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SOME ADJUNCTION PROPERTIES OF AMPLE VECTOR BUNDLES II

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Let $\mathcal{E}$ be an ample vector bundle of rank r on a projective variety X of dimension n with only log-terminal singularities. We classify pairs $(X, \mathcal{E})$ under the condition that $1 < r < n - 1$ and $K_X + (n - r) \det \mathcal{E}$ is nef but not ample. As an application, we obtain the classification of $(X, \mathcal{E})$ of c_r-sectional genus one.

Introduction.

Let X be a projective variety of dimension n with a canonical divisor K_X and let $\mathcal{E}$ be an ample vector bundle of rank r on X. The study of the positivity of $K_X + \det\mathcal{E}$ is called generalized adjunction. In recent years many authors have studied generalized adjunction (see [24], [11], [23], [26], [27] and [21] for example) under the condition $r \geq n - 1$. Although there are some results when $r = n - 2$ (see [25], [19] and [2] for example), it is generally difficult to study $K_X + \det\mathcal{E}$ in the case $1 < r < n - 1$. In this case, the author [14] considered $K_X + (n - r) \det\mathcal{E}$ instead of $K_X + \det\mathcal{E}$, and proved that $K_X + (n - r) \det\mathcal{E}$ is nef unless $(X, \mathcal{E}) \cong (\mathbb{P}^4, \mathcal{O}(1)^{\oplus 2})$ when X has only log-terminal singularities. In the present paper, as a next step, we consider the case that $K_X + (n - r) \det\mathcal{E}$ is nef but not ample.

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Theorem 2.1. *Let X be a projective variety of dimension n with only log-terminal singularities and let $\mathcal{E}$ be an ample vector bundle of rank r on X. If $1 < r < n-1$ and $K_X + (n-r)\det\mathcal{E}$ is nef but not ample, then $(X, \mathcal{E})$ is one of the following:*

(i) $(\mathbb{P}^5, \mathcal{O}(1)^{\oplus 2})$;
(ii) $(\mathbb{P}^5, \mathcal{O}(1)^{\oplus 3})$;
(iii) $(\mathbb{Q}^4, \mathcal{O}(1)^{\oplus 2})$, *where $\mathbb{Q}^4$ is a (possibly singular) hyperquadric in $\mathbb{P}^5$;*
(iv) $(\mathbb{P}_W(\mathcal{F}), \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1) \otimes \rho^*\mathcal{G})$, *where $\mathcal{F}$ and $\mathcal{G}$ are vector bundles on a smooth curve W of genus $g(W)$ such that $\operatorname{rank}\mathcal{F} = 4$, $\operatorname{rank}\mathcal{G} = 2$, $c_1(\mathcal{F}) + 2c_1(\mathcal{G}) + 2g(W) > 2$, and $\rho : \mathbb{P}_W(\mathcal{F}) \to W$ is the bundle projection.*

We note that this theorem determines Fano varieties characterized by the ample vector bundles of c_r-sectional genus one. The c_r-sectional genus is equal to the sectional genus (resp. curve genus) when $r = 1$ (resp. $r = n-1$). We refer to [13], [12] and [14] for the basic properties of c_r-sectional genus.

1. Preliminaries.

We work over the complex number field $\mathbb{C}$. Varieties are always irreducible and reduced. Line bundles are identified with the linear equivalence classes of Cartier divisors. The tensor products of line bundles are denoted additively, while we use multiplicative notation for intersection products. We denote by $L^{\oplus r}$ the direct sum of r-copies of a line bundle L. The restriction $L|_W$ of L to a variety W is written as L_W.

For a polarized variety (X, L) of dimension n, a non-negative integer $\Delta(X, L) := n + L^n - h^0(X, L)$ is called the Δ-*genus* of (X, L). We say that (X, L) is a *scroll* over a variety W if $(X, L) \cong (\mathbb{P}_W(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$ for some ample vector bundle $\mathcal{E}$ on W.

Proposition 1.1. ([6] or [5], Chap. I, 5) *Let X be a projective variety of dimension $n \geq 3$ and let L be an ample line bundle on X. If $\Delta(X, L) = 0$, then (X, L) is one of the following:*

(i) $(\mathbb{P}^n, \mathcal{O}(1))$;
(ii) $(\mathbb{Q}^n, \mathcal{O}(1))$, *where $\mathbb{Q}^n$ is a (possibly singular) hyperquadric in $\mathbb{P}^{n+1}$;*
(iii) *a scroll over $\mathbb{P}^1$;*
(iv) *a generalized cone over a smooth subvariety $W \subset X$ with $\Delta(W, L_W) = 0$.*

For generalized cones we refer to [4, (0.3)] too.

