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ON CHARACTERIZATIONS OF DUNKL-SEMICLASSICAL ORTHOGONAL POLYNOMIALS

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In this paper, the Dunkl-semiclassical orthogonal polynomials will be studied as a generalization of the Dunkl-classical ones. We obtain some characterizations for such polynomials. Moreover, an example of non-symmetric Dunkl-semiclassical orthogonal polynomials is given.

1. Introduction and preliminaries

The family of classical orthogonal polynomial constitutes the most important families of orthogonal polynomials which motives several authors to determine the characterizations of this kind of polynomials with respect to different operators.

Recently, the theory of classical orthogonal polynomial has been extended to the Dunkl operator [10]. Y.Ben Cheikh and M.Gaid [2] were the first to investigate characterization results and the classification of the Dunkl-classical symmetric orthogonal polynomials while they showed that the unique Dunkl-classical symmetric polynomials are the generalized Hermite polynomials and the generalized Gegenbauer polynomials. Short time ago , several authors ,see [3], [17] and among others, gave some characterization to the Dunkl-classical

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symmetric form and next, in 2014, Bouras and al. generalized this characterization to the non symmetric case and they also gave some new characterizations ,refer to [1] and [6], and in 2017, B.Bouras and Y.Habbachi [5] showed that there exists a unique non symmetric Dunkl-classical orthogonal polynomials.

A generalization of the family of classical orthogonal polynomials leads to semi-classical orthogonal polynomials. In fact, classical orthogonal polynomials are semi-classical of class zero (see [12], [13], [15], [16]). The aim of this paper is to generalize the results obtained by Bouras and al [1], [6] to the Dunkl-semi classical form where the symmetric case is treated by Sghaier [18] in 2017.

This paper consists of three sections. In Section 1, we deal with the general features and ingredients necessary for the sequel. In section 2, we establish four characterizations of Dunkl semi classical orthogonal polynomials. We characterize these sequences by a Dunkl-distributional equation of Pearson type determined by its associated form. We show also that Dunkl-semiclassical polynomial sequences can be characterized by a linear second order differential-difference equation. The third characterization is a first order linear difference equation with polynomial coefficients satisfied by the corresponding Stieltjes function and the last one is the so-called structure relation that the Dunkl-semiclassical polynomial sequences satisfy. Lastly, in section 3, we construct an example of Dunkl non symmetric semi-classical form of order one, which proved that the set of this last type of polynomials is not empty.

We state now some preliminary results needed for the sequel. Let $\mathcal{P}$ be the linear space of complex polynomials and let $\mathcal{P}'$ be its algebraic dual space. We denote by $\langle u, f\rangle$ the duality bracket for $u \in \mathcal{P}'$ and $f \in \mathcal{P}$. In particular, we denote by $(u)_n := \langle u, x^n\rangle$, $n \geq 0$, the moments of u.

Let us recall the definitions of some useful operations on $\mathcal{P}'$.

Definition 1.1. Let $u \in \mathcal{P}'$, $a \in \mathbb{C}^*$, $c \in \mathbb{C}$ and $g \in \mathcal{P}$. We define:

• the homothetic $h_a u$ of the form (linear functional) u

$$\langle h_a u, f\rangle = \langle u, h_a f\rangle = \langle u, f(ax)\rangle, \quad f \in \mathcal{P};$$

• the left multiplication of the form u by the polynomial g, denoted by gu, such that

$$\langle gu, f\rangle = \langle u, gf\rangle, \quad f \in \mathcal{P};$$

• the derivative of the form u, denoted by Du, such that

$$\langle Du, f\rangle = -\langle u, f'\rangle, \quad f \in \mathcal{P};$$

• the Dirac mass at the point c, denoted by δ_c, which is the element of $\mathcal{P}'$ such that

$$\langle \delta_c, f\rangle = f(c), \quad f \in \mathcal{P};$$

• and the division of the form u by $(x-c)$, denoted by $(x-c)^{-1}u$:

$$\big\langle (x-c)^{-1}u, f\big\rangle = \big\langle u, \theta_c f\big\rangle,$$

where

$$\big(\theta_c f\big)(x) = (f(x)-f(c))/(x-c), \quad f \in \mathcal{P}.$$

Then, it is straightforward to prove that for $a \in \mathbb{C}^*$, $c \in \mathbb{C}$, $f \in \mathcal{P}$ and $u \in \mathcal{P}'$, we have (see [6],[5])

$$h_{-1}(fu) = h_{-1}fh_{-1}u, \quad h_{-1}(h_{-1}(u)) = u. \tag{1}$$

$$f(h_a u) = h_a\big((h_a f)u\big). \tag{2}$$

$$h_{-1}(h_a u) = h_a(h_{-1}u). \tag{3}$$

$$h_{-1}\delta_0 = \delta_0, \quad x^{-1}\delta_0 = -\delta_0'. \tag{4}$$

$$(x-c)^{-1}(x-c)u = u - (u)_0\delta_0, \;\; u \in \mathcal{P}', \;\; c \in \mathbb{C}. \tag{5}$$

$$x^{-1}(x^{-1}(x^2u)) = u - (u)_0\delta_0 + (u)_1\delta_0', \;\; u \in \mathcal{P}'. \tag{6}$$

.

For $f \in \mathcal{P}$ and $u \in \mathcal{P}'$, the product uf is the polynomial

$$(uf)(x) = \big\langle u, \frac{xf(x)-\zeta f(\zeta)}{x-\zeta}\big\rangle.$$

The Stieltjes function of $u \in \mathcal{P}'$ is defined by

$$S(u)(z) = -\sum_{n\geq 0}\frac{(u)_n}{z^{n+1}}. \tag{7}$$

We have the following formulas [6]

$$S(\delta_0')(z) = \frac{1}{z^2}. \tag{8}$$

$$S(fu)(z) = f(z)S(u)(z) + (u\theta_0 f)(z), \;\; f \in \mathcal{P}. \tag{9}$$

A form u is called regular if there exists a sequence of polynomials $\{P_n\}_{n\geq 0}$ ($\deg P_n \leq n$) such that [8]

$$\big\langle u, P_nP_m\big\rangle = p_n\delta_{nm}, \quad p_n \neq 0, \quad n \geq 0\ ,$$

where δ_{nm} is the Kronecker symbol [11]. Then, $\deg P_n = n$, $n \geq 0$ and we can always suppose each P_n is monic. In such a case, the sequence $\{P_n\}_{n\geq 0}$ is unique.

It is said to be the sequence of monic orthogonal polynomials (MOPS) with respect to u. In the sequel, we take all regular forms u normalized i.e. $(u)_0 = 1$.

There exist a complex sequence $\{\beta_n\}_{n\geq 0}$ and a non zero complex sequence $\{\gamma_n\}_{n\geq 1}$ such that the MOPS $\{P_n\}_{n\geq 0}$ fulfils the following three-term recurrence relation [8]

$$\begin{aligned} &P_0(x) = 1, \quad P_1(x) = x - \beta_0, \\ &P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x),\ n \geq 0. \end{aligned} \tag{10}$$

If we replace β_0 by $\beta_0 + \lambda$ in (10) then we obtain a new MOPS denoted $\{P_n(.;-\lambda)\}_{n\geq 0}$ and is called co-recursive sequence of $\{P_n\}_{n\geq 0}$.

A form u is said symmetric if and only if $(u)_{2n+1} = 0$, $n \geq 0$, or, equivalently, in (10) $\beta_n = 0$, $n \geq 0$.

Let $\{P_n\}_{n\geq 0}$ be a MOPS with respect to the form u and let $\{u_n\}_{n\geq 0}$ be its dual sequence, $u_n \in \mathcal{P}'$ defined by [11]

$$\langle u_n, P_m\rangle := \delta_{n,m},\ n,\ m \geq 0.$$

Then, for any $v \in \mathcal{P}'$ satisfies $\langle v, \mathrm{P}_i\rangle = 0$ for $i \geq l$, $l \geq 1$, we have

$$v = \sum_{i=0}^{l-1} \lambda_i u_i,\ where\ \lambda_i = \langle v, P_i\rangle,\ i = 0,1,2,\ldots. \tag{11}$$

In particular, $u = u_0$. Furthermore,

$$u_i = \frac{P_i}{\langle u, P_i^2\rangle} u,\ i \geq 0.$$

For any complex number μ, the Dunkl difference operator $\mathcal{T}_\mu$ is defined by [10]

$$(\mathcal{T}_\mu f)(x) = f'(x) + 2\mu(H_{-1}f)(x),\ \ (H_{-1}f)(x) = \frac{f(x) - f(-x)}{2x},\ \ f \in \mathcal{P}.$$

Note that $\mathcal{T}_0$ is reduced to the derivative operator D. Thus, henceforth, we will assume that $\mu \neq 0$. The transposed $t_{\mathcal{T}_\mu}$ of $\mathcal{T}_\mu$ is $t_{\mathcal{T}_\mu} = -\mathcal{T}_\mu$. Thus we have

$$\langle \mathcal{T}_\mu u, f\rangle = -\langle u, \mathcal{T}_\mu f\rangle,\ \ u \in \mathcal{P}',\ \ f \in \mathcal{P}.$$

