

JORDAN ALGEBRAS OF SYMMETRIC MATRICES

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We study linear spaces of symmetric matrices whose reciprocal is also a linear space. These are Jordan algebras. We classify such algebras in low dimensions, and we study the associated Jordan loci in the Grassmannian.

1. Introduction

Let \mathbb{S}^n be the space of symmetric $n \times n$ matrices over the complex numbers \mathbb{C} . The m -dimensional subspaces $\mathcal{L} \subset \mathbb{S}^n$ are points in the Grassmannian $\text{Gr}(m, \mathbb{S}^n)$. The terms *pencil of quadrics* [6] resp. *net of quadrics* [14] are used when $m = 2, 3$. We here consider *regular* \mathcal{L} , i.e. $\mathcal{L}_{\text{inv}} = \{X \in \mathcal{L} \mid \det(X) \neq 0\}$ is nonempty. The *reciprocal variety* \mathcal{L}^{-1} is the closure in \mathbb{S}^n of the set $\{X^{-1} \mid X \in \mathcal{L}_{\text{inv}}\}$. We wish to know under which circumstances the variety \mathcal{L}^{-1} is a linear space in \mathbb{S}^n . The motivation for this question arises in optimization, namely in the theory of semidefinite programming [5, 11], and in statistics, namely from Gaussian models that are linear in covariance matrices and concentration matrices [8, 13].

The following theorem furnishes a complete answer to our question.

Theorem 1.1. For $\mathcal{L} \in \text{Gr}(m, \mathbb{S}^n)$ and $U \in \mathcal{L}_{\text{inv}}$, the following are equivalent:

- (a) The reciprocal variety \mathcal{L}^{-1} is also a linear space in \mathbb{S}^n .
- (b) \mathcal{L} is a subalgebra of the Jordan algebra $(\mathbb{S}^n, \bullet_U)$.

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(c) \mathcal{L}^{-1} equals \mathcal{L} up to congruence; namely $\mathcal{L}^{-1} = U^{-1}\mathcal{L}U^{-1}$.

Theorem 1.1 is due to Jensen [8] in the special case when \mathcal{L} contains the identity matrix $\mathbf{1}_n$. He then uses $U = \mathbf{1}_n$ for the Jordan algebra structure on \mathbb{S}^n .

Example 1.2 ($m = n = 4$). Consider the two linear subspaces of \mathbb{S}^4 given by

$$\mathcal{L}_1 = \begin{pmatrix} x & y & 0 & 0 \\ y & z & 0 & 0 \\ 0 & 0 & w & 0 \\ 0 & 0 & 0 & w \end{pmatrix} \quad \text{and} \quad \mathcal{L}_2 = \begin{pmatrix} x & y & 0 & w \\ y & z & -w & 0 \\ 0 & -w & x & y \\ w & 0 & y & z \end{pmatrix} \quad \text{with } U = \mathbf{1}_4.$$

They satisfy (a),(b),(c) in Theorem 1.1. Note that \mathcal{L}_2 appears in [8, eqn (39)]. It illustrates case (ii) in [8, Theorem 1]. The Jordan algebra property fails when $-w$ is replaced by w in \mathcal{L}_2 . The reciprocal variety is a threefold of degree five.

The proof of Theorem 1.1 is relatively short. It will be presented in Section 2, along with all relevant definitions. We also review classification results for Jordan algebras [9]. This lays the foundation for studying the *Jordan locus* $\text{Jo}(m, \mathbb{S}^n)$. This is the subvariety of the Grassmannian $\text{Gr}(m, \mathbb{S}^n)$ whose points are the subspaces \mathcal{L} that satisfy the closed conditions (a), (b) and (c) above. To be precise, let X_1, \dots, X_m represent a basis of \mathcal{L} and write $\text{adj}(U)$ for the adjoint of U . From condition (b) it can be seen that $\text{Jo}(m, \mathbb{S}^n)$ is the subvariety set-theoretically defined (in dual Stiefel coordinates) by the linear dependence of

$$X_1, X_2, \dots, X_m, X_i \text{adj}(U)X_j + X_j \text{adj}(U)X_i$$

for all matrices $U \in \mathcal{L}$ and all indices $1 \leq i < j \leq m$.

The main goal of this article is the classification of low-dimensional Jordan subalgebras of \mathbb{S}^n and the description of the corresponding Jordan loci for small n . In Section 3 we determine the loci of Jordan pencils ($m = 2$) and we study their equations. Section 4 establishes a lower bound on the codimension of any proper subalgebra of $(\mathbb{S}^n, \bullet_U)$, and it identifies the Jordan copencils ($m = 4, n = 3$). In Section 5 we classify Jordan subalgebras of dimension $m = 3$. This is applied in Section 6 to study the Jordan loci $\text{Jo}(3, \mathbb{S}^n)$. We emphasize the explicit computation of their irreducible components and defining polynomials. One finding of independent interest is the Chow matrix in Theorem 6.2 which represents the Chow form of the determinantal variety $\{X \in \mathbb{S}^n \mid \text{rk} X \leq n - 2\}$.

2. Jordan algebras

Fix an invertible matrix $U \in \mathbb{S}^n$. We define an algebra structure on \mathbb{S}^n by setting

$$X \bullet_U Y := \frac{1}{2}(XU^{-1}Y + YU^{-1}X) \quad \text{for all } X, Y \in \mathbb{S}^n. \tag{1}$$

When U is clear from context, we drop the subscript. The product \bullet is bilinear and commutative, with unit U . Associativity fails, but, setting $X^{\bullet 2} := X \bullet X$, the following weaker version holds. This is the axiom for a (unital) *Jordan algebra*:

$$X^{\bullet 2} \bullet (X \bullet Y) = X \bullet (X^{\bullet 2} \bullet Y). \tag{2}$$

The role of the unit can be played by any matrix $U \in \mathbb{S}_{\text{inv}}^n$. The resulting Jordan algebra structure is denoted by $(\mathbb{S}^n, \bullet_U)$. Especially important for applications, notably in statistics, optimization and physics, is the case when U is real and positive definite. Such algebras over \mathbb{R} are known as *Euclidean Jordan algebras*.

Remark 2.1. Over the field \mathbb{R} , the isomorphism type of the Jordan algebra $(\mathbb{S}^n, \bullet_U)$ depends on the choice of U . To see this, let $n = 2$ and write E_{ij} for the matrix units. If $U = \mathbf{1}_2 = E_{11} + E_{22}$ then $(\mathbb{S}^2, \bullet_U)$ has no nilpotent elements. However, if $U' = E_{12} + E_{21}$ then the matrices E_{11} and E_{22} are nilpotent in $(\mathbb{S}^2, \bullet_{U'})$.

We return to this issue later: over \mathbb{C} , the isomorphism type of $(\mathbb{S}^n, \bullet_U)$ is in fact independent of U . First, let us prove the result in the introduction.

Proof of Theorem 1.1. Clearly, (c) implies (a). Suppose that (a) holds, so \mathcal{L}^{-1} is a linear space. We shall prove that (b) holds. Consider invertible matrices $U, X \in \mathcal{L}$. For small $t \in \mathbb{R}$, the inverse of $X - tU$ is given by the Neumann series

$$(X - tU)^{-1} = X^{-1}(\mathbf{1}_n - tUX^{-1})^{-1} = X^{-1} \sum_{k=0}^{\infty} t^k (UX^{-1})^k \in \mathcal{L}^{-1}.$$

Since \mathcal{L}^{-1} is linear and closed, the following matrix is in \mathcal{L}^{-1} as well:

$$X^{-1}UX^{-1} = \lim_{t \rightarrow 0} \frac{1}{t} ((X - tU)^{-1} - X^{-1}) \in \mathcal{L}^{-1}.$$

Its inverse $X \bullet_U X$ is in \mathcal{L} . Since invertible X are dense in \mathcal{L} , this conclusion holds for all $X \in \mathcal{L}$. Condition (b) states that $X, Y \in \mathcal{L}$ implies $X \bullet_U Y \in \mathcal{L}$. Let $X, Y \in \mathcal{L}$. Then the Jordan algebra squares $(X + Y)^{\bullet 2}, X^{\bullet 2}, Y^{\bullet 2}$ are in \mathcal{L} , and we conclude that $X \bullet_U Y = \frac{1}{2}((X + Y)^{\bullet 2} - X^{\bullet 2} - Y^{\bullet 2})$ is also in \mathcal{L} . So (b) holds.

