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TWO CRITERIA FOR TRANSCENDENTAL SEQUENCES

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The concept of a trasnscendental sequence is defined in this contribution by means of the related transcendental series. The main results are two criteria for when certain sequences are transcendental. Several applications are presented.

1. Introduction.

It has been approximately half a century since Roth in [8] proved a very strong criterion for transcendental numbers. Several mathematicians have improved this result (for instance, see [2] or [7]). There are many theorems for transcendental series but most of them depend on arithmetical properties. An exception is the result of Duverney in [3] which proves several interesting criteria concerning transcendental numbers or Corvaja and Zannier [1]. "Also the result of Erdös in [4]" proves a criterion concerning the Liouville series. If we want to find a criterion for transcendental series which depends only on the speed of convergence and does not depend on divisibility and so on, it seems reasonable to introduce the so-called transcendental sequences.

Definition 1.1. *Let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence of positive real numbers. If for every sequence* $\{c_n\}_{n=1}^{\infty}$ *of positive integers the number* $\sum_{n=1}^{\infty} \frac{1}{a_n c_n}$ *is transcendental*

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then the sequence $\{a_n\}_{n=1}^{\infty}$ *is called transcendental. Otherwise the sequence* $\{a_n\}_{n=1}^{\infty}$ *is called algebraic.*

We can formulate the definition of transcendental and algebraic sequences in the following way. If there is a sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers such that $\sum_{n=1}^{\infty} \frac{1}{a_n c_n}$ is an algebraic number then the sequence $\{a_n\}_{n=1}^{\infty}$ is called algebraic. Otherwise the sequence $\{a_n\}_{n=1}^{\infty}$ is called transcendental.

The inspiration for this definition can be found in Erdös [4] or Erdös and Graham [5]. They defined the irrational sequences in the following way.

Definition 1.2. *Let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence of positive integers. If for every sequence* $\{c_n\}_{n=1}^{\infty}$ *of positive integers the sum of the series* $\sum_{n=1}^{\infty} \frac{1}{a_n c_n}$ *is irrational then the sequence* $\{a_n\}_{n=1}^{\infty}$ *is irrational.*

Erdös in [4] proved that the sequence $\{2^{2^n}\}_{n=1}^{\infty}$ is irrational. On the other side the sequence $\{n!\}_{n=1}^{\infty}$ is not irrational.

Recently I proved in [6] the following theorem.

Theorem 1.1. *Let* α *and* β *be two positive numbers such that* $\alpha > \beta$ *and let* $\{\frac{a_n}{b_n}\}_{n=1}^{\infty}$ *be a sequence where* a_n *and* b_n *are positive integers. If*

$$a_n \geq 2^{(3+\alpha)^n}$$

and

$$b_n \leq 2^{(3+\beta)^n}$$

hold for every large positive integer n, *then the sequence* $\{\frac{a_n}{b_n}\}_{n=1}^{\infty}$ *is transcendental.*

The present paper generalizes this result in Theorem 2.1. below. If we weaken one assumption and strenghten the other in Theorem 2.1, then we obtain Theorem 2.2. The proofs of these theorems are similar. They are both based on Roth's criterion for transcendental numbers (see [8], for instance).

2. Transcendental sequences.

Theorem 2.1. *Let* ϵ, γ *and* c *be three positive real numbers satisfying* $\gamma > 2\epsilon > 0$ *and* $1 > c > \frac{\log_2(3+2\epsilon)}{\log_2(3+\gamma)}$. *Let* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be two sequences of positive integers, with* $\{a_n\}_{n=1}^{\infty}$ *nondecreasing, such that*

$$\limsup_{n\to\infty} a_n^{\frac{1}{(3+\gamma)^n}} > 1, \tag{1}$$

(2) $$a_n > n^{1+\epsilon},$$

and

(3) $$b_n < a_n^{\frac{\epsilon}{1+\epsilon}}.2^{-(\log_2 a_n)^c}$$

hold for every sufficiently large positive integer n. Then the sequence $\{\frac{a_n}{b_n}\}_{n=1}^{\infty}$ is transcendental.