In particular, this yields $\langle \mathcal{T}_\mu u, x^n\rangle = -\mu_n\langle u, x^{n-1}\rangle = -\mu_n(u)_{n-1},\ n \geq 0$, where $(u)_{-1} = 0$ and $\mu_n = n + \mu(1 - (-1)^n),\ \ n \geq 0$.
One can see easily that

$$\mathcal{T}_\mu u = Du + 2\mu H_{-1}u,$$

where

$$\langle H_{-1}u, f\rangle = -\langle u, H_{-1}f\rangle.$$

Furthermore, we have the following formulas [6]

$$\mathcal{T}_\mu \delta_0 = (1+2\mu)\delta_0'. \tag{12}$$

$$\mathcal{T}_\mu(h_a u) = a^{-1}h_a(\mathcal{T}_\mu u), \;\; u \in \mathcal{P}', \; a \in \mathbb{C}\backslash\{0\}. \tag{13}$$

$$\mathcal{T}_\mu(fg) = f(x)(\mathcal{T}_\mu g)(x) + g(x)(\mathcal{T}_\mu f)(x) - 4\mu x(H_{-1}f)(x)(H_{-1}g)(x), \;\; f,g \in \mathcal{P}. \tag{14}$$

$$\mathcal{T}_\mu(fu) = f\mathcal{T}_\mu u + \mathcal{T}_\mu f.u + 2\mu H_{-1}f(h_{-1}u - u). \tag{15}$$

$$\mathcal{T}_\mu(fu) = f\mathcal{T}_\mu u + f'u + 2\mu H_{-1}fh_{-1}u. \tag{16}$$

$$S(\mathcal{T}_\mu u)(z) = \mathcal{T}_{-\mu}S(u)(z). \tag{17}$$

Let now a MOPS $\{\mathrm{P}_n\}_{n\geq 0}$ and let

$$\mathrm{P}_n^{[1]}(x,\mu) = \frac{\mathcal{T}_\mu \mathrm{P}_{n+1}(x)}{\mu_{n+1}}, \;\; \mu \neq -n - \frac{1}{2}, \; n \geq 0. \tag{18}$$

Let us introduce the definition of $\mathcal{T}_\mu$-classical form.

Definition 1.2. An MOPS $\{\mathrm{P}_n\}_{n\geq 0}$ is called $\mathcal{T}_\mu$-classical if $\{\mathrm{P}_n^{[1]}(.,\mu)\}_{n\geq 0}$ is also an MOPS. In this case, the form u associated to $\{\mathrm{P}_n\}_{n\geq 0}$ is called $\mathcal{T}_\mu$-classical form.

Example 1.3. Let us denote two examples of $\mathcal{T}_\mu$-classical polynomials.

1. Symmetric case [2]: The generalized Gegenbauer polynomails.

   The explicit expression of generalized Gegenbauer polynomials is given by

$$\mathrm{S}_n^{(\alpha,\mu-\frac{1}{2})}(x) = (-1)^n \frac{\gamma_\mu(n)\Gamma(\alpha+\mu+\frac{1}{2})}{2^n\Gamma(n+\alpha+\mu+\frac{1}{2})} \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k\Gamma(n-k+\alpha+\mu+\frac{1}{2})}{k!\gamma_\mu(n-2k)\Gamma(\alpha+\mu+\frac{1}{2})}(2x)^{n-2k},$$

   $n \geq 0$. where γ_μ is defined by

$$\gamma_\mu(2n) = \frac{2^{2n}n!\Gamma(n+\mu+\frac{1}{2})}{\Gamma(\mu+\frac{1}{2})} \quad \textit{and} \quad \gamma_\mu(2n+1) = \frac{2^{2n+1}n!\Gamma(n+\mu+\frac{3}{2})}{\Gamma(\mu+\frac{1}{2})}.$$

   The set $\{\mathrm{S}_n^{(\alpha,\mu-\frac{1}{2})}\}_{n\geq 0}$ is an MOPS with respect to the regular form $\mathcal{G}^{(\alpha,\mu-\frac{1}{2})}$ defined as (see[8] [9])

$$\mathcal{G}^{(\alpha,\mu-\frac{1}{2})} = |x|^{2\mu}(1-x^2)^\alpha, \quad -1 < x < 1.$$

This form is $\mathcal{T}_\mu$-classical and satisfies

$$\mathcal{T}_\mu((x^2-1)\mathcal{G}^{(\alpha,\mu-\frac{1}{2})}) = 2(\alpha+1)x\mathcal{G}^{(\alpha,\mu-\frac{1}{2})}.$$

If we apply $\mathcal{T}_\mu$ to $\mathrm{S}_n^{(\alpha,\mu-\frac{1}{2})}$, we obtain

$$\mathcal{T}_\mu(\mathrm{S}_n^{(\alpha,\mu-\frac{1}{2})}) = \mu_n \mathrm{S}_{n-1}^{(\alpha+1,\mu-\frac{1}{2})}. \tag{19}$$

The MOPS $\{\mathrm{S}_n^{(\alpha,\mu-\frac{1}{2})}\}_{n\geq 0}$ satisfies the three-term recurrence relation (10) with [4]

$$\begin{aligned} \hat{\beta}_n = 0, \quad \hat{\gamma}_{n+1} &= \tfrac{(n+1+\varepsilon_n)(n+1+2\alpha+\varepsilon_n)}{4(n+\alpha+\mu+\frac{1}{2})(n+\alpha+\mu+\frac{3}{2})}, \;\; n\geq 0, \\ \varepsilon_n &= 2\mu\tfrac{1+(-1)^n}{2}, \;\; n\geq 0. \end{aligned} \tag{20}$$

2. Non-symmetric case: The perturbed generalized Gegenbauer form.

   Bouras and al. [7] showed that the unique non-symmetric $\mathcal{T}_\mu$-classical form is the perturbed generalized Gegenbauer form. As an example, take [6]

$$v = \lambda(x-1)^{-1}\mathcal{G}^{(\alpha,\mu-\frac{1}{2})} + \delta_1, \tag{21}$$

where $\alpha \neq \frac{2\mu-1}{2}$, $\mu \neq \frac{1}{2}$ and $\lambda = -\frac{2(2\mu-\alpha)}{2\mu-2\alpha-1}$.
This form satisfies

$$\mathcal{T}_\mu\Big((x^2-1)v\Big) - \frac{1+2\mu}{\lambda+1}(x-\lambda-1)v = 0. \tag{22}$$

The MOPS corresponding to v , which we denote by $\{\widetilde{P}_n\}_{n\geq 0}$, satisfies the three-term recurrence relation

$$\begin{aligned} \widetilde{P}_0(x) = 1, \quad \widetilde{P}_1(x) = x - \beta_0, \\ \widetilde{P}_{n+2}(x) = (x-\beta^v_{n+1})\widetilde{P}_{n+1}(x) - \gamma^v_{n+1}\widetilde{P}_n(x), \; n\geq 0, \end{aligned} \tag{23}$$

with

$$\beta^v_0 = 1+\lambda, \;\; \beta^v_{n+1} = 1 + a^v_n + \frac{\hat{\gamma}_{n+1}}{a^v_n}, \;\; \gamma^v_{n+1} = -a^v_n(1+a^v_n), \;\; n\geq 0,$$

where $\hat{\gamma}_{n+1}$ is given in (20) and a^v_n is given by Maroni [14]

$$a^v_n = -\frac{S_{n+1}^{(\alpha,\mu-\frac{1}{2})}(1;-\lambda)}{S_n^{(\alpha,\mu-\frac{1}{2})}(1;-\lambda)}, \;\; n\geq 0.$$

Lemma 1.4. *[6] If $\{P_n\}_{n\geq 0}$ is $\mathcal{T}_\mu$-classical MOPS and $2|\mu| \neq 1$, then $\{P_n^{[1]}(.,\mu)\}_{n\geq 0}$ is orthogonal with respect to $\frac{K}{1-4\mu^2}(\Phi u - 2\mu h_{-1}(\Phi u))$ where K is a complex number and Φ is a non-zero monic polynomial,* $\deg(\Phi) \leq 2$.

We state now the definition of the quasi-orthogonality.

Definition 1.5. Let $u \in \mathcal{P}'$ and s a non negative integer. A MPS $\{P_n\}_{n\geq 0}$ is said to be quasi-orthogonal of order s with respect to u if

$$\begin{aligned} \langle u, P_m P_n \rangle = 0,\ \ 0 \leq m \leq n-s-1,\ \ n \geq s+1, \\ \exists\, r \geq s;\ \langle u, P_{r-s} P_r \rangle \neq 0. \end{aligned} \tag{24}$$

If $\langle u, P_{r-s} P_r \rangle \neq 0$ for any $r \geq s$, then $\{P_n\}_{n\geq 0}$ is said to be strictly quasi-orthogonal of order s with respect to u.