Assume (b) and consider any invertible $X \in \mathcal{L}U^{-1}$. Then $XU \bullet_U XU = X^2U$ is in \mathcal{L} . Next, we see that $X^3U = X^2U \bullet_U XU \in \mathcal{L}$. By iterating this, every power X^k lies in $\mathcal{L}U^{-1}$. Since the characteristic polynomial of X has nonzero constant term, the Cayley-Hamilton Theorem implies $X^{-1} \in \mathcal{L}U^{-1}$. Since invertible matrices are dense in $\mathcal{L}U^{-1}$, we get $\mathcal{L}U^{-1} = (\mathcal{L}U^{-1})^{-1} = U\mathcal{L}^{-1}$, i.e. (c) holds. \square

We next review material from the theory of abstract Jordan algebras. Already in 1934, Jordan, von Neumann and Wigner [9] established a structure theorem for Euclidean Jordan algebras. We state a variant that holds over \mathbb{C} . An *ideal* \mathcal{I} in a Jordan algebra \mathcal{A} is a subspace such that $X \bullet Y \in \mathcal{I}$ for all $X \in \mathcal{I}, Y \in \mathcal{A}$. The *radical* of \mathcal{A} is the ideal $\text{rad } \mathcal{A} := \{X \in \mathcal{A} \mid X \bullet Y \text{ is nilpotent for all } Y \in \mathcal{A}\}$.

Theorem 2.2. For a finite-dimensional complex Jordan algebra \mathcal{A} , the quotient $\mathcal{A}/\text{rad } \mathcal{A}$ is a direct sum of simple Jordan algebras. Every simple Jordan algebra of dimension ≤ 8 is isomorphic to either (\mathbb{C}, \cdot) , or to $(\mathbb{S}^3, \bullet_{\mathbf{1}_3})$, or to a *spin factor* $\mathbb{C} \times \mathbb{C}^d$, $d \geq 2$, with product $(\lambda, a) \bullet (\mu, b) = (\lambda\mu + a^\top b, \lambda b + \mu a)$ and unit $(1, 0)$.

Proof. We find this by applying known structure theorems to semi-simple Jordan algebras of dimension ≤ 8 . See [1, Ch. X, §3.3] or [10, Part I, §2.13]. \square

Theorem 2.2 will be used to classify low-dimensional Jordan algebras. First, however, we highlight the difference between \mathbb{R} and \mathbb{C} concerning Remark 2.1.

Lemma 2.3. Square roots exist in any complex Jordan subalgebra $\mathcal{L} \subset \mathbb{S}^n$ with unit U . To be precise, for every matrix $X \in \mathcal{L}_{\text{inv}}$ there exists $Y \in \mathcal{L}$ with $Y^{\bullet 2} = X$.

Proof. The subalgebra $\mathcal{B} = \text{span}\{X^{\bullet \ell} \mid \ell \geq 0\}$ of \mathcal{L} is associative as $Y \bullet Z = YU^{-1}Z = ZU^{-1}Y$ for all $Y, Z \in \mathcal{B}$. By [10, Part I, §2.13], the only associative simple Jordan algebra in \mathbb{C} . So by Theorem 2.2, the quotient $\mathcal{B}/\text{rad } \mathcal{B}$ is a direct sum of copies of \mathbb{C} . In particular, every element of $\mathcal{B}/\text{rad } \mathcal{B}$ is a square. So there exist $Z \in \mathcal{B}$ and $W \in \text{rad } \mathcal{B}$ with $Z^{\bullet 2} + W = X$. We have $X^{-1} \in U^{-1}\mathcal{B}U^{-1}$ and so, since $W \in \text{rad } \mathcal{B}$, the matrix WX^{-1} is nilpotent. Hence the matrix $X - W = Z^{\bullet 2}$ is invertible and using Theorem 1.1 we can take $Z^{\bullet -2} := U \bullet_{Z^{\bullet 2}} U \in \mathcal{B}$ which satisfies $Z^{\bullet 2} \bullet Z^{\bullet -2} = U$. Since $W \in \text{rad } \mathcal{B}$, the element $Z^{\bullet -2} \bullet W$ is nilpotent, say $(Z^{\bullet -2} \bullet W)^{\bullet p} = 0$. Using the Taylor series $\sqrt{1 + \lambda} = \sum_{\ell=0}^{\infty} a_\ell \lambda^\ell$, we see that

$$X = Z^{\bullet 2} \bullet (U + Z^{\bullet -2} \bullet W) = Z^{\bullet 2} \bullet \left(\sum_{\ell=0}^{p-1} a_\ell (Z^{\bullet -2} \bullet W)^{\bullet \ell} \right)^{\bullet 2}.$$

Hence X has a square root Y in the subalgebra \mathcal{B} of \mathcal{L} . \square

Having access to a square root within the algebra, we can find isomorphisms between (\mathcal{L}, \bullet_U) and (\mathcal{L}, \bullet_V) for different choices of matrices $U, V \in \mathcal{L}_{\text{inv}}$.

Proposition 2.4. Consider a complex Jordan algebra (\mathcal{L}, \bullet_U) as in Theorem 1.1. There is an isomorphism $(\mathcal{L}, \bullet_U) \cong (\mathcal{L}, \bullet_V)$ of Jordan algebras for all $V \in \mathcal{L}_{\text{inv}}$.

Proof. By Lemma 2.3, there exists $W \in \mathcal{L}_{\text{inv}}$ with $WU^{-1}W = V$. Consider the map

$$\phi : \mathcal{L} \rightarrow \mathcal{L}, X \mapsto VW^{-1}XW^{-1}V = 2V \bullet_W (X \bullet_W V) - (V \bullet_W V) \bullet_W X.$$

This linear automorphism of \mathcal{L} is invertible since VW^{-1} is invertible. We have $\phi(XU^{-1}X) = \phi(XW^{-1}VW^{-1}X) = VW^{-1}XW^{-1}VW^{-1}XW^{-1}V = \phi(X)V^{-1}\phi(X)$.

This shows that ϕ is a Jordan algebra isomorphism from (\mathcal{L}, \bullet_U) to (\mathcal{L}, \bullet_V) . \square

We now come to the classification of low-dimensional Jordan algebras.

Lemma 2.5. If $k = \dim(\text{rad } \mathcal{A})$, then $X^{\bullet k+1} = 0$ for all $X \in \text{rad } \mathcal{A}$.

Proof. Let $X \in \text{rad } \mathcal{A}$. As $X, X^{\bullet 2}, \dots, X^{\bullet k+1} \in \text{rad } \mathcal{A}$ are linearly dependent, the minimum polynomial of X has degree $\leq k + 1$. As X is nilpotent, $X^{\bullet k+1} = 0$. \square

Proposition 2.6. Every Jordan algebra \mathcal{A} of dimension two over \mathbb{C} is isomorphic to the Jordan algebra $\mathbb{C}\{U, X\}$ with unit U , where the product is given by

- 1: $X^{\bullet 2} = X$, when $\mathcal{A} \cong \mathbb{C} \times \mathbb{C}$; or
- 2: $X^{\bullet 2} = 0$, when $\mathcal{A}/\text{rad } \mathcal{A} \cong \mathbb{C}$.

Proof. By Theorem 2.2, either $\mathcal{A} \cong \mathbb{C} \times \mathbb{C}$ or $\mathcal{A}/\text{rad } \mathcal{A} \cong \mathbb{C}$. We seek X such that $\{U, X\}$ is a basis of \mathcal{A} . In the first case, take the idempotent $X = (1, 0)$ in $\mathbb{C} \times \mathbb{C}$. In the second case, we choose X to span $\text{rad } \mathcal{A}$. Then $X^{\bullet 2} = 0$ by Lemma 2.5. \square

Theorem 2.7. Every Jordan algebra \mathcal{A} of dimension three over \mathbb{C} is isomorphic to the Jordan algebra $\mathbb{C}\{U, X, Y\}$ with unit U , where the product is given by

- 1 (a): $X^{\bullet 2} = X, Y^{\bullet 2} = Y$ and $X \bullet Y = 0$, when $\mathcal{A} \cong \mathbb{C} \times \mathbb{C} \times \mathbb{C}$;
- (b): $X^{\bullet 2} = U, Y^{\bullet 2} = U$ and $X \bullet Y = 0$, when $\mathcal{A} \cong \mathbb{C} \times \mathbb{C}^2$;
- 2 (a): $X^{\bullet 2} = X, Y^{\bullet 2} = 0$ and $X \bullet Y = 0$ or
- (b): $X^{\bullet 2} = X, Y^{\bullet 2} = 0$ and $X \bullet Y = Y/2$, when $\mathcal{A}/\text{rad } \mathcal{A} \cong \mathbb{C} \times \mathbb{C}$; or
- 3 (a): $X^{\bullet 2} = Y, Y^{\bullet 2} = 0$ and $X \bullet Y = 0$ or
- (b): $X^{\bullet 2} = 0, Y^{\bullet 2} = 0$ and $X \bullet Y = 0$, when $\mathcal{A}/\text{rad } \mathcal{A} \cong \mathbb{C}$.

