4.3]).

Let  $\alpha = \{\alpha_1, \dots, \alpha_r\}$  be a strictly increasing subsequence of  $\{1, \dots, n\}$ . For  $n \geq 0$  the function

$$P_\alpha(n) := \begin{cases} \psi_{\{1, \dots, n\} \setminus \alpha} & \text{if } \alpha \subset \{1, \dots, n\} \\ 0 & \text{otherwise} \end{cases}$$

is a polynomial in  $n$  of degree  $\|\alpha\| = \alpha_1 + \dots + \alpha_r$ .

Finally, we provide our proof for the polynomiality of  $\phi(n, d)$ .

*Proof of [Corollary C](#).* By using [Theorem B](#) (ii) and (1) we obtain the equation

$$\phi(n, d) = \text{deg}^{m-d, d-1}(\Gamma) = \sum_{j=0}^{d-1} (-1)^j \beta(n, d-1-j).$$

Therefore, it suffices to show that

$$\beta(n, d) = \sum_{\substack{\alpha \subset \{1, \dots, n\} \\ \|\alpha\|=d}} \psi_\alpha \psi_{\alpha^c}$$

in  $n$  of degree  $d$ . Since  $\psi_\alpha$  does not depend on  $n$ , the result follows directly from [Theorem 4.1](#). □

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