Remark 1.6. 1. (24) is equivalents to

$$\begin{aligned} \langle u, x^m P_n \rangle = 0,\ \ 0 \leq m \leq n-s-1,\ \ n \geq s+1, \\ \exists\, r \geq s;\ \langle u, x_{r-s} P_r \rangle \neq 0. \end{aligned} \tag{25}$$

2. A strictly quasi-orthogonal of order zero is orthogonal.

A Dunkl-semiclassical orthogonal polynomials defined as

Definition 1.7. Let $\{P_n\}_{n\geq 0}$ be a MOPS with respect to $u \in \mathcal{P}'$. $\{P_n\}_{n\geq 0}$ is Dunkl-semiclassical if there exists a non negative integer s such that the MPS $\{P_n^{[1]}(.,\mu)\}_{n\geq 0}$ is quasi-orthogonal of order s. In this case, the form u associated to $\{P_n\}_{n\geq 0}$ is called Dunkl-semiclassical form.

Example 1.8. An example of symmetric $\mathcal{T}_\mu$-semi classical form denoted by Sghaier [18] as:

$$w = -\frac{2\lambda}{2\alpha+1}\mathcal{H}(\alpha) + (1 + \frac{2\lambda}{2\alpha+1})\delta_0,\ \ \lambda \neq \{0, \frac{-2\alpha+1}{2}\},$$

where $\mathcal{H}(\alpha)$ is the generalized Hermite form [2]. The form w is symmetric $\mathcal{T}_\mu$-semiclassical and satisfies

$$\mathcal{T}_\mu(x^2 w) + (2x^2 - 2(\alpha - \mu + 1)x)w = 0.$$

2. Some characterizations of Dunkl-semiclassical polynomials

In this section, we state some results of Dunkl-semiclassical orthogonal polynomials that are extensions of the characterizations of Dunkl classical orthogonal polynomials which have been stated by Bouras and al (see [6],[1]). In the sequel of the text, we assume that $2|\mu| \neq 1$.

We begin by the main result of this paper which give a Dunkl-distributional equation of Pearson type determined by any regular semi-classical form and a linear second order differential-difference equation of its associated Dunkl-semiclassical orthogonal polynomial sequences.

Theorem 2.1. *Let $\{\mathrm{P}_n\}_{n\geq 0}$ be a MOPS with respect to a regular form u. The following statements are equivalent*

(1) The sequence $\{\mathrm{P}_n\}_{n\geq 0}$ is Dunkl-semiclassical.

(2) There exists a non negative integers s, p, an integer $r \geq s$, three polynomials Φ (monic), $\tilde{\Phi}$, $\deg\Phi = \deg\tilde{\Phi} \leq s+2$ and Ψ, $1 \leq \deg\Psi \leq s+1$, a complex number K and a sequence of non zero complex number $\{\lambda_{n,i}\}_{n\geq s, i\geq 1}$ such that

$$x\Phi(x)u = h_{-1}(x\tilde{\Phi}(x)u) \tag{26}$$

and

$$\frac{K}{1-4\mu^2}(\Phi(x)+2\mu\tilde{\Phi}(x))\mathcal{T}_\mu^2(\mathrm{P}_{n-s+1}(x)) - \Psi(x)\mathcal{T}_\mu(\mathrm{P}_{n-s+1}(x))$$
$$-\frac{2\mu K}{1-2\mu}(\Phi(x)+\tilde{\Phi}(x))H_{-1}(\mathcal{T}_\mu(\mathrm{P}_{n-s+1}(x))) = -\mu_{n-s+1}\sum_{i=1}^{p}\lambda_{n,i}\mathrm{P}_i, \tag{27}$$

with condition

$$\frac{\Psi^{(s+1)}(0)}{(s+1)!} + \frac{K}{1-4\mu^2}\frac{\Phi^{(s+2)}(0)}{(s+2)!}(4\mu^2[r-s]-(r-s))+$$
$$\frac{2\mu K}{1-4\mu^2}\frac{\tilde{\Phi}^{(s+2)}(0)}{(s+2)!}\big([r-s]-(r-s)\big) \neq 0, \tag{28}$$

where

$$[r-s] = \frac{1-(-1)^{r-s}}{2}.$$

(3) There exist non negative integers s, p, an integer $r \geq s$, polynomials Φ (monic), B and Ψ with $\deg(\Phi) \leq s+2$, $\deg(B) = \deg(\Phi)+1$ *and*

$\deg(\Psi) = p$, $1 \le p \le s+1$, *fulfilling*

$$\begin{aligned}&\frac{\Psi^{(s+1)}(0)}{(s+1)!} + \frac{K}{1-4\mu^2}\frac{\Phi^{(s+2)}(0)}{(s+2)!}(4\mu^2[r-s]-(r-s))+ \\ &\frac{2\mu K}{1-4\mu^2}\frac{B^{(s+3)}(0)}{(s+3)!}\big([r-s]-(r-s)\big) \neq 0,\end{aligned} \tag{29}$$

such that the regular form u associated to $\{\mathrm{P}_n\}_{n\ge 0}$ *satisfies*

$$\mathcal{T}_\mu\big(\Phi u - 2\mu h_{-1}(\Phi u)\big) + \frac{1-4\mu^2}{K}\Psi u = 0 \tag{30}$$

$$x\Phi(x)u = h_{-1}(B(x)u). \tag{31}$$

(4) *There exist a complex number K and three polynomials* Φ *(monic),* $\tilde{\Phi}$ *and* Ψ *with* $\deg(\Phi) = \deg(\tilde{\Phi}) \le s+2$ *and* $\deg(\Psi) = p$, $1 \le p \le s+1$, *fulfilling* (26) *and* (28) *such that the associated regular form u satisfies*

$$\mathcal{T}_\mu\big((\Phi + 2\mu\tilde{\Phi})u\big) - 2\mu(1+2\mu)\big\langle u, \Phi + \tilde{\Phi}\big\rangle \delta_0' + \frac{1-4\mu^2}{K}\Psi u = 0. \tag{32}$$

Proof. ► (1) ⇒ (2). By hypothesis, there exists a non negative integer s such that the MPS $\{\mathrm{P}_n^{[1]}(.,\mu)\}_{n\ge 0}$ is quasi-orthogonal of order s. Denote by $w(\mu)$ the form associated to $\{\mathrm{P}_n^{[1]}(.,\mu)\}_{n\ge 0}$.
For all $n \ge s$, we have

$$\big\langle \mathcal{T}_\mu\big(\mathrm{P}_{n-s}^{[1]}(.,\mu)w(\mu)\big), \mathrm{P}_{m+1}\big\rangle = -\mu_{m+1}\big\langle w(\mu), \mathrm{P}_{n-s}^{[1]}\mathrm{P}_m^{[1]}\big\rangle,\ m \ge 0.$$

According of (24),

$$\big\langle w(\mu), \mathrm{P}_{n-s}^{[1]}(.,\mu)\mathrm{P}_m^{[1]}(.,\mu)\big\rangle = 0,\ \ m \ge n+1.$$

Since $w(\mu)$ non null form (because of the existence of $r \ge s$ such that $\big\langle w(\mu), \mathrm{P}_{r-s}^{[1]}(.,\mu)\mathrm{P}_r^{[1]}(.,\mu)\big\rangle \neq 0$), there exists an integer p, $1 \le p \le n+1$, such that

$$\big\langle w(\mu), \mathrm{P}_{n-s}^{[1]}(.,\mu)\mathrm{P}_{p-1}^{[1]}(.,\mu)\big\rangle \neq 0,\ \ \big\langle w(\mu), \mathrm{P}_{n-s}^{[1]}(.,\mu)\mathrm{P}_m^{[1]}(.,\mu)\big\rangle = 0,\ \ \forall\, m \ge p.$$

Therefore,

$$\big\langle \mathcal{T}_\mu\big(\mathrm{P}_{n-s}^{[1]}(.,\mu)(.,\mu)w(\mu)\big), \mathrm{P}_p\big\rangle \neq 0,\ \big\langle \mathcal{T}_\mu\big(\mathrm{P}_{n-s}^{[1]}(.,\mu)w(\mu)\big), \mathrm{P}_m\big\rangle = 0,\ m \ge p+1.$$

So, the orthogonality of $\{\mathrm{P}_n\}_{n\geq 0}$ leads to

$$\mathcal{T}_\mu\big(\mathrm{P}^{[1]}_{n-s}(.,\mu)w(\mu)\big) = -\sum_{i=1}^{p}\mu_i\frac{\big\langle w(\mu),\mathrm{P}^{[1]}_{n-s}(.,\mu)\mathrm{P}^{[1]}_{i-1}(.,\mu)\big\rangle}{\big\langle u,\mathrm{P}_i^2\big\rangle}\mathrm{P}_i u.$$

Put $\lambda_{n,i} = \mu_i\frac{\big\langle w(\mu),\mathrm{P}^{[1]}_{n-s}(.,\mu)\mathrm{P}^{[1]}_{i-1}(.,\mu)\big\rangle}{\big\langle u,\mathrm{P}_i^2\big\rangle}$. Then,

$$\begin{aligned}&\mathrm{P}^{[1]}_{n-s}(.,\mu)\mathcal{T}_\mu w(\mu)+\mathcal{T}_\mu(\mathrm{P}^{[1]}_{n-s}(.,\mu))w(\mu)\\&+2\mu H_{-1}\mathrm{P}^{[1]}_{n-s}(.,\mu)\big(h_{-1}w(\mu)-w(\mu)\big) = -\sum_{i=1}^{p}\lambda_{n,i}\mathrm{P}_i u,\quad n\geq s.\end{aligned}\tag{33}$$