Lemma 2.1. *Let ϵ_4, δ and δ_2 be three positive real numbers such that $(3+\delta_2)^{1+\epsilon_4} < 3+\delta$. Suppose that $\{a_n\}_{n=1}^{\infty}$ is a nondecreasing sequence of positive integers such that there exist infinitely many positive integers i and j with $a_i > 2^{(3+\delta)^i}$ and $a_j < 2^j$. Let k be a sufficiently large positive integer such that $a_k > 2^{(3+\delta)^k} > 2^{k(3+\delta_2)^{k(1+\epsilon_4)}}$ and let m be the greatest positive integer less then k such that $a_m < 2^m$. Then there is a positive integer $t = t(k, \epsilon_4, \delta, \delta_2)$ not greater than k such that*

(4) $$\prod_{a_n < 2^{m(3+\delta_2)^{t+k\epsilon_4}}} a_n \le 2^{\frac{m}{2+\delta_2}(3+\delta_2)^{t+k\epsilon_4}}.$$

Proof of Lemma 2.1. Denote by M the number of a_n such that $a_n \le 2^{m(3+\delta_2)^{k\epsilon_4}}$. Let P_j $(j = 0, 1, 2, \ldots, k)$ be the number of a_n such that $a_n \in (2^{m(3+\delta_2)^{k\epsilon_4}}, 2^{m(3+\delta_2)^{j+k\epsilon_4}}]$ and let $Q_j = j - P_j - M$ $(j = 0, 1, \ldots, k)$. It follows that $Q_0 = -M$, Q_j is an integer, and $Q_{j+1} - Q_j \le 1$ $(j = 0, 1, \ldots, k)$. From the definition of Q_n and the fact that $2^{m(3+\delta_2)^{k(1+\epsilon_4)}} < 2^{k(3+\delta_2)^{k(1+\epsilon_4)}} < 2^{(3+\delta)^k} < a_k$ we obtain that $Q_k = k - P_k - M \ge 1$. It follows that there is a least positive integer $t > M$ such that $Q_t = t - P_t - M = 1$. Thus $Q_{t-1} = 0$ and there is no a_n such that $a_n \in (2^{m(3+\delta_2)^{t-1+k\epsilon_4}}, 2^{m(3+\delta_2)^{t+k\epsilon_4}}]$. In addition, for every $v = 1, 2, \ldots, t$ the number of a_n such that $a_n \in (2^{m(3+\delta_2)^{t-v+k\epsilon_4}}, 2^{m(3+\delta_2)^{t+k\epsilon_4}}]$ is less then v (otherwise the number t will not be the least in the sence defined above) and the number of a_n such that $a_n \in (2^{m(3+\delta_2)^{k\epsilon_4}}, 2^{m(3+\delta_2)^{t+k\epsilon_4}}]$ is equal to $t - M - 1$. It follows that

$$\prod_{a_n \le 2^{m(3+\delta_2)^{t+k\epsilon_4}}} a_n = \prod_{n=1}^{M} a_n \prod_{a_n \in (2^{m(3+\delta_2)^{k\epsilon_4}}, 2^{m(3+\delta_2)^{t+k\epsilon_4}}]} a_n \le$$

$$\prod_{n=M+1}^{t-1} 2^{m(3+\delta_2)^{n+k\epsilon_4}} \le \prod_{n=0}^{t-1} 2^{m(3+\delta_2)^{n+k\epsilon_4}} =$$

$$2^{m\frac{(3+\delta_2)^{t+k\epsilon_4}-(3+\delta_2)^{k\epsilon_4}}{2+\delta_2}} \le 2^{\frac{m}{(2+\delta_2)}(3+\delta_2)^{t+k\epsilon_4}},$$

so (4) holds.