Proof. By Theorem 2.2, there are four possible cases to consider:

$$\mathcal{A} \cong \mathbb{C} \times \mathbb{C} \times \mathbb{C}, \mathcal{A} \cong \mathbb{C} \times \mathbb{C}^2, \mathcal{A}/\text{rad } \mathcal{A} \cong \mathbb{C} \times \mathbb{C} \text{ or } \mathcal{A}/\text{rad } \mathcal{A} \cong \mathbb{C}.$$

We seek X, Y such that $\{U, X, Y\}$ is a basis of \mathcal{A} . In the first case, the choices $X = (1, 0, 0)$ and $Y = (0, 1, 0)$ in $\mathbb{C} \times \mathbb{C} \times \mathbb{C}$ have the desired properties. In the second case, we can choose X, Y to be elements of order two with product zero. Two such elements in $\mathbb{C} \times \mathbb{C}^2$ are $(0, e_1), (0, e_2)$. So, case 1(a) or case 1(b) holds.

Now suppose $\mathcal{A}/\text{rad } \mathcal{A} \cong \mathbb{C} \times \mathbb{C}$. Let $X \in \mathcal{A}$ be a preimage of $(1, 0)$, so $X^{\bullet 2} \equiv X \pmod{\text{rad } \mathcal{A}}$. Choose Y to span $\text{rad } \mathcal{A}$. Then $Y^{\bullet 2} = 0$ and $X \bullet Y \in \text{rad } \mathcal{A}$. Here we use that $\text{rad } \mathcal{A}$ is an ideal. So $X^{\bullet 2} = X + \lambda Y$ and $X \bullet Y = \mu Y$ for some $\lambda, \mu \in \mathbb{C}$.

We determine all $\lambda, \mu \in \mathbb{C}$ such that the commutative bilinear form \bullet satisfies

$$Z^{\bullet 2} \bullet (Z \bullet W) = Z \bullet (Z^{\bullet 2} \bullet W)$$

with $Z = c_1U + c_2X + c_3Y$ and $W = d_1U + d_2X + d_3Y$ for all $c_1, c_2, c_3, d_1, d_2, d_3 \in \mathbb{C}$. Using computer algebra, we find that either $\mu = 0, \mu = 1$ or $(\lambda, \mu) = (0, 1/2)$. The first two cases are isomorphic via the base change that replaces X by $U - X$ or $X \pm \lambda Y$. From this we can conclude that case 2(a) holds or case 2(b) holds.

If $\mathcal{A}/\text{rad } \mathcal{A} \cong \mathbb{C}$, we choose X, Y to span $\text{rad } \mathcal{A}$. In this case, either the square of every element in $\text{rad } \mathcal{A}$ is 0, or we can choose X, Y such that $X^{\bullet 2} = Y$, since X is nilpotent. In the first case, we find that $X^{\bullet 2} = Y^{\bullet 2} = X \bullet Y = 0$. In the second case, we find that $Y^{\bullet 2} = X \bullet Y = 0$ as $X^{\bullet 3} = 0$. So case 3(a) or case 3(b) holds. \square

3. Jordan Pencils

We now embark on the study of the Jordan locus $\text{Jo}(m, \mathbb{S}^n)$ in $\text{Gr}(m, \mathbb{S}^n)$. It is convenient to work in the sub-Grassmannian $\text{Gr}_1(m, \mathbb{S}^n)$ of subspaces \mathcal{L} that contain the identity matrix $\mathbf{1}_n$. The *restricted Jordan locus* is the intersection

$$\text{Jo}_1(m, \mathbb{S}^n) = \text{Gr}_1(m, \mathbb{S}^n) \cap \text{Jo}(m, \mathbb{S}^n). \tag{3}$$

We next examine the variety (3) for $m = 2$. Pencils in $\text{Gr}_1(2, \mathbb{S}^n)$ have the form $\mathcal{L} = \mathbb{C}\{\mathbf{1}_n, X\}$, where $\text{trace}(X) = 0$. This identifies $\text{Gr}_1(2, \mathbb{S}^n)$ with the projective space $\mathbb{P}^{\binom{n+1}{2}-2}$ of traceless symmetric matrices. With these conventions, \mathcal{L} is a Jordan algebra if and only if the minimal polynomial of X has degree two.

Proposition 3.1. The restricted Jordan locus $\text{Jo}_1(2, \mathbb{S}^n)$ is a variety with $\lfloor n/2 \rfloor$ irreducible components V_i for $1 \leq i \leq n/2$. The component V_i has codimension $\binom{i+1}{2} + \binom{n-i+1}{2} - 2$. It is parametrized by diagonalizable matrices that have two distinct eigenvalues of multiplicities i and $n - i$. This is case 1 in Proposition 2.6.

Proof. Following [6], regular pencils are classified by their Segre symbols σ . By [6, Theorem 3.2], a pencil \mathcal{L} has $\text{deg}(\mathcal{L}^{-1}) = 1$ if and only if its Segre symbol is $\sigma = [(1, \dots, 1), (1, \dots, 1)]$ or $\sigma = [(2, \dots, 2, 1, \dots, 1)]$. The varieties V_i are the closures of the former strata. These contain the latter in their closure, by [6, Theorem 5.1]. The formula for $\text{codim}(V_i)$ appears in [6, Proposition 5.4]. \square

Remark 3.2. Nondiagonalizable pencils in $\text{Jo}_1(2, \mathbb{S}^n)$ have only one eigenvalue and Jordan blocks of sizes one or two. This is case 2 in Proposition 2.6.

Equations defining V_i are computed as follows. Fix a symmetric $n \times n$ matrix of unknowns, $X = (x_{ij})$. Consider the entries of the product $(X - a\mathbf{1}_n)(X - b\mathbf{1}_n)$, the $(i + 1)$ -minors of $X - a\mathbf{1}_n$, and the $(n - i + 1)$ -minors of $X - b\mathbf{1}_n$. By eliminating a and b from these equations, we obtain homogeneous polynomials in the entries of X that cut out V_i set-theoretically. We discuss these varieties for $n \leq 5$. All pencils of binary quadrics ($n = 2$) are Jordan: $\text{Jo}_1(2, \mathbb{S}^2) = V_1 = \text{Gr}_1(2, \mathbb{S}^2)$.

We thus start with Jordan pencils of plane conics ($n = 3$).

Example 3.3 ($m = 2, n = 3$). The variety $\text{Jo}_1(2, \mathbb{S}^3) = V_1$ is irreducible of codimension 2 and degree 4. Its ideal is generated by the following seven cubics:

$$\begin{aligned} & x_{11}x_{13}x_{22} - x_{11}x_{13}x_{33} - x_{12}^2x_{13} + x_{12}x_{22}x_{23} - x_{12}x_{23}x_{33} + x_{13}^3 - x_{13}x_{22}^2 + x_{13}x_{22}x_{33}, \\ & x_{11}x_{12}x_{22} - x_{11}x_{12}x_{33} - x_{12}^3 + x_{12}x_{13}^2 - x_{12}x_{22}x_{33} + x_{12}x_{23}^2 + x_{13}x_{22}x_{23} - x_{13}x_{23}x_{33}, \\ & x_{11}^2x_{23} - x_{11}x_{12}x_{13} - x_{11}x_{22}x_{23} - x_{11}x_{23}x_{33} + x_{12}x_{13}x_{22} + x_{13}^2x_{23} + x_{22}x_{23}x_{33} - x_{23}^3, \\ & x_{12}^2x_{23} - x_{12}x_{13}x_{22} + x_{12}x_{13}x_{33} - x_{13}^2x_{23}, \quad x_{11}x_{13}x_{23} - x_{12}x_{13}^2 + x_{12}x_{23}^2 - x_{13}x_{22}x_{23}, \\ & x_{11}x_{12}x_{23} - x_{12}^2x_{13} - x_{12}x_{23}x_{33} + x_{13}x_{23}^2, \quad x_{11}^2x_{22} - x_{11}^2x_{33} - x_{11}x_{12}^2 + x_{11}x_{13}^2 \\ & \quad - x_{11}x_{22}^2 + x_{11}x_{23}^2 + x_{12}^2x_{22} - x_{12}^2x_{33} + x_{22}^2x_{33} - x_{22}x_{23}^2 - x_{22}x_{33}^2 + x_{23}^2x_{33}. \end{aligned}$$

These are the expressions in the sum of squares representation of the discriminant of the characteristic polynomial of X , seen in [12, page 97]. Indeed, V_1 is also the Zariski closure of all real matrices in \mathbb{S}^3 with a double eigenvalue.