For $n=s$ and $n=s+1$, the equation (33) becomes

$$\mathcal{T}_\mu w(\mu) = -\sum_{i=1}^{p}\lambda_{s,i}\mathrm{P}_i u.\tag{34}$$

$$\mathrm{P}^{[1]}_1(.,\mu)\mathcal{T}_\mu w(\mu)+w(\mu)+2\mu h_{-1}w(\mu) = -\sum_{i=1}^{p}\lambda_{s+1,i}\mathrm{P}_i u.\tag{35}$$

Substitution of (34) into (35) gives

$$w(\mu)+2\mu h_{-1}w(\mu) = K\Phi u,\tag{36}$$

where

$$K\Phi = \mathrm{P}^{[1]}_1(.,\mu)\sum_{i=1}^{p}\lambda_{s,i}\mathrm{P}_i-\sum_{i=1}^{p}\lambda_{s+1,i}\mathrm{P}_i.\tag{37}$$

K is a normalization factor. Applying h_{-1} to (37), we get

$$h_{-1}(w(\mu))+2\mu w(\mu) = h_{-1}(K\Phi u).\tag{38}$$

Elimination of $h_{-1}(w(\mu))$ between (36) and (38) gives

$$w(\mu) = \frac{K}{1-4\mu^2}(\Phi u-2\mu h_{-1}(\Phi u)).\tag{39}$$

Applying $\mathcal{T}_\mu$ to the last equation and according to (34) we get

$$\mathcal{T}_\mu(\Phi u-2\mu h_{-1}(\Phi u))+\frac{1-4\mu^2}{K}\Psi u = 0,$$

where

$$\Psi = \sum_{i=1}^{p}\lambda_{s,i}\mathrm{P}_i.\tag{40}$$

Now, take $n=s+2$ in (33), we get

$$\mathrm{P}_2^{[1]}(.,\mu)\mathcal{T}_\mu w(\mu)+\mathcal{T}_\mu \mathrm{P}_2^{[1]}(.,\mu)w(\mu)+2\mu H_{-1}\mathrm{P}_2^{[1]}(.,\mu)(h_{-1}w(\mu)-w(\mu)) = -\sum_{i=1}^{p}\lambda_{s+2,i}\mathrm{P}_i u.$$

Taking into account (34) and (39), we get

$$-\frac{2\mu K}{1-4\mu^2}(\mathcal{T}_\mu(\mathrm{P}_2^{[1]}(.,\mu))-(1+2\mu)H_{-1}\mathrm{P}_2^{[1]}(.,\mu))h_{-1}(\Phi u) = (\mathrm{P}_2^{[1]}(.,\mu)\sum_{i=1}^{p}\lambda_{s,i}\mathrm{P}_i - \frac{K}{1-4\mu^2}\Phi\mathcal{T}_\mu(\mathrm{P}_2^{[1]}(.,\mu))+\frac{2\mu K}{1-2\mu}\Phi H_{-1}\mathrm{P}_2^{[1]}(.,\mu)-\sum_{i=1}^{p}\lambda_{s+2,i}\mathrm{P}_i)u. \tag{41}$$

Since $\mathcal{T}_\mu(\mathrm{P}_2^{[1]}(.,\mu))-(1+2\mu)H_{-1}\mathrm{P}_2^{[1]}(.,\mu)=2x$, the application of the operator h_{-1} to (41) gives

$$x\Phi u = h_{-1}(Bu), \tag{42}$$

where

$$B(x)=\frac{1-4\mu^2}{4\mu K}\Big(\mathrm{P}_2^{[1]}(.,\mu)\sum_{i=1}^{p}\lambda_{s,i}\mathrm{P}_i-\frac{K}{1-4\mu^2}\Phi\mathcal{T}_\mu(\mathrm{P}_2^{[1]}(.,\mu))+ \frac{2\mu K}{1-2\mu}\Phi H_{-1}\mathrm{P}_2^{[1]}(.,\mu)-\sum_{i=1}^{p}\mathrm{P}_i\lambda_{s+2,i}\Big). \tag{43}$$

(Which proved $(1)\Rightarrow(3)$).

To prove (27), applying the operator h_{-1} to (42) and taking into account (1) we get

$$-x\Phi(-x)h_{-1}u = B(x)u. \tag{44}$$

Elimination of $h_{-1}u$ between (42) and (44) gives

$$-x^2\Phi(x)\Phi(-x)u = B(x)B(-x)u.$$

Hence,

$$-x^2\Phi(x)\Phi(-x) = B(x)B(-x).$$

So, $\deg(B)=\deg(\Phi)+1$ and $B(0)=0$. Therefore, we can write

$$B(x)=x\tilde{\Phi}(x), \tag{45}$$

with $\tilde{\Phi}$ is a polynomial such that $\deg\tilde{\Phi}=\deg\Phi$.
Substitution of (45) in (42), we get (26).

On the other hand, owing to (42), the multiplication of (39) by x given

$$xw(\mu)=\frac{K}{1-4\mu^2}(x\Phi(x)+2\mu B(x))u. \tag{46}$$

On account again of (42) and by application of the operator h_{-1} to (46), we get

$$xh_{-1}w(\mu) = -\frac{K}{1-4\mu^2}(B(x)+2\mu x\Phi(x))u. \tag{47}$$

From (34), (46) and (47), the multiplication of (33) by x get

$$\frac{K}{1-4\mu^2}(x\Phi(x)+2\mu B(x))\mathcal{T}_\mu(\mathrm{P}^{[1]}_{n-s}(.,\mu)) - x\Psi(x)\mathrm{P}^{[1]}_{n-s}(.,\mu) - \frac{2\mu K}{1-2\mu}(x\Phi(x)+B(x))H_{-1}\mathrm{P}^{[1]}_{n-s}(.,\mu) = -\sum_{i=1}^{p}\lambda_{n,i}x\mathrm{P}_i, \tag{48}$$

with Ψ as in (40). i.e.,

$$\frac{K}{1-4\mu^2}(x\Phi(x)+2\mu B(x))\mathcal{T}^2_\mu(\mathrm{P}_{n-s+1}(x)) - x\Psi(x)\mathcal{T}_\mu(\mathrm{P}_{n-s+1}(x)) - \frac{2\mu K}{1-2\mu}(x\Phi(x)+B(x))H_{-1}(\mathcal{T}_\mu(\mathrm{P}_{n-s+1}(x))) = -\mu_{n-s+1}\sum_{i=1}^{p}\lambda_{n,i}x\mathrm{P}_i, \quad n\geq s. \tag{49}$$

Using (45) and after simplification by x in the last equation we obtain (27) .

▶ (2) $\Rightarrow$ (3). From (27), we have

$$\frac{K}{1-4\mu^2}(\Phi(x)+2\mu\tilde{\Phi}(x))\mathcal{T}_\mu(\mathrm{P}^{[1]}_{n-s}(.,\mu)) - \Psi(x)\mathrm{P}^{[1]}_{n-s}(.,\mu) - \frac{2\mu K}{1-2\mu}(\Phi(x)+\tilde{\Phi}(x))H_{-1}\mathrm{P}^{[1]}_{n-s}(.,\mu) = -\sum_{i=1}^{p}\lambda_{n,i}\mathrm{P}_i. \tag{50}$$

Since $\sum_{i=1}^{p}\lambda_{n,i}\langle u,\mathrm{P}_i\rangle = 0$ for all $n\geq s$,

$$\begin{aligned} 0 = \; & \Big\langle u, \tfrac{K}{1-4\mu^2}(\Phi(x)+2\mu\tilde{\Phi}(x))\mathcal{T}_\mu(\mathrm{P}^{[1]}_{n-s}(.,\mu)) \\ & -\Psi(x)\mathrm{P}^{[1]}_{n-s}(.,\mu) - \tfrac{2\mu K}{1-2\mu}(\Phi(x)+\tilde{\Phi}(x))H_{-1}\mathrm{P}^{[1]}_{n-s}(.,\mu)\Big\rangle \\ & = -\Big\langle \tfrac{K}{1-4\mu^2}\mathcal{T}_\mu\Big((\Phi(x)+2\mu\tilde{\Phi}(x))u\Big) + \Psi(x)u - \\ & \tfrac{2\mu K}{1-2\mu}H_{-1}\Big((\Phi(x)+\tilde{\Phi}(x))u\Big), \mathrm{P}^{[1]}_{n-s}(.,\mu)\Big\rangle, \quad n\geq s. \end{aligned} \tag{51}$$

By the fact that $\{\mathrm{P}^{[1]}_n(.,\mu)\}_{n\geq 0}$ is a basis sequence, we get

$$\begin{aligned} & \frac{K}{1-4\mu^2}\mathcal{T}_\mu\Big((\Phi(x)+2\mu\tilde{\Phi}(x))u\Big) + \Psi(x)u - \\ & \frac{2\mu K}{1-2\mu}H_{-1}\Big((\Phi(x)+2\mu\tilde{\Phi}(x))u\Big) = 0. \end{aligned} \tag{52}$$

On the other hand, applying the operator h_{-1} to (26) and multiplying the result by x^{-1}, we obtain

$$\tilde{\Phi}(x)u = -h_{-1}(\Phi(x)u) + \langle u, \Phi(x) + \tilde{\Phi}(x) \rangle \delta_0. \tag{53}$$

Substituting (53) into (52) and taking into account (12) and the fact that $H_{-1}\delta_0 = \delta_0'$ and $H_{-1}(v - h_{-1}v) = 0$, $v \in \mathcal{P}'$, we get (30).
Finally, if we put $B(x) = x\tilde{\Phi}(x)$, one can see easily that (26) is equivalent to (31) and condition (28) is equivalent to (29). This completes the proof.