Proof of Theorem 2.1. Let $\{c_n\}_{n=1}^{\infty}$ be a sequence of positive integers. If we order the sequence $\{a_n c_n\}_{n=1}^{\infty}$ so that it is nondecreasing, then the new sequence and the new sequence $\{b_n\}_{n=1}^{\infty}$ will satisfy conditions (1)-(3) also. So it suffices to prove that the series $\beta = \sum_{n=1}^{\infty} \frac{b_n}{a_n}$ is a transcendental number where $\{a_n\}_{n=1}^{\infty}$ is a nondecreasing sequence of positive integers. To prove this we find a positive integer n for every $\delta_1 > 0$ such that

$$(5) \qquad (\prod_{j=1}^{n} a_j)^{2+\epsilon_2} \sum_{j=1}^{\infty} \frac{b_{n+j}}{a_{n+j}} < \delta_1$$

where $\epsilon_2 > 0$ and does not depend on n. From this and Roth's theorem (see [8], for example) we obtain that the number β is transcendental. Let $\epsilon_1 = \frac{\epsilon}{1+\epsilon}$. Let δ and δ_2 be two positive real numbers such that $\gamma > \delta > \delta_2 > 2\epsilon$ and $1 > c > \frac{\log_2(3+\delta_2)}{\log_2(3+\delta)} > \frac{\log_2(3+2\epsilon)}{\log_2(3+\delta)} > \frac{\log_2(3+2\epsilon)}{\log_2(3+\gamma)}$. From this and (1) we obtain

$$(6) \qquad \limsup_{n\to\infty} a_n^{\frac{1}{(3+\delta)^n}} = \limsup_{n\to\infty} (a_n^{\frac{1}{(3+\gamma)^n}})^{(\frac{3+\gamma}{3+\delta})^n} = \infty.$$

Now the proof falls into two parts. First assume that

$$(7) \qquad a_n > 2^n$$

for every sufficiently large n. From (6) we obtain

$$a_{n+1}^{\frac{1}{(3+\delta)^{n+1}}} > \max_{k=1,2,\dots,n} a_k^{\frac{1}{(3+\delta)^k}}$$

for infinitely many n. This implies

$$(8) \qquad a_{n+1} > (\max_{k=1,2,\dots,n} a_k^{\frac{1}{(3+\delta)^k}})^{(3+\delta)^{n+1}} > (\max_{k=1,2,\dots,n} a_k^{\frac{1}{(3+\delta)^k}})^{(2+\delta)\sum_{j=1}^{n}(3+\delta)^j} =$$

$$(\prod_{j=1}^{n}(\max_{k=1,2,\dots,n} a_k^{\frac{1}{(3+\delta)^k}})^{(3+\delta)^j})^{2+\delta} > (\prod_{j=1}^{n} a_j)^{2+\delta}.$$

Now let $\epsilon_3 = \frac{1}{2}(\frac{\delta}{2+\delta} + \epsilon_1)$. Then $\epsilon_3 > \epsilon_1$. From this, (3), and (7) we obtain

(9)
$$\sum_{j=1}^{\infty} \frac{b_{n+j}}{a_{n+j}} < \sum_{j=1}^{\infty} \frac{a_{n+j}^{\epsilon_1}}{a_{n+j}} = \sum_{n=1}^{\infty} \frac{1}{a_{n+j}^{1-\epsilon_1}} =$$

$$\sum_{n+j<\log_2 a_{n+1}} \frac{1}{a_{n+j}^{1-\epsilon_1}} + \sum_{n+j\geq\log_2 a_{n+1}} \frac{1}{a_{n+j}^{1-\epsilon_1}} <$$

$$\frac{\log_2 a_{n+1}}{a_{n+1}^{1-\epsilon_1}} + \sum_{n+j\geq\log_2 a_{n+1}} \frac{1}{2^{(n+j)(1-\epsilon_1)}} <$$

$$\frac{\log_2 a_{n+1}}{a_{n+1}^{1-\epsilon_1}} + \frac{n}{a_{n+1}^{1-\epsilon_1}} < \frac{1}{a_{n+1}^{1-\epsilon_3}}$$

for sufficiently large n. Put $\epsilon_2 = \frac{1}{4}(2+\delta)(\frac{\delta}{2+\delta} - \epsilon_1)$. Then $\epsilon_2 > 0$. From this, (8) and (9) we obtain