Example 3.4 ($m = 2, n = 4$). We compute $\text{Jo}_1(2, \mathbb{S}^4)$ in Macaulay2 as follows:

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R = QQ[a, b, x11, x12, x13, x14, x22, x23, x24, x33, x34, x44];
X = matrix {{x11, x12, x13, x14}, {x12, x22, x23, x24},
            {x13, x23, x33, x34}, {x14, x24, x34, x44}};
I = eliminate({a, b}, minors(1, (X-a)*(X-b)))
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The ideal I is generated by 30 cubics. It is the intersection of two prime ideals, corresponding to $\text{Jo}_1(2, \mathbb{S}^4) = V_1 \cup V_2$. The component V_1 has codimension 5 and degree 8. Its prime ideal is generated by 10 quadrics, including $x_{13}x_{24} - x_{12}x_{34}$, $x_{14}x_{23} - x_{12}x_{34}$ and $x_{13}x_{23} - x_{14}x_{24} - x_{12}x_{33} + x_{12}x_{44}$. The component V_2 has codimension 4 and degree 6. Its prime ideal is generated by 9 quadrics.

Example 3.5 ($m = 2, n = 5$). The ideal of $\text{Jo}_1(2, \mathbb{S}^5) = V_1 \cup V_2$ is generated by 81 cubics. The variety V_1 has codimension 9 and degree 16. Its ideal is generated by 35 quadrics, including ten 2×2 -minors, like $x_{12}x_{34} - x_{13}x_{24}$. The variety V_2 has codimension 7 and degree 40. Its ideal is generated by 95 cubics, e.g. $x_{12}x_{13}x_{35} - x_{13}x_{15}x_{23} + x_{22}x_{23}x_{35} - x_{23}^2x_{25} - x_{23}x_{34}x_{45} - x_{23}x_{35}x_{55} + x_{24}x_{34}x_{35} + x_{25}x_{35}^2$.

We now observe that $\text{Jo}(m, \mathbb{S}^n)$ is the orbit of (3) under the congruence action by $\text{GL}(n)$. This implies the following result in the Grassmannian $\text{Gr}(2, \mathbb{S}^n)$.

Corollary 3.6. The Jordan locus $\text{Jo}(2, \mathbb{S}^n)$ has $\lfloor n/2 \rfloor$ irreducible components. Using the notation from [6, Section 5], these components are the Grassmann strata $\overline{\text{Gr}}_\sigma$ that are associated with the Segre symbols $\sigma = [(1, \dots, 1), (1, \dots, 1)]$.

Remark 3.7. Jordan algebras can be viewed as a nonabelian generalization of *partition matroids*. These are the matroids that are direct sums of uniform matroids. Indeed, the diagonalizable types in Corollary 3.6 correspond to the rank-2 partition matroids, and those in Theorem 5.1 to the rank-3 partition matroids.

4. Jordan Copencils

This section concerns Jordan algebras of low codimension. These are quite rare:

Theorem 4.1. Let $\mathcal{L} \subset \mathbb{S}^n$ be a proper Jordan subalgebra. Then $\text{codim}(\mathcal{L}) \geq n - 1$.

To prove this theorem, we need the following result and some lemmas.

Theorem 4.2 (Peirce decomposition, [10, Part II, Chapter 8]). Let X_1, \dots, X_d be orthogonal idempotents in \mathcal{L} with $U = X_1 + \dots + X_d$. Then $\mathcal{L} = \bigoplus_{1 \leq i \leq j \leq d} \mathcal{X}_{i,j}$ where $\mathcal{X}_{i,i} = \{Y \in \mathcal{L} \mid Y = X_i \bullet Y\}$ and $\mathcal{X}_{i,j} = \{Y \in \mathcal{L} \mid Y = 2X_i \bullet Y = 2X_j \bullet Y\}$ for $i < j$.

Here, being *orthogonal* means that $X_i \bullet X_j = 0$ whenever $i \neq j$. An idempotent X is called *primitive* if there are no nonzero orthogonal idempotents X_1 and X_2 with $X = X_1 + X_2$. We denote by J_n the $n \times n$ matrix with ones on its anti-diagonal.

Lemma 4.3 ([3, Lemma 1]). If $X, Y \in \mathbb{S}^n$ are similar, then they are orthogonally congruent.

Lemma 4.4. If the unit U of a Jordan subalgebra $\mathcal{L} \subset \mathbb{S}^n$ is primitive, then \mathcal{L} is congruent to $\mathbb{C}J_n \oplus \mathcal{L}'$ where \mathcal{L}' consists of upper anti-triangular matrices.

Proof. After congruence, we may assume $U = \mathbf{1}_n$. Let $X \in \mathcal{L}$. By considering the Jordan canonical form of X , adding multiples of $\mathbf{1}_n$ to X , and taking powers, we see that $\mathbf{1}_n$ cannot be primitive if X has distinct eigenvalues. So $\mathcal{L} = \mathbb{C}\mathbf{1}_n \oplus \mathcal{L}'$ where $\mathcal{L}' \subset \mathcal{L}$ is the subset of nilpotent matrices. By Lemma 4.3, we see that $XY + YX \in \mathcal{L}'$ for $X, Y \in \mathcal{L}'$. So \mathcal{L}' is anti-triangularizable by [3, Theorem 9]. The transformation given there takes the unit matrix $\mathbf{1}_n$ to the matrix J_n . \square

Two spaces $\mathcal{L}, \mathcal{L}' \subset \mathbb{S}^n$ are congruent if $\mathcal{L}' = P\mathcal{L}P^\top$ for some $P \in GL(n)$. We say that $\mathcal{L}, \mathcal{L}'$ are *orthogonally congruent* if P lies in the orthogonal group $O(n)$.

Lemma 4.5. If $\mathcal{L}, \mathcal{L}' \subset \mathbb{S}^n$ are similar, then they are orthogonally congruent.

Proof. We have $\mathcal{L}P = P\mathcal{L}'$ for some $P \in GL(n)$. Consider $A \in \mathcal{L}$ and $B \in \mathcal{L}'$ with $AP = PB$. By the proof of [3, Lemma 1], we get $AU = UB$ for $U = (PP^\top)^{-1/2}P$. The matrix U is orthogonal, and it maps \mathcal{L} into \mathcal{L}' under congruence. \square

Lemma 4.6. Let $X \in \mathbb{S}^n$ be an idempotent matrix of rank r . Then X is orthogonally congruent to $\text{Diag}(\mathbf{1}_r, \mathbf{0}_{n-r})$. If $X = \text{Diag}(\mathbf{1}_r, \mathbf{0}_{n-r})$ and $Y \in \mathbb{S}^n$ is a matrix with $XY + YX = 0$, then $Y = \text{Diag}(\mathbf{0}_r, Z)$ for some $Z \in \mathbb{S}^{n-r}$.

Proof. This follows from Lemma 4.3. \square

Proof of Theorem 4.1. Either \mathcal{L} contains a primitive idempotent of rank > 1 , or the unit U of \mathcal{L} is a sum of orthogonal rank-1 idempotents. We shall examine the associated Peirce decompositions (Theorem 4.2). These have $d = 2$ and $d = n$.