► (3) ⇒ (1). Put

$$v = \frac{K}{1-4\mu^2}(\Phi u - 2\mu h_{-1}(\Phi u)) \tag{54}$$

Let us prove that the MPS $\{\mathrm{P}_n^{[1]}(.,\mu)\}_{n\geq 0}$ is quasi-orthogonal of order s with respect to v.
By (16), we have

$$\begin{aligned}\mu_{n+1}\langle v, \mathrm{P}_m \mathrm{P}_n^{[1]}(.,\mu)\rangle &= -\langle \mathcal{T}_\mu(\mathrm{P}_m v), \mathrm{P}_{n+1}\rangle \\ &= -\langle \mathrm{P}_m \mathcal{T}_\mu v + \mathrm{P}_m' v + 2\mu H_{-1}\mathrm{P}_m h_{-1} v, \mathrm{P}_{n+1}\rangle.\end{aligned}$$

But, by hypothesis

$$\mathcal{T}_\mu v = -\Psi u, \tag{55}$$

with Ψ is a polynomial of degree p, $1 \leq p \leq s+1$. Then,

$$\mu_{n+1}\langle v, \mathrm{P}_m \mathrm{P}_n^{[1]}(.,\mu)\rangle = \langle \mathrm{P}_m \Psi u - \mathrm{P}_m' v - 2\mu H_{-1}\mathrm{P}_m h_{-1} v, \mathrm{P}_{n+1}\rangle.$$

For $m \leq n-s-1$, $n \geq s+1$. By the fact that $\{\mathrm{P}_n\}_{n\geq 0}$ is orthogonal with respect to u, we obtain

$$\mu_{n+1}\langle v, \mathrm{P}_m \mathrm{P}_n^{[1]}(.,\mu)\rangle = -\langle v, \mathrm{P}_m'(x)\mathrm{P}_{n+1}(x) + 2\mu H_{-1}\mathrm{P}_m(x)\mathrm{P}_{n+1}(-x)\rangle.$$

Taking into account (54), the orthogonality of $\{\mathrm{P}_n\}_{n\geq 0}$ with respect to u and the fact that $\deg(\Phi) \leq s+2$, we get

$$\mu_{n+1}\langle v, \mathrm{P}_m \mathrm{P}_n^{[1]}(.,\mu)\rangle = -\frac{2\mu K}{(1-4\mu^2)}\langle u, \Phi(x)\mathrm{P}_{n+1}(-x)(\mathrm{P}_m'(-x) - H_{-1}\mathrm{P}_m(-x))\rangle.$$

It is easy to see that $\mathrm{P}_m'(-x) - H_{-1}\mathrm{P}_m(-x) = x\mathrm{Q}(x)$, where Q is a polynomial of degree less than or equal to $m-2$. So

$$\mu_{n+1}\langle v, \mathrm{P}_m \mathrm{P}_n^{[1]}(.,\mu)\rangle = -\frac{2\mu K}{(1-4\mu^2)}\langle u, x\Phi(x)\mathrm{Q}(x)\mathrm{P}_{n+1}(-x)\rangle.$$

Using (31), we get

$$\mu_{n+1}\left\langle v,\mathrm{P}_m\mathrm{P}_n^{[1]}(.,\mu)\right\rangle = -\frac{2\mu K}{(1-4\mu^2)}\left\langle u,B(x)\mathrm{Q}(-x)\mathrm{P}_{n+1}(x)\right\rangle.$$

Since $\deg(B)\leq s+3$, according to the orthogonality of $\{\mathrm{P}_n\}_{n\geq 0}$ with respect to u, we get

$$\left\langle v,\mathrm{P}_m\mathrm{P}_n^{[1]}(.,\mu)\right\rangle = 0.$$

Suppose now that for all $r\geq s$ we have

$$\left\langle v,\mathrm{P}_{r-s}\mathrm{P}_r^{[1]}(.,\mu)\right\rangle = 0.$$

By following

$$-\frac{1}{\mu_{r+1}}\left\langle \mathrm{P}_{r-s}\mathcal{T}_\mu v+\mathrm{P}'_{r-s}v+2\mu H_{-1}\mathrm{P}_{r-s}h_{-1}v,\mathrm{P}_{r+1}\right\rangle = 0. \tag{56}$$

Using (55) and the orthogonality of $\{\mathrm{P}_n\}_{n\geq 0}$ with respect to u, we get

$$\left\langle \mathrm{P}_{r-s}\mathcal{T}_\mu v,\mathrm{P}_{r+1}\right\rangle = -\frac{\Psi^{(s+1)}(0)}{(s+1)!}p_{r+1}, \tag{57}$$

where $p_{r+1}=\left\langle u,\mathrm{P}_{r+1}^2\right\rangle$.

From (54), we obtain

$$\begin{aligned}&\left\langle \mathrm{P}'_{r-s}v,\mathrm{P}_{r+1}\right\rangle = \\ &\frac{K}{1-4\mu^2}\Big(\frac{\Phi^{(s+2)}(0)}{(s+2)!}(r-s)p_{r+1}-2\mu\left\langle u,\Phi(x)\mathrm{P}'_{r-s}(-x)\mathrm{P}_{r+1}(-x)\right\rangle\Big),\end{aligned} \tag{58}$$

and

$$\begin{aligned}\left\langle 2\mu H_{-1}\mathrm{P}_{r-s}h_{-1}v,\mathrm{P}_{r+1}\right\rangle = \frac{2\mu K}{1-4\mu^2}\Big(&\left\langle u,\Phi(x)H_{-1}\mathrm{P}_{r-s}(-x)\mathrm{P}_{r+1}(-x)\right\rangle \\ &-2\mu\frac{\Phi^{(s+2)}(0)}{(s+2)!}[r-s]p_{r+1}\Big).\end{aligned} \tag{59}$$

Substitution of (57), (58) and (59) into (56) gives

$$\begin{aligned}&\left(\frac{\Psi^{(s+1)}(0)}{(s+1)!}+\frac{\Phi^{(s+2)}(0)}{(s+2)!}\frac{K}{1-4\mu^2}(4\mu^2[r-s]-(r-s))\right)\frac{p_{r+1}}{\mu_{r+1}} \\ &-\frac{2\mu K}{\mu_{r+1}(1-4\mu^2)}\left\langle u,\Phi(x)\big(H_{-1}\mathrm{P}_{r-s}(-x)-\mathrm{P}'_{r-s}(-x)\big)\mathrm{P}_{r+1}(-x)\right\rangle = 0.\end{aligned} \tag{60}$$

Putting $H_{-1}\mathrm{P}_{r-s}(-x)-\mathrm{P}'_{r-s}(-x)=x\mathrm{Q}(x)$, where Q is a polynomial of degree $r-s-2$ and of leading coefficient $(-1)^{r-s-1}([r-s]-(r-s))$. Using (31) we obtain (for $r\geq s$)

$$\left\langle u,\Phi(x)(H_{-1}\mathrm{P}_{r-s}(-x)-\mathrm{P}'_{r-s}(-x))\mathrm{P}_{r+1}(-x)\right\rangle = -\frac{B^{(s+3)}(0)}{(s+3)!}\big([r-s]-(r-s)\big)p_{r+1}.$$

Then, (60) becomes

$$\Bigg(\frac{\Psi^{(s+1)}(0)}{(s+1)!}+\frac{K}{1-4\mu^2}\frac{\Phi^{(s+2)}(0)}{(s+2)!}\big(4\mu^2[r-s]-(r-s)\big) +\frac{2\mu K}{1-4\mu^2}\frac{B^{(s+3)}(0)}{(s+3)!}\big([r-s]-(r-s)\big)\Bigg)\frac{p_{r+1}}{\mu_{r+1}}=0,$$

which contradicts (29). Then, there exists a $r\geq s$ such that $\left\langle v,\mathrm{P}_{r-s}\mathrm{P}_r\right\rangle\neq 0$ and by following $\mathrm{P}_n^{[1]}(.,\mu)$ is quasi-orthogonal of order s with respect to v.

.

▶ (3) ⇔ (4). By virtue of (45) and (5), (31) becomes

$$\Phi(x)u=-h_{-1}((\tilde{\Phi})u)+\left\langle u,\Phi+\tilde{\Phi}\right\rangle\delta_0.$$

Applying operator h_{-1} to the last equation and taking into account the fact that $h_{-1}\delta_0=\delta_0$, we obtain

$$h_{-1}(\Phi(x)u)=-\tilde{\Phi}u+\left\langle u,\Phi+\tilde{\Phi}\right\rangle\delta_0. \tag{61}$$

Hence, according of (12) and (61), we can easily deduce the equivalence between (30) and (32).