The following characterization of special varieties is also useful.

Proposition 1.2. ([9]; see also [18, (2.1)], [4, (1.3)] and [14, (2.3)].) *Let X be a projective variety of dimension n with only log-terminal singularities and let L be an ample line bundle on X. Then we have the following:*

(i) $K_X + (n+1)L$ *is always nef;*
(ii) *if* $K_X + nL$ *is not nef, then* $(X, L) \cong (\mathbb{P}^n, \mathcal{O}(1))$*;*
(iii) *if* $K_X + nL = \mathcal{O}_X$*, then* $(X, L) \cong (\mathbb{Q}^n, \mathcal{O}(1))$*;*
(iv) *if* $K_X + (n-1)L$ *is not nef, then* $\Delta(X, L) = 0$ *or* (X, L) *is a scroll over a smooth curve.*

We need a relative version of (1.2).

Proposition 1.3. ([1, (2.1)]; see also [3].) *Let X be a projective variety with only log-terminal singularities and let L be an ample line bundle on X. Suppose that there exists a birational contraction morphism $f : X \to W$ of an extremal ray R of X. Let τ be the rational number such that $(K_X + \tau L)R = 0$ and let F be an irreducible component of a non-trivial fiber of f. Then we have the following:*

(i) $\dim F \geq \lfloor \tau \rfloor$ *(the integral part of* τ*);*
(ii) *if* $\dim F \leq \tau + 1$*, then* $\Delta(F, L_F) = 0$.

2. Theorem and its proof.

Theorem 2.1. *Let X be a projective variety of dimension n with only log-terminal singularities and let $\mathcal{E}$ be an ample vector bundle of rank r on X. If $1 < r < n-1$ and $K_X + (n-r)\det\mathcal{E}$ is nef but not ample, then $(X, \mathcal{E})$ is one of the following:*

(i) $(\mathbb{P}^5, \mathcal{O}(1)^{\oplus 2})$*;*
(ii) $(\mathbb{P}^5, \mathcal{O}(1)^{\oplus 3})$*;*
(iii) $(\mathbb{Q}^4, \mathcal{O}(1)^{\oplus 2})$*, where* $\mathbb{Q}^4$ *is a (possibly singular) hyperquadric in* $\mathbb{P}^5$*;*
(iv) $(\mathbb{P}_W(\mathcal{F}), \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1) \otimes \rho^*\mathcal{G})$*, where* $\mathcal{F}$ *and* $\mathcal{G}$ *are vector bundles on a smooth curve* W *of genus* $g(W)$ *such that* $\operatorname{rank}\mathcal{F} = 4$, $\operatorname{rank}\mathcal{G} = 2$, $c_1(\mathcal{F}) + 2c_1(\mathcal{G}) + 2g(W) > 2$*, and* $\rho : \mathbb{P}_W(\mathcal{F}) \to W$ *is the bundle projection.*

Remark 2.1.1. First we note that the above case (iv) really exists. Let W be a smooth curve of positive genus and let $\mathcal{F}$ (resp. $\mathcal{G}$) be a semistable vector bundle of rank 4 (resp. 2) on W with $c_1(\mathcal{F}) + 2c_1(\mathcal{G}) > 0$. We set $X := \mathbb{P}_W(\mathcal{F})$ and $\mathcal{E} := \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1) \otimes \rho^*\mathcal{G}$, where $\rho : \mathbb{P}_W(\mathcal{F}) \to W$ is the bundle projection. Then $\mathcal{E}$ is ample by an argument similar to that in [8, (2.6)]. We see that $K_X + 2$

$\det\mathcal{E} = \rho^*(K_W + \det\mathcal{F} + 2\det\mathcal{G})$ is nef but not ample since $\deg(K_W + \det\mathcal{F} + 2\det\mathcal{G}) > 0$. Thus $(X, \mathcal{E})$ is an example of (2.1; iv) with $g(W) > 0$. We easily see that $(\mathbb{P}^1 \times \mathbb{P}^3, \mathcal{O}(1,1)^{\oplus 2})$ is an example of (2.1; iv) with $g(W) = 0$.