$$(\prod_{j=1}^{n} a_j)^{2+\epsilon_2} \sum_{j=1}^{\infty} \frac{b_{n+j}}{a_{n+j}} < (\prod_{j=1}^{n} a_j)^{2+\epsilon_2} \frac{1}{a_{n+1}^{1-\epsilon_3}} \leq$$

$$(\prod_{j=1}^{n} a_j)^{2+\epsilon_2} (\prod_{j=1}^{n} a_j)^{(2+\delta)(\epsilon_3-1)} = (\prod_{j=1}^{n} a_j)^{2+\epsilon_2+(\epsilon_3-1)(2+\delta)} =$$

$$(\prod_{j=1}^{n} a_j)^{-\frac{1}{4}(2+\delta)(\frac{\delta}{2+\delta}-\epsilon_1)}.$$

Therefore (5) follows for infinitely many sufficiently large n.

Now assume

(10)
$$a_n < 2^n$$

for infinitely many n. From (6) we obtain that there is a positive integer k such that

(11)
$$a_k > 2^{(3+\delta)^k}.$$

Let m be the greatest positive integer less than k such that (10) holds. Let $\epsilon_4 = \frac{1}{2}(c\frac{\log_2(3+\delta)}{\log_2(3+\delta_2)} - 1)$. Then $\epsilon_4 > 0$ and we have

(12)
$$(3+\delta_2)^{1+\epsilon_4} = (3+\delta_2)^{\frac{1}{2}(c\frac{\log_2(3+\delta)}{\log_2(3+\delta_2)}+1)} =$$

$$= \sqrt{(3+\delta)^c(3+\delta_2)} < (3+\delta)^c < 3+\delta.$$

From Lemma 2.1 we obtain that there is a positive integer $t = t(k, \epsilon_4, \delta, \delta_2)$ such that $k \geq t$ and

$$\prod_{a_n \leq 2^{m(3+\delta_2)^{t+k\epsilon_4}}} a_n \leq 2^{\frac{m}{2+\delta_2}(3+\delta_2)^{t+k\epsilon_4}}. \tag{13}$$

Let $\epsilon_6 = \frac{1}{2}(1-\epsilon_1)$. Then we have

$$\sum_{a_n \geq 2^{m(3+\delta_2)^{t+k\epsilon_4}}} \frac{b_n}{a_n} = \sum_{a_k \geq a_n \geq 2^{m(3+\delta_2)^{t+k\epsilon_4}}} \frac{b_n}{a_n} + \sum_{a_k^{\epsilon_6} \geq n > k} \frac{b_n}{a_n} + \sum_{a_k^{\epsilon_6} < n} \frac{b_n}{a_n}. \tag{14}$$

First we estimate the first two summands on the right-hand side of equation (14). From (3), (11), and (12) we obtain

$$\sum_{a_k \geq a_n \geq 2^{(3+\delta_2)^{t+k\epsilon_4}}} \frac{b_n}{a_n} + \sum_{a_k^{\epsilon_6} \geq n > k} \frac{b_n}{a_n} \leq \sum_{a_k \geq a_n \geq 2^{m(3+\delta_2)^{t+k\epsilon_4}}} \frac{1}{a_n^{1-\epsilon_1}} + \sum_{a_k^{\epsilon_6} \geq n > k} \frac{1}{a_n^{1-\epsilon_1}} \leq \tag{15}$$

$$k2^{-(1-\epsilon_1)m(3+\delta_2)^{t+k\epsilon_4}} + a_k^{\epsilon_6-(1-\epsilon_1)} \leq k2^{-(1-\epsilon_1)m(3+\delta_2)^{t+k\epsilon_4}} + 2^{-\frac{1-\epsilon_1}{2}(3+\delta)^k} \leq$$

$$(k+1)2^{-(1-\epsilon_1)m(3+\delta_2)^{t+k\epsilon_4}}.$$

Now we estimate the third summand on the right-hand side of equation (14). From (2), (3), (11), and (12) we obtain