□

Remark 2.2. 1. According of (30) and (40), we have

$$\begin{aligned}0&=\Big\langle\frac{K}{1-4\mu^2}\mathcal{T}_\mu\big(\Phi u-2\mu h_{-1}(\Phi u)\big)+\sum_{i=1}^{p}\lambda_{s,i}\mathrm{P}_i u,\mathrm{P}_1\Big\rangle\\&=-K\left\langle u,\Phi\right\rangle+\lambda_{s,1}\left\langle u,\mathrm{P}_1\right\rangle\\&=-K(a_{s+2}(u)_{s+2}+a_{s+1}(u)_{s+1}+\ldots+a_1(u)_1+a_0)+1+2\mu.\end{aligned}$$

So,

$$K=\frac{1+2\mu}{a_{s+2}(u)_{s+2}+a_{s+1}(u)_{s+1}+\ldots+a_1(u)_1+a_0}.$$

2. Notice that, in the above proof, if we take $s=0$, we get all equations and relations achieved in [6] and [11].

Corollary 2.3. *Let u be a regular form. Then u is $\mathcal{T}_\mu$-semi classical if and only if there exist three polynomials Φ (monic),$\tilde{\Phi}$, $\deg(\Phi) = \deg(\tilde{\Phi}) \leq s+2$ and Ψ, $\deg(\Psi) = p$, $1 \leq p \leq s+1$, fulfilling* (26) *and* (28) *such that*

$$\langle u, \Psi \rangle = 0, \ \ \langle u, \Phi \rangle = \frac{\langle u, x\Psi \rangle}{K} \tag{62}$$

and

$$x^2 \mathcal{T}_\mu \big((\Phi + 2\mu\tilde{\Phi})u \big) + \frac{1-4\mu^2}{K} x^2 \Psi u = 0. \tag{63}$$

Proof. Suppose that the regular form u is $\mathcal{T}_\mu$-semi classical. Then, (32) holds and gives immediately (62). On the other hand, by multiplication of (32) by x^2, we get (63).

Conversely, multiplying (63) by x^{-1} two times and taking into account (6) and (62), we get (32).

□

Notice that in the symmetric case, Sghaier [18] has characterized all $\mathcal{T}_\mu$-semi classical forms by $\mathcal{T}_\mu$-discributional equation of Pearson type. In the following proposition, we will also give the characterization of symmetric $\mathcal{T}_\mu$-semi classical forms by $\mathcal{T}_\mu$-second order differential-difference equation which is immediately from Theorem 2.1.

Proposition 2.4. *Let $\{\mathrm{P}_n\}_{n\geq 0}$ be a symmetric MOPS with respect to a regular form u. The following statement are equivalent*

(1) The sequence $\{\mathrm{P}_n\}_{n\geq 0}$ is Dunkl-semiclassical.

(2) There exist a non negative integer s, two polynomials Φ (monic), $\deg\Phi \leq s+2$ and Ψ, $1 \leq \deg\Psi \leq s+1$, a complex number K and a sequence of non zero complex number $\{\lambda_{n,i}\}_{n\geq s, i\geq 1}$ such that

$$\begin{aligned} &\frac{K}{1-4\mu^2}(\Phi(x) - 2\mu\Phi(-x))\mathcal{T}_\mu^2(\mathrm{P}_{n-s}(x)) - \Psi(x)\mathcal{T}_\mu(\mathrm{P}_{n-s}(x)) \\ &\quad - \frac{2\mu K}{1-2\mu}(\Phi(x) - \Phi(-x))H_{-1}(\mathcal{T}_\mu(\mathrm{P}_{n-s}(x))) = -\mu_{n-s+1}\sum_{i=1}^{p}\lambda_{n,i}\mathrm{P}_i, \end{aligned} \tag{64}$$

with condition

$$\begin{aligned} &\frac{\Psi^{(s+1)}(0)}{(s+1)!} + \frac{K}{1-4\mu^2}\frac{\Phi^{(s+2)}(0)}{(s+2)!}(4\mu^2[r-s] - (r-s)) + \\ &\frac{2\mu K}{1-4\mu^2}\frac{2(-1)^{s+2}\Phi^{(s+2)}(0)}{(s+3)!}\big([r-s] - (r-s)\big) \neq 0. \end{aligned} \tag{65}$$

Proof. Since $\{P_n\}_{n\geq 0}$ is Dunkl-symmetric MOPS, $u = h_{-1}u$. By following

$$x\Phi(x) = h_{-1}(-x\Phi(-x)u), \ \ \Phi \in \mathcal{P}.$$

Putting $\tilde{\Phi}(x) = -\Phi(-x)$, we get (26) and we can easily see that (27) is equivalent to (64) and the condition (28) becomes as (65). Hence, according to Theorem 2.1, we obtain the desired equivalence.

□

Let $\{P_n\}_{n\geq 0}$ be an MOPS with respect to the regular form u. Consider the sequence $\{\hat{P}_n(x) = a^{-n}P_n(ax)\}_{n\geq 0}$, $a \neq 0$. One can easily see that $\{\hat{P}_n\}_{n\geq 0}$ is an MOPS with respect the regular form $\hat{u}$ defined as

$$\hat{u} = h_{a^{-1}}u. \tag{66}$$

Moreover, we have the following result

Proposition 2.5. *If u is a $\mathcal{T}_\mu$-semi classical form , then $\hat{u} = h_{a^{-1}}u$ is also for every $a \neq 0$.*

Proof. By hypotheses, there exist three polynomial Φ (monic), $\deg(\Phi) \leq s+1$, B, $\deg(B) = \deg(\Phi)+1$, and Ψ, $\deg(\Psi) = p$, $1 \leq p \leq s+1$ such that u satisfies (29)-(31). From (2) and (66), we get

$$\Psi u = \Psi h_a \hat{u} = h_a\big((h_a\Psi)\hat{u}\big) = a^{d-1}h_a(\widehat{\Psi}\hat{u}), \ \widehat{\Psi}(x) = a^{1-d}\Psi(ax), \ \ d = \deg(\Phi).$$

Similarly,

$$\Phi u = \Phi h_a \hat{u} = h_a\big((h_a\Phi)\hat{u}\big) = a^{d}h_a(\widehat{\Phi}\hat{u}), \ \ \widehat{\Phi}(x) = a^{-d}\Psi(ax).$$

Then,

$$\Psi u = a^{d-1}h_a(\widehat{\Psi}\hat{u}), \ \ \Phi u = a^{d}h_a(\widehat{\Phi}\hat{u}). \tag{67}$$

Using (3), (13) and (67), we get

$$\mathcal{T}_\mu\left(\Phi u - 2\mu h_{-1}(\Phi u)\right) = a^{d-1}h_a\mathcal{T}_\mu\left(\widehat{\Phi}\hat{u} - 2\mu h_{-1}(\widehat{\Phi}\hat{u})\right). \tag{68}$$

Substitution of (67) and (68) in (30), we get

$$\mathcal{T}_\mu\left(\widehat{\Phi}\hat{u} - 2\mu h_{-1}(\widehat{\Phi}\hat{u})\right) + \frac{1-4\mu^2}{K}\widehat{\Psi}\hat{u} = 0$$

Using again (2) and (66), we get

$$Bu = Bh_a\hat{u} = h_a(h_aB\hat{u}) = a^{d+1}h_a(\widehat{B}\hat{u}), \ \ \widehat{B} = a^{-1-d}B(ax).$$

Substitution of the last result and (67) in (32), we obtain

$$x\widehat{\Phi}(x)\hat{u} = h_{-1}(\widehat{B}(x)\hat{u}).$$

Moreover, by (29), we have

$$\frac{\widehat{\Psi}^{(s+1)}(0)}{(s+1)!} + \frac{K}{1-4\mu^2}\frac{\widehat{\Phi}^{(s+2)}(0)}{(s+2)!}\big(4\mu^2[r-s]-(r-s)\big)+ \\ \frac{2\mu K}{1-4\mu^2}\frac{\widehat{B}^{(s+3)}(0)}{(s+3)!}\big([r-s]-(r-s)\big) \neq 0.$$

Then the desired result.

□

The third important characterization of the Dunkl-semiclassical forms is given in terms of a non-homogeneous first order linear $\mathcal{T}_\mu$-difference equation that its Stieltjes series satisfies.