Corollary 2.2. *Let X be a projective variety of dimension n with only log-terminal singularities and let $\mathcal{E}$ be an ample vector bundle of rank r on X. If $1 < r < n-1$ and $K_X + (n-r)\det\mathcal{E} = \mathcal{O}_X$, then $(X, \mathcal{E}) \cong (\mathbb{P}^5, \mathcal{O}(1)^{\oplus 2})$, $(\mathbb{P}^5, \mathcal{O}(1)^{\oplus 3})$ or $(\mathbb{Q}^4, \mathcal{O}(1)^{\oplus 2})$.*

Proof. We see that $(X, \mathcal{E})$ is one of the four cases in (2.1). The case (iv) is excluded since $K_X + (n-r)\det\mathcal{E} = \rho^*(K_W + \det\mathcal{F} + 2\det\mathcal{G})$ and $\deg(K_W + \det\mathcal{F} + 2\det\mathcal{G}) > 0$. □

Proof of Theorem 2.1. Let $(X, \mathcal{E})$ be a pair with the condition of (2.1). Then There exists an extremal ray R of X such that $(K_X + (n-r)\det\mathcal{E})R = 0$. Let $\rho : X \to W$ be the contraction morphism of R. We set $Y := \mathbb{P}_X(\mathcal{E})$, $L := \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ and denote by $p : Y \to X$ the bundle projection. We note that the relative Picard number $\rho(Y/W) = 2$ and $-K_Y = rL - p^*(K_X + \det\mathcal{E})$ is $(\rho \circ p)$-ample. Then there exist an extremal ray R' of Y, which is different from the extremal ray that corresponds to p, and the relative contraction morphism $f : Y \to Z$ of R' over W that makes the following comutative diagram:

$$\begin{array}{ccc} \mathbb{P}_X(\mathcal{E}) = Y & \xrightarrow{f} & Z \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow[\rho]{} & W. \end{array}$$

(2.3) We set $E := \{y \in Y \mid f \text{ is not an isomorphism at } y\}$. With the length estimate in [15, Theorem 1], we can take a rational curve $C \subset E$ which belongs to R' such that

$$-K_Y \cdot C \leq 2(\dim E - \dim f(E));$$

moreover, we have this inequality strict if f is birational. Since $p|_F : F \to X$ is a finite morphism for every fiber F of $f|_E : E \to f(E)$, we see that $\dim E - \dim f(E) \leq n$, hence $-K_Y \cdot C \leq 2n$.

(2.4) On the other hand, since $p^*(K_X + (n-r)\det\mathcal{E}) \cdot C = 0$ and $1 < r < n-1$, we have

$$\begin{aligned} -K_Y \cdot C &= (rL - p^*(K_X + \det\mathcal{E}))C \\ &= r \cdot LC + (n-r-1)(p^*\det\mathcal{E})C \\ &\geq r \cdot LC + (n-r-1)r \\ &= r(LC-1) + (n-r)r \\ &\geq 2(LC-1) + 2(n-2), \end{aligned}$$

hence $LC \leq 3$ by (2.3).

(2.5) Case $LC = 3$.

By (2.3) and (2.4), we get $-K_Y \cdot C = 2n$, $r = 2$, $(p^* \det\mathcal{E})C = 2$ and f is of fiber type. We set $A := L - p^* \det\mathcal{E}$. Then $AC = 1$, hence A is f-ample and A_F is ample for a general fiber F of f. Since $(K_Y + 2nA)C = 0$, we see that $K_F + 2n \cdot A_F = \mathcal{O}_F$, which is impossible because of (1.2).

(2.6) Case $LC = 2$.

By (2.4), we get $-K_Y \cdot C \geq 2n - 2$, hence $(K_Y + (n-1)L)C \leq 0$.

(2.6.1) If f is birational, then $-K_Y \cdot C < 2(\dim E - \dim f(E)) = 2n$ by (2.3), hence $\dim F = n$ for some irreducible component F of a fiber of f. By (1.3), we get $\Delta(F, L_F) = 0$. Then there exists a rational curve $C' \subset F$ such that $L_F \cdot C' = 1$ because of (1.1). By replacing C with C', we may consider this case as the case $LC = 1$.