Suppose that \mathcal{L} contains a primitive idempotent X_1 of rank $r > 1$ and take $X_2 = U - X_1$. Then $\mathcal{L} = \mathcal{X}_{1,1} \oplus \mathcal{X}_{1,2} \oplus \mathcal{X}_{2,2}$. By Lemmas 4.4 and 4.6, we assume that $X_1 = \text{Diag}(J_r, \mathbf{0}_{n-r})$, $X_2 = \text{Diag}(\mathbf{0}_r, \mathbf{1}_{n-r})$ and $\mathcal{X}_{1,1}$ consists of upper anti-triangular $r \times r$ matrices. So $\mathcal{X}_{1,1}$ has codimension $\geq r - 1$ in \mathbb{S}^r . Elements in $\mathcal{X}_{1,2}$ are of the form $Z = \begin{pmatrix} \mathbf{0}_r & V^\top \\ V & \mathbf{0}_{n-r} \end{pmatrix}$ with $Z^{\bullet 2} = \text{Diag}(V^\top V, VJ_rV^\top)$. It follows that $V^\top V \in \mathcal{X}_{1,1}$. So, the last column v of V satisfies $v^\top v = 0$. Hence v is contained in a subspace $\mathcal{Y} \subset \mathbb{C}^{n-r}$ of codimension $c \geq (n-r)/2$. We have $\mathcal{X}_{2,2} = \text{Diag}(\mathbf{0}_r, \mathcal{Z})$ where $\mathcal{Z} \subset \mathbb{S}^{n-r}$ consists of matrices that map \mathcal{Y} into itself. Therefore \mathcal{Z} has codimension $\geq c(n-r-c)$. So \mathcal{L} has codimension $\geq r - 1 + c + c(n-r-c) \geq n - 1$.

If all primitive idempotents of \mathcal{L} have rank 1, then $U = X_1 + \dots + X_n$ for rank-1 orthogonal idempotents X_1, \dots, X_n . By Lemma 4.6, we may assume $X_i = E_{ii}$. We note that the spaces of the Peirce decomposition are either $\mathcal{X}_{i,j} = \mathbb{C}\{E_{i,j} + E_{j,i}\}$ or $\mathcal{X}_{i,j} = \{0\}$. Since \mathcal{L} is a proper subalgebra, at least one of these spaces is $\{0\}$, say $\mathcal{X}_{1,2} = \{0\}$. Since $2(E_{i,j} + E_{j,i}) \bullet (E_{i,k} + E_{k,i}) = E_{k,j} + E_{j,k}$, this implies either $\mathcal{X}_{1,i} = \{0\}$ or $\mathcal{X}_{2,i} = \{0\}$ for each $i \in \{3, \dots, n\}$. Therefore $\text{codim}(\mathcal{L}) \geq n - 1$. \square

A linear subspace $\mathcal{L} \subset \mathbb{S}^n$ is said to be a *copencil* if $\text{codim}(\mathcal{L}) = 2$, i.e. the orthogonal complement $\mathcal{L}^\perp = \{X \in \mathbb{S}^n \mid \text{trace}(XZ) = 0 \text{ for } Z \in \mathcal{L}\}$ is a pencil. We say that \mathcal{L} is a *Jordan copencil* if \mathcal{L} is a copencil that satisfies the equivalent conditions (a), (b) and (c) from Theorem 1.1. Theorem 4.1 implies:

Corollary 4.7. There are no Jordan copencils unless $n \leq 3$. Every copencil with $n = 2$ is a Jordan copencil since it is the span of one invertible matrix U in \mathbb{S}^2 .

It remains to study $n = 3$. We represent Jordan copencils by the variety

$$\{(X, Y) \in \mathbb{S}^3 \times \mathbb{S}^3 \mid (\mathbb{C}\{X, Y\})^\perp \text{ is a point in } \text{Jo}(4, \mathbb{S}^3)\}. \tag{4}$$

Proposition 4.8. There are two congruence orbits of Jordan copencils in \mathbb{S}^3 :

$$\mathcal{L}_1 = \begin{pmatrix} x & 0 & 0 \\ 0 & y & w \\ 0 & w & z \end{pmatrix} \quad \text{and} \quad \mathcal{L}_2 = \begin{pmatrix} x & y & w \\ y & z & 0 \\ w & 0 & 0 \end{pmatrix}.$$

The orbit of \mathcal{L}_2 is in the closure of the orbit of \mathcal{L}_1 . These orbits have codimensions 5 and 4, so the Jordan locus $\text{Jo}(4, \mathbb{S}^3)$ is irreducible of codimension 4. The prime ideal of (4) has degree 21 and is generated by 4 cubics and 15 quartics.

Proof. Theorem 2.2 gives a finite list of cases to examine. By multiple applications of Lemma 4.6, we see that $\mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C}$ cannot be embedded into \mathbb{S}^3 . By Theorem 5.3, $\mathbb{C} \times \mathbb{C}^2$ cannot be embedded into \mathbb{S}^3 . And, a computation shows that the Jordan algebra $\mathbb{C}\{X, Y\}$ with zero bilinear form \bullet cannot be embedded into \mathbb{S}^3 . Using the help of a computer as in the proof of Theorem 2.7, we generate a finite list of Jordan algebras that do not contain any of these Jordan algebras as a subalgebra. Using Lemma 4.6, we find that all embeddings of these Jordan algebras are congruent to \mathcal{L}_1 or \mathcal{L}_2 .

The last two statements are proved by computer algebra. Note that \mathcal{L}_1^\perp and \mathcal{L}_2^\perp are singular pencils. The four cubics in the ideal for \mathcal{L}_1 are the four coefficients of $\det(xX + yY)$. The 15 quartics can also be derived using linear algebra. For the containment of orbits, we check that \mathcal{L}_2 satisfies these 19 equations. \square

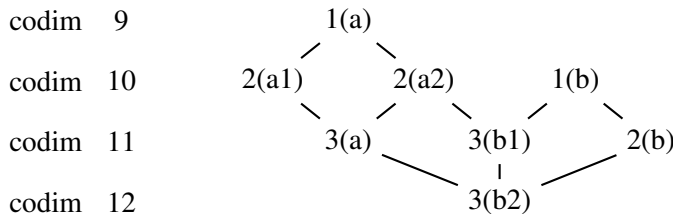
5. Three-Dimensional Jordan Algebras

The conditions in Theorem 1.1 are invariant under congruence. Ideally, one would classify Jordan subalgebras of \mathbb{S}^n up to congruence and describe the poset of orbit closures. We tackle this problem for $m = 3$. The following two theorems are our main results in Section 5. The first gives a complete solution for 4×4 matrices. The second gives a solution for Jordan subalgebras without a radical.

Theorem 5.1. Every Jordan net in \mathbb{S}^4 is congruent to one of the following nets:

- 1 (a) : $\text{Diag}(x\mathbf{1}_2, y, z)$;
- (b) : the Kronecker product of $\mathbf{1}_2$ with \mathbb{S}^2 ; (see \mathcal{L}_2 in Example 6.9)
- 2 (a1): $\text{Diag}(xJ_2 + yE_{11}, z\mathbf{1}_2)$;
- (a2): $\text{Diag}(xJ_3 + yE_{11}, z)$;
- (b) : $\text{Diag}(xJ_2, yJ_2) + z(E_{13} + E_{31})$;
- 3 (a) : $x\text{Diag}(J_3, 1) + y(E_{12} + E_{21}) + zE_{11}$;
- (b1): $xJ_4 + yE_{11} + zE_{22}$; or
- (b2): $xJ_4 + yE_{11} + z(E_{13} + E_{31})$.

The numberings of the congruence classes are the same as in Theorem 2.7. Their orbit closures are subvarieties in the Grassmannian $\text{Gr}(3, \mathbb{S}^4)$ of dimension 21. The codimensions of these Jordan loci and their containments are as follows:



Corollary 5.2. The Jordan locus $\text{Jo}(3, \mathbb{S}^4)$ has two irreducible components.

Jordan algebras of types 1(a) and 1(b) are of primary interest also for $n \geq 5$.

Theorem 5.3. Every Jordan net in \mathbb{S}^n of type 1(a) is congruent to a diagonal net $\text{Diag}(x\mathbf{1}_{k_1}, y\mathbf{1}_{k_2}, z\mathbf{1}_{k_3})$ for $k_1 \geq k_2 \geq k_3 \geq 1$ with sum n . Jordan nets of type 1(b) exist only for even n . They are congruent to the Kronecker product of $\mathbf{1}_{n/2}$ with \mathbb{S}^2 .