Proposition 2.6. *The form u is $\mathcal{T}_\mu$-semi classical and satisfies* (26), (28) *and* (32) *if and only if there exist three polynomials A (monic), C, D such that the Stieltjes formal series $S(u)$ satisfies*

$$A(z)\mathcal{T}_{-\mu}S(u)(z) = -2\mu(H_{-1}A)(z)S(h_{-1}u)(z) + C(z)S(u)(z) + D(z) \\ +2\mu(2\mu+1)\frac{\langle u, A+(1-2\mu)\tilde{\Phi}\rangle}{z^2}, \tag{69}$$

and

$$z\Phi(z)S(u)(z) - (u\Phi)(z) = -z\tilde{\Phi}(-z)S(h_{-1}u)(z) + (h_{-1}u\tilde{\Phi})(-z) \tag{70}$$

with

$$-\frac{C^{(s+1)}(0)}{(s+1)!} + (4\mu^2[r-s]-(r-2))\frac{A^{(s+2)}(0)}{(s+2)!} + \\ 2\mu\frac{\tilde{\Phi}^{(s+2)}(0)}{(s+2)!}(1-4\mu^2)[r-s] \neq 0. \tag{71}$$

Proof. Necessity. If u is a $\mathcal{T}_\mu$-semi classical form, then (32) holds and from (16), (32) becomes

$$(\Phi+2\mu\tilde{\Phi})\mathcal{T}_\mu u + (\Phi'+2\mu\tilde{\Phi}')u + 2\mu H_{-1}(\Phi+2\mu\tilde{\Phi})h_{-1}u \\ -2\mu(1+2\mu)\langle u, \Phi+\tilde{\Phi}\rangle\delta_0' + \frac{1-4\mu^2}{K}\Psi u = 0.$$

From the linearity of S, we obtain

$$\begin{aligned} &S((\Phi+2\mu\tilde{\Phi})\mathcal{T}_\mu u)(z)+S((\Phi'+2\mu\tilde{\Phi}')u)(z) \\ &\quad +2\mu S(H_{-1}(\Phi+2\mu\tilde{\Phi})h_{-1}u)(z)-2\mu(1+2\mu)\langle u,\Phi+\tilde{\Phi}\rangle S(\delta_0')(z) \\ &\quad +\tfrac{1-4\mu^2}{K}S(\Psi u)(z)=0. , \end{aligned} \tag{72}$$

According to (9) and (17), we have

$$S((\Phi+2\mu\tilde{\Phi})\mathcal{T}_\mu u)(z)=(\Phi+2\mu\tilde{\Phi})(z)\mathcal{T}_{-\mu}S(u)(z)+(\mathcal{T}_\mu u\theta_0(\Phi+2\mu\tilde{\Phi}))(z),$$

$$S((\Phi'+2\mu\tilde{\Phi}')u)(z)=(\Phi'+2\mu\tilde{\Phi}')(z)S(u)(z)+(u\theta_0(\Phi'+2\mu\tilde{\Phi}'))(z),$$

$$\begin{aligned} S(H_{-1}(\Phi+2\mu\tilde{\Phi})h_{-1}u)(z)&=H_{-1}(\Phi+2\mu\tilde{\Phi})(z)S(h_{-1}u)(z) \\ &\quad +((h_{-1}u)\theta_0(H_{-1}(\Phi+2\mu\tilde{\Phi}))(z), \end{aligned}$$

$$S(\Psi u)(z)=\Psi(z)S(u)(z)+(u\theta_0\Psi)(z).$$

Therefore, owing to (8), (72) gives (69) with

$$\begin{cases} A(z)=\Phi(z)+2\mu\tilde{\Phi}(z) \\ C(z)=-\Big(\Phi'(z)+2\mu\tilde{\Phi}'(z)+\frac{1-4\mu^2}{K}\Psi(z)\Big) \\ D(z)=-(\mathcal{T}_\mu u\theta_0(\Phi+2\mu\tilde{\Phi}))(z)-(u\theta_0(\Phi'+2\mu\tilde{\Phi}'))(z) \\ \qquad -2\mu((h_{-1}u)\theta_0(H_{-1}(\Phi+2\mu\tilde{\Phi}))(z)-\frac{1-4\mu^2}{K}(u\theta_0\Psi)(z). \end{cases} \tag{73}$$

Using (9), we can prove that (26) is equivalent to (70) and taking into account (73), we can see that condition (28) can be written as (71).

Sufficiency. If u is regular such that its Stieltjes series $S(u)$ satisfies (69) and (71). Using (8), (9) and (17), (69) becomes

$$\begin{aligned} &S\big(A\mathcal{T}_\mu u+2\mu(H_{-1}A)h_{-1}u-Cu-2\mu(2\mu+1)\langle u,A+(1-2\mu)\tilde{\Phi}\rangle\delta_0'\big)(z) \\ &\quad =(\mathcal{T}_\mu u\theta_0 A)(z)+2\mu((h_{-1}u)\theta_0(H_{-1}A))(z)-(u\theta_0 C)(z)+D(z). \end{aligned}$$

Then

$$\begin{cases} S(A\mathcal{T}_\mu u+2\mu(H_{-1}A)h_{-1}u-Cu-2\mu(2\mu+1)\langle u,A+(1-2\mu)\tilde{\Phi}\rangle\delta_0')(z)=0 \\ D(z)=-(\mathcal{T}_\mu u\theta_0 A)(z)-2\mu((h_{-1}u)\theta_0(H_{-1}A))(z)+(u\theta_0 C)(z). \end{cases}$$

Putting $\Phi(x)=A(x)-2\mu\tilde{\Phi}(x)$ and $\Psi(x)=-\frac{K}{1-4\mu^2}(A'(x)+C(x))$. So, by virtue of (14), it is easy to see that

$$\mathcal{T}_\mu\big((\Phi+2\mu\tilde{\Phi})u\big)-2\mu(1+2\mu)\langle u,\Phi+\tilde{\Phi}\rangle\delta_0'+\frac{1-4\mu^2}{K}\Psi u=0.$$

□

We arrive now at a other characterization of Dunkl-semiclassical orthogonal polynomials which is the so-called structure relation.

Proposition 2.7. *Let* $\{\mathrm{P}_n\}_{n\geq 0}$ *be a MOPS with respect to a regular form u. The following statement are equivalent*

1. *The MOPS is* $\mathcal{T}_\mu$*-semi classical.*

2. *There exist a complex number K and three polynomials* Φ *(monic),* $\tilde{\Phi}$ *and* Ψ*,* $\deg(\Phi) = \deg(\tilde{\Phi}) \leq s+2$*,* $\deg(\Psi) = p$*,* $1 \leq p \leq s+1$ *fulfilling* (28) *and* (62) *such that*

$$(\Phi(x)+2\mu\tilde{\Phi}(x))\mathcal{T}_\mu(x^2\mathrm{P}_n(x)) = \sum_{k=n-s-3}^{n+d+1} \chi_{n,k}\mathrm{P}_k(x),\ d=\deg(\Phi), \quad (74)$$

where

$$\chi_{n,k} = 0,\ 0 \leq n \leq s+3,\ k<0,$$
$$\chi_{n,0} = \frac{1-4\mu^2}{K}\left\langle u, x^2\Psi(x)\mathrm{P}_n(x)\right\rangle,\ \ 0 \leq n \leq s+3 \quad (75)$$

and

$$x\Phi(x)\mathrm{P}_n(-x) = \sum_{k=n-s-3}^{n+d+1} \theta_{n,k}\mathrm{P}_k(x) \quad (76)$$

Proof. ► $(1) \Rightarrow (2)$. We always have

$$(\Phi(x)+2\mu\tilde{\Phi}(x))\mathcal{T}_\mu(x^2\mathrm{P}_n(x)) = \sum_{k=0}^{n+d+1} \chi_{n,k}\mathrm{P}_k(x), \quad (77)$$

with $d = \deg(\Phi) \leq s+2$ and

$$\chi_{n,k} = \frac{\left\langle u, (\Phi(x)+2\mu\tilde{\Phi}(x))\mathcal{T}_\mu(x^2\mathrm{P}_n(x))\mathrm{P}_k(x)\right\rangle}{\left\langle u, \mathrm{P}_k^2(x)\right\rangle},\ \ 0 \leq k \leq n+d+1,\ n \geq 0.$$

By virtue of (14), we get

$$\begin{aligned}\chi_{n,k}\left\langle u, \mathrm{P}_k^2(x)\right\rangle = \big\langle (\Phi(x)+2\mu\tilde{\Phi}(x))u, &\mathcal{T}_\mu(x^2\mathrm{P}_n(x)\mathrm{P}_k(x)) \\ &-x^2\mathrm{P}_n(x)\mathcal{T}_\mu(\mathrm{P}_k(x)) + 4\mu x H_{-1}\mathrm{P}_k(x)H_{-1}(x^2\mathrm{P}_n(x))\big\rangle\ .\end{aligned} \quad (78)$$

Using (63), the orthogonality of $\{\mathrm{P}_n\}_{n\geq 0}$ with respect to u and the fact that $\deg(\Psi) \leq s+1$, we get

$$\begin{aligned}\left\langle (\Phi(x)+2\mu\tilde{\Phi}(x))u, \mathcal{T}_\mu(x^2\mathrm{P}_n(x)\mathrm{P}_k(x))\right\rangle &= \frac{1-4\mu^2}{k}\left\langle u, x^2\Psi(x)\mathrm{P}_n(x)\mathrm{P}_k(x)\right\rangle \\ &= 0,\ \ k \leq n-s-4,\ n \geq s+4. \quad (79)\end{aligned}$$

Again the orthogonality of $\{P_n\}_{n\geq 0}$ and the fact that $\deg(\Phi+2\mu\tilde{\Phi})\leq s+2$ give

$$\begin{aligned}&\left\langle(\Phi(x)+2\mu\tilde{\Phi}(x))u,x^2P_n(x)\mathcal{T}_\mu(P_k(x))\right\rangle\\&\quad=\left\langle u,x^2(\Phi(x)+2\mu\tilde{\Phi}(x))\mathcal{T}_\mu(P_k(x))P_n(x)\right\rangle\\&\quad=0,\ \ k\leq n-s-4,\ n\geq s+4.\end{aligned}\tag{80}$$