(2.6.2) If f is of fiber type and $(K_Y + (n-1)L)C < 0$, then $K_F + (n-1)L_F$ is not nef for a general fiber F of f. From (2.3) and (1.2), we see that dim $F = n - 1$ or n, and then $\Delta(F, L_F) = 0$ or (F, L_F) is a scroll over a curve. As in (2.6.1), both cases are considered as the case $LC = 1$.

(2.6.3) If f is of fiber type and $(K_Y + (n-1)L)C = 0$, then $r = (p^* \det\mathcal{E})C = 2$ by (2.4). Since $K_F + (n-1)L_F = \mathcal{O}_F$ for a general fiber F of f, by (1.2), $(F, L_F) \cong (\mathbb{P}^{n-2}, \mathcal{O}(1))$, $(\mathbb{Q}^{n-1}, \mathcal{O}(1))$, or $\dim F = n$ (i.e. f is a Del Pezzo fibration). The former two cases can be considered as the case $LC = 1$ as before. We show that the last case does not occur in the following.

Since $W = \rho(p(F))$ is a point, we have $K_X + (n-2)\det\mathcal{E} = \mathcal{O}_X$. Then a general member X' of $\det|\mathcal{E}|$ is irreducible and reduced with only log-terminal singularities by [22, (2.2)]. We set $Y' := p^{-1}(X')$ and $\mathcal{E}' := \mathcal{E}|_{X'}$. We have $Y' \cong \mathbb{P}_{X'}(\mathcal{E}')$, $L_{Y'} = \mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1)$, $K_{X'} + (n-3)\det\mathcal{E}' = \mathcal{O}_{X'}$ and

$$K_{Y'} + (n-2)L_{Y'} = (n-4)(L_{Y'} - (p|_{Y'})^* \det\mathcal{E}').$$

Since $LC = (p^* \det\mathcal{E})C$, we infer that $f|_{Y'} : Y' \to Z$ is also a Del Pezzo fibration. By repeating this procedure, we may assume that $n = 4$. Then $K_Y + 3L = f^*B$ for some $B \in \text{Pic } Z$. Since $K_Y = -2L - p^* \det\mathcal{E}$, for a fiber $l \cong \mathbb{P}^1$ of p, we have $f|_l{}^*B = L_l = \mathcal{O}_{\mathbb{P}^1}(1)$. We note that $Z \cong \mathbb{P}^1$ since $Z = f(l)$. Then we get $\deg B = F \cdot l = 1$. It follows that $p|_F : F \to X$ is a finite morphism of degree 1, hence $p|_F$ is an isomorphism by Zariski's Main Theorem. Then we see that $K_X + 3A = \mathcal{O}_X$ for some ample line bundle A on X and this is a contradiction since $K_X + 2\det\mathcal{E} = \mathcal{O}_X$.

(2.7) Case $LC = 1$.

By (2.4), we get $-K_Y \cdot C \geq 2n - 4 \geq n$, hence $(K_Y + nL)C \leq 0$.

(2.7.1) If f is birational, let F be an irreducible component of a non-trivial fiber of f. Since $\dim F \leq n$ by (2.3), we get $(K_Y + nL)C \geq 0$ from (1.3), hence $(K_Y+nL)C=0$, dim $F=n=4$ and $r=2$ by (2.4). Then we infer that $F\cap$ Sing $Y=\emptyset$ or dim $(F\cap \text{Sing } Y)=3$ from an argument that is similar to the proof of [1, (3.1)]. In the former case, X is smooth since Sing $Y=p^{-1}$(Sing X) and $p|_F : F\to X$ is surjective. Since $W=\rho(p(F))$ is a point, we see that $K_X+2\det\mathscr{E}=\mathscr{O}_X$ and this case does not occur because of [13, (1.7)]. The latter case does not occur either since codim(Sing X) ≥ 2.

(2.7.2) If f is of fiber type and $(K_Y+nL)C<0$, then K_F+nL_F is not nef for a general fiber F of f. Since dim $F\leq n$, we get $(F, L_F)\cong(\mathbb{P}^n, \mathscr{O}(1))$ by (1.2). Let U be a smooth open subset of Z such that $f^{-1}(z)\cong\mathbb{P}^n$ for every $z\in U$. We set $V:=f^{-1}(U)$; then $f|_V : V\to U$ is a smooth morphism. It follows that V is smooth and X is also smooth. Then we get $(X,\mathscr{E})\cong(\mathbb{P}^5,\mathscr{O}(1)^{\oplus 2})$ or $(\mathbb{P}^5,\mathscr{O}(1)^{\oplus 3})$ by the proof of [13, (1.7)].