We note that Theorem 5.3 does not account for all irreducible components of $\text{Jo}(3, \mathbb{S}^n)$ when $n \geq 5$. At present we do not know these decompositions.

Proposition 5.4. The Jordan net $\mathcal{L}_* = uJ_5 + x(E_{15} + E_{51}) + y(E_{14} + E_{41})$ in \mathbb{S}^5 has type 2(b). It is not in the closure of the diagonalizable locus 1(a) in $\text{Jo}(3, \mathbb{S}^5)$.

Proof. For any linear space $\mathcal{L} \subset \mathbb{S}^n$ we consider $\tau(\mathcal{L}) := \min\{\text{rk}X \mid X \in \mathcal{L} \setminus \{0\}\}$. This invariant equals 1 for all Jordan algebras \mathcal{L} of type 1(a), since $k_3 = 1$ for $n = 5$. For the special net above, we see that $\tau(\mathcal{L}_*) = 2$. It now suffices to note that $\{\mathcal{L} \in \text{Gr}(m, \mathbb{S}^n) \mid \tau(\mathcal{L}) \leq k\}$ is a closed set because it is the projection along $\mathbb{P}(\mathbb{S}^n)$ of the incidence variety $\{(\mathcal{L}, X) \in \text{Gr}(m, \mathbb{S}^n) \times \mathbb{P}(\mathbb{S}^n) \mid X \in \mathcal{L}, \text{rk}X \leq k\}$. \square

The situation is easier if we restrict to the case of interest in applications. Let \mathcal{L} be a real Jordan subalgebra of symmetric matrices whose unit U is positive definite. Then \mathcal{L} is *formally real*, i.e. if $X_1^{\bullet 2} + \dots + X_k^{\bullet 2} = 0$, then $X_1, \dots, X_k = 0$. Formally real Jordan algebras have zero radical [9] and this persists if we view them as Jordan algebras over \mathbb{C} . We define the *Euclidean Jordan locus* $\text{EJo}(3, \mathbb{S}^n)$ to be the closure in $\text{Gr}(3, \mathbb{S}^n)$ of the set of such Jordan algebras. For our study with $m = 3$, we get that formally real Jordan nets have type 1(a) or 1(b).

Corollary 5.5. The number of irreducible components of $\text{EJo}(3, \mathbb{S}^n)$ is the coefficient of the t^n in the generating function

$$t^3 / ((1-t)(1-t^2)(1-t^3)) + t^2 / (1-t^2).$$

Proof. The first summand is the generating function of triples of $a, b, c \geq 0$ with $a + 2b + 3c = n - 3$. These correspond to triples of $k_1 \geq k_2 \geq k_3 \geq 1$ with sum n via

$$(a, b, c) \mapsto (a + b + c + 1, b + c + 1, c + 1).$$

The second generating function accounts for the Jordan nets of type 1(b). \square

In the remainder of this section we present the proofs of our two theorems.

Proof of Theorem 5.3. Let \mathcal{L} be a Jordan net of type 1(a). Then \mathcal{L} is congruent to $\text{Diag}(x\mathbf{1}_{k_1}, y\mathbf{1}_{k_2}, z\mathbf{1}_{k_3})$. This is seen by two applications of Lemma 4.6. Suppose \mathcal{L} has unit $U = \mathbf{1}_n$ and type 1(b). After applying an orthogonal congruence, we may assume that $X = \text{Diag}(\mathbf{1}_r, -\mathbf{1}_{n-r})$ for some $1 \leq r \leq n - 1$. The condition $XY + YX = 0$ now implies $Y = \begin{pmatrix} 0 & P \\ P^\top & 0 \end{pmatrix}$ for some $P \in \mathbb{C}^{r \times (n-r)}$. The condition $Y^2 = \mathbf{1}_n$ implies $PP^\top = \mathbf{1}_r$ and $P^\top P = \mathbf{1}_{n-r}$. This is only possible when $2r = n$. By applying the orthogonal congruence $\text{Diag}(P^\top, \mathbf{1}_r)$, we get the desired form. \square

Proof of Theorem 5.1. We proceed case-by-case, starting from the classification of Jordan algebras in Theorem 2.7. Let U, X, Y be a basis of \mathcal{L} , where we assume $U = \mathbf{1}_4$. The types 1(a) and 1(b) are already covered by the previous proof.

Suppose that \mathcal{L} has type 2(a). We can assume $X = \text{Diag}(\mathbf{1}_{4-k}, \mathbf{0}_k)$ and $Y = \text{Diag}(\mathbf{0}_{4-k}, Z)$ with $Z^2 = \mathbf{0}_k$ for some $k \in \{2, 3\}$. Up to orthogonal congruence, there is only one $Z \in \mathbb{S}^k$ with $Z^2 = \mathbf{0}_k \neq Z$, as there is only one option for its Jordan canonical form. This uses Lemma 4.3. So, we get two classes of this type.

Suppose that \mathcal{L} has type 2(b). Then we can assume that $X = \text{Diag}(\mathbf{1}_k, \mathbf{0}_{4-k})$. The condition $XY + YX = Y$ implies $Y = \begin{pmatrix} 0 & P \\ P^\top & 0 \end{pmatrix}$ for some matrix $P \in \mathbb{C}^{k \times (4-k)}$. The condition $Y^2 = \mathbf{0}_4$ implies $PP^\top = \mathbf{0}_k$ and $P^\top P = \mathbf{0}_{4-k}$. As P is nonzero, this is only possible when $k = 2$. So $P = ab^\top$ for some nonzero vectors $a, b \in \mathbb{C}^2$ with $a^\top a = b^\top b = 0$. Such matrices Y form a single orbit of $\text{Diag}(\text{O}(2), \text{O}(2))$. So we get one class of this type. Suppose that \mathcal{L} has type 3(a). Then $X^2 = Y$ and $X^3 = 0$. By Lemma 4.3 and appealing to Jordan canonical forms as above, there is a unique X up to orthogonal congruence. So we get one class of this type.

Consider \mathcal{L} of type 3(b). We have $X^2 = Y^2 = XY + YX = 0$. Since X, Y or $X + Y$ has rank 2, we may assume $\text{rk} X = 2$. One can show, using Lemma 4.3, that we may assume $\text{rk} Y = 1$. Using Lemma 4.5 and conjugating \mathcal{L} in $\mathbb{C}^{n \times n}$, we get

$$X = \begin{pmatrix} 0 & \mathbf{1}_2 \\ 0 & 0 \end{pmatrix} \text{ and } Y = \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix} \text{ where } \text{rk} Z = 1.$$

By conjugating with a matrix $\text{Diag}(P, P)$ and scaling, we find $Z = E_{22}$ or $Z = E_{12}$. So we get two classes. In one of them, $\text{rad} \mathcal{L}$ is spanned by two rank-1 matrices.

To identify the poset of inclusions, consider the associated nets of quadrics $(a, b, c, d)\mathcal{L}(a, b, c, d)^T$. In this notation, type 1(a) is written as $x(a^2 + b^2) + yc^2 + zd^2$. Replacing (a, b, c, d) by $(a, b, c, c + td)$ in 1(a), where $t \in \mathbb{R} \setminus \{0\}$, we get

$$x(a^2 + b^2) + y'c^2 + z'(2cd + td^2)$$

for $y' = y + z$ and $z' = tz$. As $t \rightarrow 0$, we obtain type 2(a1). Similarly:

- We get 1(a) \rightarrow 2(a2) using $(c + t^2a, tb, c, d)$.
- We get 1(b) \rightarrow 2(b) using $(-a, b, i(a + tc), i(b + td))$.
- We get 1(b) \rightarrow 3(b1) using (a, tb, tc, d) .
- We get 2(a1) \rightarrow 3(a) using $(a, b, a + tb + t^2c, td)$.
- We get 2(a2) \rightarrow 3(a) using $(a, td, t(tc - b), a + tb)$.
- We get 2(a2) \rightarrow 3(b1) using $(a, d + tb, c, d)$.
- We get 2(b) \rightarrow 3(b2) using $(c + ta, d + tb, c, d)$.
- We get 3(a) \rightarrow 3(b2) using $(a, b, tc, i(b - td))$.
- We get 3(b1) \rightarrow 3(b2) using $(a, a + tb, c, -c + td)$.