$$\sum_{a_k^{\epsilon_6} < n} \frac{b_n}{a_n} \leq \sum_{a_k^{\epsilon_6} < n} \frac{1}{a_n^{1-\epsilon_1} 2^{(\log_2 a_n)^c}} \leq \sum_{a_k^{\epsilon_6} < n} \frac{1}{n^{(1-\epsilon_1)(1+\epsilon)} 2^{(\log_2(n^{1+\epsilon}))^c}} \leq \tag{16}$$

$$\int_{a_k^{\epsilon_6}}^{\infty} \frac{dx}{x 2^{(\log_2 x)^c}} \leq 2^{-\frac{1}{2}(\log_2 a_k^{\epsilon_6})^c} \leq 2^{-\frac{1}{2}(\log_2 2^{\epsilon_6(3+\delta)^k})^c} = 2^{-\frac{1}{2}\epsilon_6^c(3+\delta)^{ck}} \leq$$

$$2^{-(1-\epsilon_1)m(3+\delta_2)^{t+k\epsilon_4}}.$$

From (14), (15), and (16) we obtain

$$\sum_{2^{m(3+\delta_2)^{t+k\epsilon_4}} \leq a_n} \frac{b_n}{a_n} \leq 2k2^{-(1-\epsilon_1)m(3+\delta_2)^{t+k\epsilon_4}}. \tag{17}$$

Put $\epsilon_2 = \frac{1}{2}(\delta_2 - 2\epsilon_1 - \epsilon_1\delta_2) = \frac{1}{2}(\frac{\delta_2}{2+\delta_2} - \epsilon_1)(2+\delta_2)$. Then $\epsilon_2 > 0$. From this, (13), and (17) we obtain

$$\Big(\prod_{a_k < 2^{m(3+\delta_2)^{t+k\epsilon_4}}} a_n\Big)^{2+\epsilon_2} \sum_{a_n > 2^{m(3+\delta_2)^{t+k\epsilon_4}}} \frac{b_n}{a_n} \leq 2^{\frac{2+\epsilon_2}{2+\delta_2}m(3+\delta_2)^{t+k\epsilon_4}} 2k2^{-(1-\epsilon_1)m(3+\delta_2)^{t+k\epsilon_4}} =$$

$$2k2^{m(\frac{2+\epsilon_2}{2+\delta_2}-(1-\epsilon_1))(3+\delta_2)^{t+k\epsilon_4}} = 2k2^{-m\frac{\epsilon_2}{2+\delta_2}(3+\delta_2)^{t+k\epsilon_4}},$$

so (5) follows for infinitely many large n. The proof of Theorem 2.1 is complete.

Theorem 2.2. *Let A_1, A_2, c_1 and γ be four positive real numbers satisfying $A_1 > A_2 > 0$ and $1 > c_1 > \frac{\log_2 3}{\log_2(3+\gamma)}$. Let $L_0(x) = x$, $L_{j+1}(x) = \log_2(L_j(x))$ for every $j = 0, 1, \ldots$, and assume that s is a nonnegative integer. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of positive integers, with $\{a_n\}_{n=1}^{\infty}$ nondecreasing, such that*

$$\limsup_{n\to\infty} L_s(a_n)^{\frac{1}{(3+\gamma)^n}} > 1, \tag{18}$$

$$a_n > \Big(\prod_{j=0}^{s} L_j(n)\Big)2^{A_1 L_{s+1}^{c_1}(n)}, \tag{19}$$

and

$$b_n < 2^{A_2 L_{s+1}^{c_1}(a_n)} \tag{20}$$

hold for every sufficiently large positive integer n. Then the sequence $\{\frac{a_n}{b_n}\}$ is transcendental.

Proof. As in Theorem 2.1 it suffices to prove that the series $\beta = \sum_{n=1}^{\infty} \frac{b_n}{a_n}$ is a transcendental number for the nondecreasing sequence $\{a_n\}_{n=1}^{\infty}$ of positive integers. To establish this we find a positive integer n for every $\delta_1 > 0$ such that (5) holds.

First of all let us assume

$$a_n > 2^n$$

for every sufficiently large n. Then from Theorem 2.1 we immediately obtain that the number β is transcendental.