(2.7.3) If f is of fiber type and $(K_Y+nL)C=0$, then $n=4$ and $r=2$ by (2.4). Since $K_F+4L_F=\mathscr{O}_F$ for a general fiber F of f, we get $(F, L_F)\cong(\mathbb{P}^3,\mathscr{O}(1))$ or $(\mathbb{Q}^4,\mathscr{O}(1))$ by (1.2). We note that every fiber of f is not contained in Sing $Y=p^{-1}$(Sing X) because codim(Sing X) ≥ 2.

In the case that $(F, L_F)\cong(\mathbb{P}^3,\mathscr{O}(1))$, there are only a finite number of 4-dimensional fibers of f since dim $Y=5$. By taking a general slicing of $f: Y\to Z$ as in [2, (2.4)], we get a surjective morphism $f': Y'\to Z'$ such that Z' is a smooth curve and a general fiber of f' is $\mathbb{P}^3$. Then $(Y', L_{Y'})\cong(\mathbb{P}_{Z'}(\mathscr{F}'),\mathscr{O}_{\mathbb{P}(\mathscr{F}')}(1))$ for some vector bundle $\mathscr{F}'$ on Z' by the proof of [4, (1.4)]. It follows that Y' is smooth. Since Y' is a Cartier divisor on Y, we see that Y is smooth along Y', hence X is smooth. From [25, Prop. 1.1'] we infer that dim $W=1$ and $(X,\mathscr{E})\cong(\mathbb{P}_W(\mathscr{F}),\mathscr{O}_{\mathbb{P}(\mathscr{F})}(1)\otimes\rho^*\mathscr{G})$ for some vector bundles $\mathscr{F}$ and $\mathscr{G}$ on W. Since $K_X+2\det\mathscr{E}=\rho^*(K_W+\det\mathscr{F}+2\det\mathscr{G})$ is nef, we have

$$0\leq(K_X+2\det\mathscr{E})^4=2g(W)-2+c_1(\mathscr{F})+2c_1(\mathscr{G}).$$

We also have $0<c_2(\mathscr{E})^2=c_1(\mathscr{F})+2c_1(\mathscr{G})$. Then we find that $c_1(\mathscr{F})+2c_1(\mathscr{G})+2g(W)>2$ by an argument in the proof of [13, (1.7)], thus this is the case (iv) of (2.1).

In the case that $(F, L_F)\cong(\mathbb{Q}^4,\mathscr{O}(1))$, we have dim $Z=1$ and $K_X+2\det\mathscr{E}=\mathscr{O}_X$ since W is a point. We have $K_Y+4L=f^*B$ for some $B\in$ Pic Z. Since $K_Y=-2L-p^*\det\mathscr{E}$, for a fiber $l\cong\mathbb{P}^1$ of p, we get $f|_l{}^*B=2L_l=\mathscr{O}_{\mathbb{P}^1}(2)$. Then we see that $Z\cong\mathbb{P}^1$, deg $B\leq 2$ and $F\cdot l\leq 2$. If $F\cdot l=2$, then deg $B=1$ and $K_Y+4L=\mathscr{O}_Y(F)$. From an exact sequence $0\to\mathscr{O}_Y(-F)\to\mathscr{O}_Y\to\mathscr{O}_F\to 0$, we get the following exact sequence

$$0\to p_*\mathscr{O}_Y(-F)\to p_*\mathscr{O}_Y\to p_*\mathscr{O}_F\to R^1p_*\mathscr{O}_Y(-F)\to R^1p_*\mathscr{O}_Y.$$