The codimensions were found by direct computation. It remains to show that there is no inclusion of orbit closures when there is no edge in our Hasse diagram. The arrow $1(a) \rightarrow 2(b)$ is missing because τ cannot go up in limits; see Proposition 5.4. The absence of other inclusions is certified by explicit polynomials that vanish on one orbit, but not on another. For example, the polynomial for the orbit 1(b) displayed in (7) does not vanish on the orbits 1(a), 2(a1), 2(a2) and 3(a). Similarly, $4p_{012}p_{146} - 4p_{013}p_{145} - p_{014}p_{056} - p_{014}p_{126} + p_{014}p_{135} - p_{014}p_{234} + p_{024}p_{046} - p_{034}p_{045}$ vanishes on 2(a1) but not on 3(b1). \square

6. Nets of Quadrics

In this section we offer a more detailed study of Jordan loci in the case $m = 3$.

Proposition 6.1. The Jordan locus $\text{Jo}(3, \mathbb{S}^3)$ consists of nets that are generated by three rank-1 conics in \mathbb{P}^2 . This is an irreducible variety of codimension 3 in the Grassmannian $\text{Gr}(3, \mathbb{S}^3)$, and it has degree 57 in the Plücker embedding in \mathbb{P}^{19} . Its prime ideal is generated by 62 quadrics in the 20 Plücker coordinates.

Proof. Here we use the Wall’s classification [14] of nets of conics. For each net \mathcal{L} in his list, we computed \mathcal{L}^{-1} . The surface \mathcal{L}^{-1} is linear only for Wall’s types E, G and H. According to [14, Figure 5], the strata G and H are contained in the closure of stratum E, which has codimension 3, by [14, Table 2]. We see in [14, Table 1] that E corresponds to nets spanned by rank-1 conics. Hence the closure of stratum E is an irreducible variety. It equals $\text{Jo}(3, \mathbb{S}^3)$. The results about the ideal and degree of this Jordan locus are in [2, Proposition 3.1]. \square

The ideal in Proposition 6.1 simplifies greatly if we restrict to $\text{Jo}_1(3, \mathbb{S}^3)$. Namely, for the net $\mathcal{L} = \text{span}\{\mathbf{1}_3, X, Y\}$, with $X = (x_{ij})$ and $Y = (y_{ij})$, we obtain

$$\begin{aligned} & \langle (x_{11}y_{12} - x_{12}y_{11}) + (x_{12}y_{22} - x_{22}y_{12}) + (x_{13}y_{23} - x_{23}y_{13}), \\ & (x_{11}y_{23} - x_{23}y_{11}) + (x_{12}y_{13} - x_{13}y_{12}) + 2(x_{22}y_{23} - x_{23}y_{22}), \\ & 2(x_{11}y_{13} - x_{13}y_{11}) + (x_{12}y_{23} - x_{23}y_{12}) - (x_{13}y_{22} - x_{22}y_{13}) \rangle. \end{aligned} \tag{5}$$

This prime ideal of codimension 3 defines the Jordan locus in $\text{Gr}_1(3, \mathbb{S}^3)$.

We now assume $n \geq 3$. For any net $\mathcal{L} \in \text{Gr}(3, \mathbb{S}^n)$, the reciprocal \mathcal{L}^{-1} is a surface in $\mathbb{P}(\mathbb{S}^n) = \mathbb{P}(\binom{n+1}{2})^{-1}$. This is a linear plane precisely when \mathcal{L} is a Jordan algebra. We now examine the other extreme case, when \mathcal{L} is generic in $\text{Gr}(3, \mathbb{S}^n)$.

Theorem 6.2. The following conditions are equivalent for a net of quadrics \mathcal{L} :

1. Every nonzero complex matrix in \mathcal{L} has rank $\geq n - 1$.
2. The linear span of the surface \mathcal{L}^{-1} is the ambient projective space $\mathbb{P}(\mathbb{S}^n)$.
3. \mathcal{L}^{-1} is linearly isomorphic to the $(n - 1)$ st Veronese embedding of \mathbb{P}^2 .
4. The Chow matrix of \mathcal{L} is invertible.

To explain this result, we must first define the *Chow matrix*. Fix a basis X, Y, Z of \mathcal{L} , and let x, y, z be unknowns. The Chow matrix is square of size $\binom{n+1}{2} \times \binom{n+1}{2}$, and its entries are homogeneous polynomials of degree $n - 1$ in the entries of X, Y, Z . The columns are labeled by the monomials of degree $n - 1$ in x, y, z , and the rows are labeled by pairs (i, j) where $1 \leq i \leq j \leq n$. Consider the entry in position (i, j) of the adjoint matrix of $xX + yY + zZ$. This is a polynomial of degree $n - 1$ in x, y, z . Its coefficients form the row indexed (i, j) of the Chow matrix. The entry in a given column is the coefficient of the monomial label.

Example 6.3 ($n = 3$). The Chow matrix of three plane conics has format 6×6 :

$$\begin{pmatrix} x_{22}x_{33} - x_{23}^2 & x_{22}y_{33} - 2x_{23}y_{23} + x_{33}y_{22} & x_{22}z_{33} - 2x_{23}z_{23} + x_{33}z_{22} & y_{22}y_{33} - y_{23}^2 & \cdots & \cdots \\ x_{13}x_{23} - x_{12}x_{33} & x_{13}y_{23} - x_{12}y_{33} + x_{23}y_{13} - x_{33}y_{12} & x_{13}z_{23} - x_{12}z_{33} + x_{23}z_{13} - x_{33}z_{12} & y_{13}y_{23} - y_{12}y_{33} & \cdots & \cdots \\ x_{12}x_{23} - x_{13}x_{22} & x_{12}y_{23} - x_{13}y_{22} - x_{22}y_{13} + x_{23}y_{12} & x_{12}z_{23} - x_{13}z_{22} - x_{22}z_{13} + x_{23}z_{12} & y_{12}y_{23} - y_{13}y_{22} & \cdots & \cdots \\ x_{11}x_{33} - x_{13}^2 & x_{11}y_{33} - 2x_{13}y_{13} + x_{33}y_{11} & x_{11}z_{33} - 2x_{13}z_{13} + x_{33}z_{11} & y_{11}y_{33} - y_{13}^2 & \cdots & \cdots \\ x_{12}x_{13} - x_{11}x_{23} & x_{12}y_{13} - x_{11}y_{23} + x_{13}y_{12} - x_{23}y_{11} & x_{12}z_{13} - x_{11}z_{23} + x_{13}z_{12} - x_{23}z_{11} & y_{12}y_{13} - y_{11}y_{23} & \cdots & \cdots \\ x_{11}x_{22} - x_{12}^2 & x_{11}y_{22} - 2x_{12}y_{12} + x_{22}y_{11} & x_{11}z_{22} - 2x_{12}z_{12} + x_{22}z_{11} & y_{11}y_{22} - y_{12}^2 & \cdots & \cdots \end{pmatrix}.$$

The row labels $(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)$ refer to the entries of the adjoint of $xX + yY + zZ$. The column labels are the monomials $x^2, xy, xz, y^2, yz, z^2$. The determinant of the Chow matrix is an irreducible homogeneous polynomial of degree 12 with 22659 terms. It vanishes if and only if the net contains a rank-1 matrix. Thus, it equals the Chow form [7] of the Veronese surface in \mathbb{P}^5 .

The *Chow form* of a projective variety is a hypersurface in the Grassmannian. Its points are linear spaces of complementary dimension that unexpectedly intersect the variety. If the linear space is expressed as the row space of a matrix then the entries of the matrix are the *dual Stiefel coordinates* of that linear space.

Lemma 6.4. Consider the net $\mathcal{L} = \{xX + yY + zZ\}$ spanned by $X, Y, Z \in \mathbb{S}^n$. The Chow matrix is singular if and only if $\mathcal{L} \setminus \{\mathbf{0}_n\}$ contains a matrix of rank $\leq n - 2$. So the determinant of the Chow matrix is the Chow form, written in dual Stiefel coordinates, of the subvariety of $\mathbb{P}(\mathbb{S}^n)$ defined by the $(n - 1) \times (n - 1)$ -minors.