Now assume (10) for infinitely many n. Let δ be a positive real number such that $1 > c_1 > \frac{\log_2 3}{\log_2(3+\delta)} > \frac{\log_2 3}{\log_2(3+\gamma)}$. From this and from (18) we obtain

$$\limsup_{n\to\infty} L_s^{\frac{1}{(3+\delta)^n}}(a_n) = \limsup_{n\to\infty} L_s^{(\frac{1}{(3+\gamma)^n})(\frac{3+\gamma}{3+\delta})^n}(a_n) = \infty. \tag{21}$$

From (21) we obtain that there is a positive integer k such that

$$L_s(a_k) > 2^{(3+\delta)^k} \tag{22}$$

and k is sufficiently large. Let m be the greatest positive integer less then k such that (10) holds. Let δ_2 be a positive real number such that $1 > c_1 > \frac{\log_2(3+\delta_2)}{\log_2(3+\delta)} > \frac{\log_2 3}{\log_2(3+\delta)}$. Let $\epsilon_4 = \frac{1}{2}(c_1 \frac{\log_2(3+\delta)}{\log_2(3+\delta_2)} - 1)$. Then $\epsilon_4 > 0$ and we have

$$(3+\delta_2)^{1+\epsilon_4} = (3+\delta_2)^{\frac{1}{2}(c_1 \frac{\log_2(3+\delta)}{\log_2(3+\delta_2)}+1)} = \tag{23}$$

$$\sqrt{(3+\delta_2)(3+\delta)^{c_1}} < (3+\delta)^{c_1} < 3+\delta.$$

Thus all the conditions of Lemma 2.1 are satisfied. Now Lemma 2.1 implies that there is a positive integer $t = t(\epsilon_4, \delta, \delta_2)$ not greater than k such that

$$\prod_{a_n < 2^{m(3+\delta_2)^{t+k\epsilon_4}}} a_n \le 2^{\frac{m}{2+\delta_2}(3+\delta_2)^{t+k\epsilon_4}}. \tag{24}$$

We also have

$$\sum_{a_n \ge 2^{m(3+\delta_2)^{t+k\epsilon_4}}} \frac{b_n}{a_n} \le \tag{25}$$

$$\sum_{a_k \ge a_n \ge 2^{m(3+\delta_2)^{t+k\epsilon_4}}} \frac{b_n}{a_n} + \sum_{k<n\le\sqrt{a_k}} \frac{b_n}{a_n} + \sum_{n>\sqrt{a_k}} \frac{b_n}{a_n}.$$

First we estimate the first two summands on the right hand side of inequality (25). From (20), (22), (23), and the fact that the function $2^{A_2 L_{s+1}^{c_1}(x)} x^{-1}$ is decreasing for sufficiently large x we obtain

$$\sum_{a_k \ge a_n \ge 2^{m(3+\delta_2)^{t+k\epsilon_4}}} \frac{b_n}{a_n} + \sum_{k<n\le\sqrt{a_k}} \frac{b_n}{a_n} \le \tag{26}$$

$$\sum_{a_k \ge a_n \ge 2^{m(3+\delta_2)^{t+k\epsilon_4}}} \frac{2^{A_2 L_{s+1}^{c_1}(a_n)}}{a_n} + \sum_{k<n\le\sqrt{a_k}} \frac{2^{A_2 L_{s+1}^{c_1}(a_n)}}{a_n} \le$$

$$\frac{k 2^{A_2 L_{s+1}^{c_1}(2^{m(3+\delta_2)^{t+k\epsilon_4}})}}{2^{m(3+\delta_2)^{t+k\epsilon_4}}} + \frac{\sqrt{a_k}\, 2^{A_2 L_{s+1}^{c_1}(a_k)}}{a_k} \le \frac{k 2^{A_2 L_s^{c_1}(m(3+\delta_2)^{t+k\epsilon_4})}}{2^{m(3+\delta_2)^{t+k\epsilon_4}}} + \frac{1}{\sqrt[3]{a_k}} \le$$