Then we have $p_*\mathscr{O}_Y=\mathscr{O}_X$, $R^1p_*\mathscr{O}_Y=0$, $p_*\mathscr{O}_Y(-F)=p_*(p^*\det\mathscr{E}-2L)=0$, and

$$R^1p_*\mathscr{O}_Y(-F)=R^1p_*\mathscr{O}_Y(K_Y-p^*K_X)\cong\mathscr{O}_X$$

since $p : Y = \mathbb{P}_X(\mathcal{E}) \to X$ is a projective bundle. Hence we get an exact sequence $0 \to \mathcal{O}_X \to p_*\mathcal{O}_F \to \mathcal{O}_X \to 0$. It follows that $\chi(\mathcal{O}_F) = \chi(p_*\mathcal{O}_F) = 2\chi(\mathcal{O}_X)$, which is a contradiction. Thus $F \cdot l = 1$ and $p|_F : F \to X$ is a finite morphism of degree 1. By Zariski's Main Theorem, $p|_F$ is an isomorphism and $\det\mathcal{E} = \mathcal{O}_{\mathbb{Q}}(2)$. We see that $\mathcal{E}$ is split because general fibers F of f are disjoint sections of p. Since $\mathcal{E}$ is ample, we get $\mathcal{E} \cong \mathcal{O}_{\mathbb{Q}}(1)^{\oplus 2}$ and $(X, \mathcal{E}) \cong (\mathbb{Q}^4, \mathcal{O}(1)^{\oplus 2})$. This completes the proof of (2.1). □

3. Application to c_r-sectional genus.

Definition 3.1. ([14, (3.1)]) Let X be a normal projective variety of dimension n and let $\mathcal{E}$ be an ample vector bundle of rank $r < n$ on X. The *c_r-sectional genus* $g(X, \mathcal{E})$ of a pair $(X, \mathcal{E})$ is defined by the formula

$$2g(X, \mathcal{E}) - 2 := (K_X + (n - r)c_1(\mathcal{E}))c_1(\mathcal{E})^{n-r-1}c_r(\mathcal{E}),$$

where K_X is a canonical divisor of X.

When $r = 1$, $g(X, \mathcal{E})$ is equal to the *sectional genus* of a polarized variety $(X, \mathcal{E})$. We refer to [5] and [4] for the general property of sectional genus. When $r = n-1$, $g(X, \mathcal{E})$ is equal to the *curve genus* of a generalized polarized variety $(X, \mathcal{E})$. We refer to [17], [16] and [20] for the property of curve genus. When $1 < r < n - 1$, basic properties of $g(X, \mathcal{E})$ have been studied in [13] and [12] in the case that X is smooth. For singular X, we have the following.

Proposition 3.2. ([14, (3.2) and (3.5)]) *Let $(X, \mathcal{E})$ be as in* (3.1). *Then $g(X, \mathcal{E})$ is an integer and non-negative when X has only log-terminal singularities; moreover, if $g(X, \mathcal{E}) = 0$ and $1 < r < n - 1$, then $(X, \mathcal{E}) \cong (\mathbb{P}^4, \mathcal{O}(1)^{\oplus 2})$.*

Remark 3.2.1. We also have classification results for $(X, \mathcal{E})$ with $g(X, \mathcal{E}) = 0$ and either $r = 1$ or $r = n - 1$ (see, e.g., [14, (3.3) and (3.4)]).

As an application of (2.2), we obtain the following.

Theorem 3.3. *Let $(X, \mathcal{E})$ be as in* (3.1). *Suppose that X has only log-terminal singularities, $g(X, \mathcal{E}) = 1$ and $1 < r < n - 1$. Then $(X, \mathcal{E}) \cong (\mathbb{P}^5, \mathcal{O}(1)^{\oplus 2})$, $(\mathbb{P}^5, \mathcal{O}(1)^{\oplus 3})$ or $(\mathbb{Q}^4, \mathcal{O}(1)^{\oplus 2})$.*

Proof. From the assumption, we see that $(K_X+(n-r)c_1(\mathcal{E}))c_1(\mathcal{E})^{n-r-1}c_r(\mathcal{E}) = 0$. Since $\mathcal{E}$ is ample, we get $K_X+(n-r)\det\mathcal{E} = \mathcal{O}_X$ and the conclusion follows by (2.2). □

Remark 3.3.1. When $g(X, \mathcal{E}) = 1$ and $r = 1$, except scrolls over an elliptic curve, $(X, \mathcal{E})$ is a Del Pezzo variety and the classification has been obtained by [7] and [10]. When $g(X, \mathcal{E}) = 1$ and $r = n - 1$, the classification of $(X, \mathcal{E})$ is given by [20] (see also [23]) in the case that X is smooth; for singular X, however, the classification is yet to be studied.

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