Proof. If X has rank $\leq n - 2$ then the coefficient of x^{n-1} is zero in every entry of the adjoint of $xX + yY + zZ$, so the first row of the Chow matrix is zero. Such a rank drop happens whenever some matrix in \mathcal{L} has rank $\leq n - 2$ since the determinant of the Chow matrix is invariant of the choice of basis in \mathcal{L} . That determinant is a nonzero homogeneous polynomial in the entries of X, Y, Z of degree $(n - 1) \binom{n+1}{2}$. We already argued that it vanishes on the hypersurface defined by the Chow form of the rank- $(n - 2)$ variety. That variety has degree $\binom{n+2}{3}$, so its Chow form in dual Stiefel coordinates is an irreducible polynomial of degree $3 \binom{n+2}{3}$. This irreducible polynomial divides our determinant. Since $(n - 1) \binom{n+1}{2} = 3 \binom{n+2}{3}$, the two polynomials agree up to a nonzero constant. \square

Remark 6.5. The Chow matrix is new and interesting even for $n = 4$. It gives the Chow form for symmetric 4×4 matrices of rank ≤ 2 . A formula for that Chow form was the main result in [7, Section 3]. Our construction is much simpler.

Proof of Theorem 6.2. Lemma 6.4 says that conditions (1) and (4) are equivalent. The surface \mathcal{L}^{-1} is parametrized by the entries of the adjoint of $xX + yY + zZ$. By inverting the Chow matrix, we can express every monomial of degree $n - 1$ in x, y, z as a linear combination of those entries. Hence \mathcal{L}^{-1} is linearly isomorphic to the Veronese surface, so (4) implies (3). Clearly (3) implies (2). Finally, if (4) fails then \mathcal{L}^{-1} lies in a hyperplane in $\mathbb{P}(\mathbb{S}^n)$, so (2) fails. \square

Corollary 6.6. Given any regular net \mathcal{L} of symmetric $n \times n$ matrices, the rank of its Chow matrix equals the dimension of the linear span of \mathcal{L}^{-1} inside \mathbb{S}^n .

Proof. A linear form vanishes on \mathcal{L}^{-1} if and only if it vanishes on the adjoint of $xX + yY + zZ$. These are the vectors in the left kernel of the Chow matrix. \square

Every regular subspace $\mathcal{L} \subset \mathbb{S}^n$ generates a unique Jordan subalgebra $\text{Jor}(\mathcal{L})$ inside \mathbb{S}^n . The dimension of $\text{Jor}(\mathcal{L})$ is bounded below by the rank of the Chow matrix of \mathcal{L} . However, this bound is not tight. To see this, consider the net

$$\mathcal{L} = \begin{pmatrix} x+y & z & z & 0 \\ z & x-y & 0 & z \\ z & 0 & x & z \\ 0 & z & z & x \end{pmatrix}. \tag{6}$$

Here, $\text{Jor}(\mathcal{L}) = \mathbb{S}^4$ but the 10×10 Chow matrix has rank 8. Its left kernel gives the two linear forms $z_{14} - z_{23} - z_{33} + z_{44}$ and $2z_{12} - z_{13} - z_{24}$ which vanish on \mathcal{L}^{-1} .

We now return to the polynomial equations that vanish on the Jordan locus in the Grassmannian. The following result is immediate from Corollary 6.6.

Proposition 6.7. Using dual Stiefel coordinates on $\text{Gr}(m, \mathbb{S}^n)$, the Jordan locus $\text{Jo}(m, \mathbb{S}^n)$ is cut out by the $(m + 1) \times (m + 1)$ -minors of the Chow matrix.

Proof. For any regular $\mathcal{L} \in \text{Gr}(m, \mathbb{S}^n)$, the reciprocal variety \mathcal{L}^{-1} is irreducible of dimension m and contained in its span, which is itself an irreducible variety. These two varieties coincide if and only if the latter has dimension $\leq m$. \square

Remark 6.8. These 4×4 -minors do not generate the prime ideal of $\text{Jo}(3, \mathbb{S}^n)$. For instance, if $n = 3$ and we fix $\mathbf{1}_3 \in \mathcal{L}$ then Proposition 6.7 yields 30 quartics in x_{ij}, y_{ij} . These cut out the same variety $\text{Jo}_1(3, \mathbb{S}^3)$ as the three quadrics in (5).

We know from Theorem 5.1 that $\text{Jo}(3, \mathbb{S}^4)$ has two irreducible components, of types 1(a) and 1(b). They are given by the nets \mathcal{L}_1 and \mathcal{L}_2 in the next example.

Example 6.9. We consider the congruence orbits of the following nets in \mathbb{S}^4 :

$$\mathcal{L}_1 = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & z \end{pmatrix}, \quad \mathcal{L}_2 = \begin{pmatrix} x & y & 0 & 0 \\ y & z & 0 & 0 \\ 0 & 0 & x & y \\ 0 & 0 & y & z \end{pmatrix}, \quad \mathcal{L}_3 = \begin{pmatrix} 0 & 0 & x & y \\ 0 & -2x & -y & z \\ x & -y & -2z & 0 \\ y & z & 0 & 0 \end{pmatrix}.$$

They have codimensions 9, 10 and 9 in $\text{Gr}(3, \mathbb{S}^4)$. The first two fill out $\text{Jo}(3, \mathbb{S}^4)$.

It is interesting to compare the non-diagonalizable nets \mathcal{L}_2 and \mathcal{L}_3 . Both represent double conics in \mathbb{P}^2 , since $\det(\mathcal{L}_2) = \det(\mathcal{L}_3) = (xz - y^2)^2$. But, while \mathcal{L}_2^{-1} is linear, the Chow matrix of \mathcal{L}_3 is invertible, so \mathcal{L}_3^{-1} is a Veronese surface. The congruence orbit of \mathcal{L}_3 is the smooth variety of codimension 9 studied in [4]. It is essentially the Hilbert scheme of twisted cubic curves in \mathbb{P}^3 .

We end this section by reporting a challenging computation. It is predicated on the idea that Plücker coordinates are best when working with Grassmannians.

Example 6.10. The 21-dimensional Grassmannian $\text{Gr}(3, \mathbb{S}^4)$ lives in \mathbb{P}^{119} . This space has $\binom{10}{3} = 120$ dual Plücker coordinates p_{ijk} , one for each 3×3 -minor of

$$\begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} & u_{22} & u_{23} & u_{24} & u_{33} & u_{34} & u_{44} \\ x_{11} & x_{12} & x_{13} & x_{14} & x_{22} & x_{23} & x_{24} & x_{33} & x_{34} & x_{44} \\ y_{11} & y_{12} & y_{13} & y_{14} & y_{22} & y_{23} & y_{24} & y_{33} & y_{34} & y_{44} \end{pmatrix}.$$

We computed the quadrics that vanish on subvarieties of $\text{Gr}(3, \mathbb{S}^4)$. Modulo Plücker relations, there are 4950 linearly independent quadrics. Each quadric has the form $\sum \lambda_{lmn}^{ijk} p_{ijk} p_{lmn}$ where the λ_{lmn}^{ijk} are unknown coefficients and $i \leq l, j \leq m, k \leq n$. For each of the three orbits in Example 6.9, we set up a linear system of equations in the λ_{lmn}^{ijk} whose solutions are quadrics that vanish on that variety. We then solved these equations using MapLe. Here are the results:

We found a basis of 85 quadrics for the orbit of \mathcal{L}_1 . One representative is

$$p_{124}p_{789} - p_{125}p_{589} + p_{125}p_{679} - p_{126}p_{678} + p_{127}p_{569} - p_{128}p_{568} - p_{234}p_{579} \\ + p_{234}p_{678} + p_{235}p_{479} + p_{235}p_{568} - p_{236}p_{478} - p_{236}p_{567} - p_{237}p_{459} + p_{238}p_{458}.$$

We found a basis for 189 quadrics for the orbit of \mathcal{L}_2 . One representative is

$$p_{012}p_{018} - p_{012}p_{026} - p_{012}p_{035} - 2p_{012}p_{123} - p_{013}p_{017} + 2p_{013}p_{025} - p_{023}p_{024}. \quad (7)$$

Each of these 189 quadrics also vanishes on \mathcal{L}_3 . But, there are 918 quadrics vanishing on the orbit of \mathcal{L}_3 . One representative is $8p_{012}p_{457} - 4p_{045}p_{057} + p_{047}^2$. To see that \mathcal{L}_3 is not contained in \mathcal{L}_2 , we can use the quartics in Proposition 6.7.

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