$$2^{-\epsilon_7 m(3+\delta_2)^{t+k\epsilon_4}},$$

where $\epsilon_7 = \frac{1}{2}(1+\frac{2}{2+\delta_2})$. Now we estimate the third summand on the right hand side of inequality (25). From (19), (20), (22), (23), and the fact that the function $x^{-1}2^{A_2 L_{s+1}^{c_1}(x)}$ is decreasing for sufficiently large x we obtain

$$\sum_{\sqrt{a_k}<n} \frac{b_n}{a_n} \le \sum_{\sqrt{a_k}<n} \frac{2^{A_2 L_{s+1}^{c_1}(a_n)}}{a_n} \le \sum_{\sqrt{a_k}<n} \frac{2^{A_2 L_{s+1}^{c_1}((\prod_{j=0}^{s} L_j(n))2^{A_1 L_{s+1}^{c_1}(n)})}}{(\prod_{j=0}^{s} L_j(n))2^{A_1 L_{s+1}^{c_1}(n)}} \le \tag{27}$$

$$\sum_{\sqrt{a_k}<n} \frac{1}{(\prod_{j=0}^{s} L_j(n))2^{\frac{A_1-A_2}{2}L_{s+1}^{c_1}(n)}} \leq \int_{\sqrt{a_k}}^{\infty} \frac{dx}{(\prod_{j=0}^{s} L_j(x))2^{\frac{A_1-A_2}{3}L_{s+1}^{c_1}(x)}} \leq$$

$$\frac{1}{2^{\frac{A_1-A_2}{4}L_{s+1}^{c_1}(\sqrt{a_k})}} \leq \frac{1}{2^{\frac{A_-A_2}{4}2^{-c_1}(3+\delta)^{c_1 k}}} \leq 2^{-\epsilon_7 m(3+\delta_2)^{t+k\epsilon_4}}.$$

From (25), (26), and (27) we obtain

$$\sum_{2^{m(3+\delta_2)^{t+k\epsilon_4}} \leq a_n} \frac{b_n}{a_n} \leq 2.2^{-\epsilon_7 m(3+\delta_2)^{t+k\epsilon_4}}. \tag{28}$$

Put $\epsilon_2 = \frac{1}{4}\delta_2$. Then (24) and (28) imply

$$\left(\prod_{a_n<2^{m(3+\delta_2)^{t+k\epsilon_4}}} a_n\right)^{2+\epsilon_2} . \sum_{a_n>2^{m(3+\delta_2)^{t+k\epsilon_4}}} \frac{b_n}{a_n} \leq 2^{\frac{2+\epsilon_2}{2+\delta_2}m(3+\delta_2)^{t+k\epsilon_4}} .2.2^{-\epsilon_7 m(3+\delta_2)^{t+k\epsilon_4}} =$$

$$2.2^{(\frac{2+\epsilon_2}{2+\delta_2}-\epsilon_7)m(3+\delta_2)^{t+k\epsilon_4}} = 2.2^{-\frac{\delta_2}{4(2+\delta_2)}m(3+\delta_2)^{t+k\epsilon}},$$

so (5) follows.

3. Examples and comments.

Corollary 1. *Let ϵ, γ be two positive real numbers such that $1 > \epsilon$ and $\gamma > 2\epsilon$. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of positive integers such that*

$$a_n > 2^{(3+\gamma)^n},$$

and

$$b_n < a_n^{\frac{\epsilon}{1+\epsilon}}$$

hold for every sufficiently large positive integer n. Then the sequence $\{\frac{a_n}{b_n}\}_{n=1}^{\infty}$ is transcendental.

Proof. This is an immediate consequence of Theorem 2.1.

Remark 1. From Corollary 1 we immediately obtain Theorem 1.1 (see [6] also).

Corollary 2. *Let γ be a positive real number. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of positive integers, with $\{a_n\}_{n=1}^{\infty}$ is nondecreasing, such that*

$$\limsup_{n\to\infty}(\log_2 a_n)^{\frac{1}{(3+\gamma)^n}} > 1,$$

$$a_n > n\log_2^2 n,$$

and

$$b_n < \log_2\log_2 n$$

hold for every sufficiently large n. Then the sequence $\{\frac{b_n}{a_n}\}_{n=1}^{\infty}$ is transcendental.

Proof. This is an immediate consequence of Theorem 2.2.

Remark 2. From Theorem 2.1 and Theorem 2.2 we can derive the relationship between conditions (1) and (2) or between conditions (18) and (19). If we, for example, weaken condition (2) by (11), then we must strengthen condition (1) by (18). Similary condition (3) depends on (1) and (2).

Example 1. The sequences

$$\{(\frac{28}{3})^{4^n}\}_{n=1}^{\infty}, \{(\frac{26}{5})^{5^n}\}_{n=1}^{\infty}, \{\frac{9^{4^n}+1}{2^{4^n}+1}\}_{n=1}^{\infty}, \{\frac{4^{6^n}+2}{2^{6^n}+4}\}_{n=1}^{\infty}, \text{and}\{\frac{17^{5^n}+n!}{4^{5^n}+n^n}\}_{n=1}^{\infty}$$

are transcendental. This is an immediate consequence of Theorem 2.1.

Example 2. Let $a_1 = 2$,

$$a_k = 2^{2^{4^2}} + k - 2, k = 2, 3, \ldots, 2^{2^{4^2}}2^{-3.4^2} = n_1 - 1,$$

$$a_k = 2^{2^{4^{n_1}}} + k - n_1, k = n_1, \ldots, 2^{2^{4^{n_1}}}2^{-3.4^{n_1}} = n_2 - 1,$$

$$a_k = 2^{2^{4^{n_2}}} + k - n_2, k = n_2, \ldots, 2^{2^{4^{n_2}}}2^{-3.4^{n_2}} = n_3 - 1,$$

and so on. Then the sequences

$$\{[\![\frac{a_n}{\log_2\log_2 n}]\!]\}_{n=1}^{\infty}, \{[\![\frac{a_n+1}{\sqrt{\log_2\log_2 n+2}}]\!]\}_{n=1}^{\infty}, \text{and}\{[\![\frac{a_n+\sqrt{\log_2 n}}{\log_2\log_2 n+1}]\!]\}_{n=1}^{\infty},$$

where $[\![x]\!]$ is the greatest integer not greater than x, are transcendental. This is an immediate consequence of Theorem 2.2.

Example 3. Let $\{G_n\}_{n=1}^{\infty}$ be the linear recurrence sequence of the k-th order such that $G_1, G_2, \dots, G_k, b_0, \dots, b_k$ are positive integers and for every positive integer n, $G_{n+k} = G_n b_0 + G_{n+1} b_1 + \dots + G_{n+k-1} b_{k-1}$. If the roots $\alpha_1, \dots, \alpha_s$ of the equation $x^k = b_0 + b_1 x + \dots + b_{k-1} x^{k-1}$ satisfy $|\alpha_1| \geq |\alpha_2| \geq \dots \geq |\alpha_s|$, and $|\alpha_1| > 1$, and α_1/α_j is not a root of unity for every $j = 2, 3, \dots, s$, then the sequences $\{\frac{G_{4^n+1}}{G_{4^n}}\}_{n=1}^{\infty}$, $\{\frac{G_{4^{n+1}+1}}{G_{4^n+1}}\}_{n=1}^{\infty}$, and $\{\frac{G_{5^n+1}}{G_{5^n+3}}\}_{n=1}^{\infty}$ are transcendental. This is an immediate consequence of Corollary 1 and the inequalities

$$|\alpha_1|^{n(1-\epsilon)} < G_n < |\alpha_1|^{n(1+\epsilon)}$$

which can be found in [9], for instance.

Open Problems. It is not known if there are positive integers K_1 and K_2 greater than one such that the sequence $\{K_1^{3^n} + K_2\}_{n=1}^{\infty}$ is transcendental. Similarly, we do not know if there are positive integers K_1 and K_2 greater than one such that the sequence $\{K_1^{3^n} + K_2\}_{n=1}^{\infty}$ is algebraic.

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