On the other hand,

$$\begin{aligned}&\left\langle(\Phi(x)+2\mu\tilde{\Phi}(x))u,4\mu xH_{-1}P_k(x)H_{-1}(x^2P_n(x))\right\rangle=\\&\qquad\left\langle(\Phi(x)+2\mu\tilde{\Phi}(x))u,2\mu x^2H_{-1}P_k(x)P_n(x)\right\rangle\\&\qquad\quad-\left\langle(\Phi(x)+2\mu\tilde{\Phi}(x))u,2\mu x^2H_{-1}P_k(x)P_n(-x)\right\rangle.\end{aligned}$$

One can easily see that

$$\left\langle(\Phi(x)+2\mu\tilde{\Phi}(x))u,2\mu x^2H_{-1}P_k(x)P_n(x)\right\rangle=0,\ k\leq n-s-4,\ n\geq s+4.$$

Then, from (26) and the fact that $\{P_n\}_{n\geq 0}$ is orthogonal with respect to u, we get

$$\begin{aligned}&\left\langle(\Phi(x)+2\mu\tilde{\Phi}(x))u,4\mu xH_{-1}P_k(x)H_{-1}(x^2P_n(x))\right\rangle\\&\quad=-\left\langle(\Phi(x)+2\mu\tilde{\Phi}(x))u,2\mu x^2H_{-1}P_k(x)P_n(-x)\right\rangle\\&\quad=\left\langle(\tilde{\Phi}(x)+2\mu\Phi(x))u,2\mu x^2H_{-1}P_k(-x)P_n(x)\right\rangle\\&\quad=0,\ \ k\leq n-s-4,\ n\geq s+4.\end{aligned}\tag{81}$$

Substitution of (79), (80) and (81) into (78) gives

$$\chi_{n,k}\left\langle u,P_k^2(x)\right\rangle=0,\ k\leq n-s-4,\ \ n\geq s+4$$

Since $\left\langle u,P_k^2(x)\right\rangle\neq 0$, $k\geq 0$, then

$$\chi_{n,k}=0,\ k\leq n-s-4,\ \ n\geq s+4$$

and (74) holds.

For $0\leq n\leq s+3$, (63) gives

$$\chi_{n,0}=-\left\langle x^2\mathcal{T}_\mu\left((\Phi(x)+2\mu\tilde{\Phi}(x))u\right),P_n(x)\right\rangle=\frac{1-4\mu^2}{K}\left\langle u,x^2\Psi(x)P_n(x)\right\rangle.$$

In the same way, to prove (76), we write

$$x\Phi(x)P_n(-x)=\sum_{k=0}^{n+d+1}\theta_{n,k}P_k(x),$$

with

$$\theta_{n,k}=\frac{\big\langle u,x\Phi(x)\mathrm{P}_n(-x)\mathrm{P}_k(x)\big\rangle}{\big\langle u,\mathrm{P}_k^2(x)\big\rangle}.$$

By following, we use (26), we get

$$\begin{aligned}\theta_{n,k}\big\langle u,\mathrm{P}_k^2(x)\big\rangle&=\big\langle u,x\Phi(x)\mathrm{P}_k(x)\mathrm{P}_n(-x)\big\rangle\\&=\big\langle u,x\tilde{\Phi}\mathrm{P}_k(-x)\mathrm{P}_n(x)\big\rangle\\&=0,\quad k\leq n-s-4.\end{aligned}$$

So, (76) follows.

► $(2)\Rightarrow(1)$. For $n\geq s+4$, owing to (74), we have

$$\begin{aligned}\Big\langle x^2\mathcal{T}_\mu\big((\Phi(x)+2\mu\tilde{\Phi}(x))u\big),\mathrm{P}_n(x)\Big\rangle&=-\big\langle u,(\Phi(x)+2\mu\tilde{\Phi}(x))\mathcal{T}_\mu(x^2\mathrm{P}_n(x))\big\rangle\\&=-\sum_{k=n-s-3}^{n+d+1}\chi_{n,k}\big\langle u,\mathrm{P}_k(x)\big\rangle=0.\end{aligned}$$

On account of (11), there exists a polynomial R, $\deg(\mathrm{R})\leq s+3$, such that

$$x^2\mathcal{T}_\mu\big((\Phi(x)+2\mu\tilde{\Phi}(x))u\big)=\mathrm{R}(x)u. \tag{82}$$

Therefore, by virtue of (74) and (75), we get

$$\begin{aligned}\Big\langle u,\big(R(x)+\frac{1-4\mu^2}{K}x^2\Psi(x)\big)\mathrm{P}_n(x)\Big\rangle=&\Big\langle x^2\mathcal{T}_\mu\big((\Phi(x)+2\mu\tilde{\Phi}(x))u,\mathrm{P}_n(x)\Big\rangle\\&+\frac{1-4\mu^2}{K}\big\langle u,x^2\Psi(x)\mathrm{P}_n(x)\big\rangle\\&=-\chi_{n,0}+\frac{1-4\mu^2}{K}\big\langle u,x^2\Psi(x)\mathrm{P}_n(x)\big\rangle\\&=0,\quad 0\leq n\leq s+3.\end{aligned}$$

Hence,

$$R(x)=-\frac{1-4\mu^2}{K}x^2\Psi(x). \tag{83}$$

Substituting (83) in (82), we obtain (63).

On the other hand, from (76), we have

$$\begin{aligned}\big\langle h_{-1}(x\Phi(x)u),\mathrm{P}_n(x)\big\rangle&=\big\langle u,x\Phi(x)\mathrm{P}_n(-x)\big\rangle\\&=\sum_{k=n-d-1}^{n+d+1}\theta_{n,k}\big\langle u,\mathrm{P}_k(x)\big\rangle\\&=0,\quad n\geq d+2.\end{aligned}$$

Using (11), we get

$$h_{-1}(x\Phi(x)u) = B(x)u,$$

with B is a polynomial such that $\deg(B) = d+1$. Owing (45), we obtain (26) which completes the proof. $\square$

A regular form u fulfilling (30) and (31) satisfies an infinity of equation of the same type. In fact, let us multiply (30) by $\mathcal{A}$, where $\mathcal{A}$ is a non zero even polynomial, we obtain

$$\mathcal{T}_\mu\big(\mathcal{A}\Phi u - 2\mu h_{-1}(\mathcal{A}\Phi u)\big) - \mathcal{A}'\Phi u + 2\mu\mathcal{A}' h_{-1}(\Phi u) + \frac{1-4\mu^2}{K}\mathcal{A}\Psi u = 0. \quad (84)$$

Since $\mathcal{A}$ is an even polynomial, $\mathcal{A}'$ is an odd polynomial. Then, there exists a polynomial $\widetilde{\mathcal{A}}$, $\deg(\widetilde{\mathcal{A}}) = \deg(\mathcal{A}') - 1$, such that $\mathcal{A}'(x) = x\widetilde{\mathcal{A}}(x)$. So (84) becomes

$$\mathcal{T}_\mu\big(\mathcal{A}\Phi u - 2\mu h_{-1}(\mathcal{A}\Phi u)\big) + \frac{1-4\mu^2}{K}\Big(\mathcal{A}\Psi - \frac{k}{1-4\mu^2}(\mathcal{A}'\Phi + 2\mu\widetilde{\mathcal{A}}B)\Big)u = 0.$$

Moreover,

$$x\mathcal{A}(x)\Phi(x)u = h_{-1}(\mathcal{A}(x)B(x)u)$$

Then, for any pair (Φ,Ψ) satisfying (30) we associate the positive integer $s = \max\{\deg(\Phi)-2, \deg(\Psi)-1\}$. Putting

$$\mathfrak{h}(u) := \{s = \max\{\deg(\Phi)-2, \deg(\Psi)-1\},\ (30)-(31)\ hold\}.$$

This leads to the following definition:

Definition 2.8. Let u be a $\mathcal{T}_\mu$-semi classical regular form. The non negative integer s defined by

$$s = \min \mathfrak{h}(u)$$

is called the class of u.
The corresponding MOPS $\{\mathrm{P}_n\}_{n\geq 0}$ will be said to be of class s.

Remark 2.9. 1. If $s = 0$, the form u is called $\mathcal{T}_\mu$-classical [6].

2. Here arises the questions: Is the pair $\{\phi,\psi\}$ which realizes the minimum of $\mathfrak{h}(u)$ is unique? If it is unique, how we know whether the integer s associated with a pair $\{\phi,\psi\}$ is the minimum of $\mathfrak{h}(u)$?

3. An example of non-symmetric Dunkl-semiclassical orthogonal polynomials

Let us define the form u as

$$u = \eta x^{-1} v + \delta_0, \;\; \eta \in \mathbb{C}^* \backslash \{1+\lambda\}, \tag{85}$$

where v defined by (21). The form u is regular, except in a discrete set (see [14]). The MOPS corresponding to u, which we denote by $\{P_n\}_{n\geq 0}$, satisfies the three-term recurrence relation (10) with

$$\beta_0 = \eta, \;\; \beta_{n+1} = a_n + \frac{\gamma^v_{n+1}}{a_n}, \;\; \gamma_{n+1} = -a_n(a_n - \beta^v_n), \tag{86}$$

where (refer to [14])

$$a_n = -\frac{\tilde{P}_{n+1}(0;-\eta)}{\tilde{P}_n(0;-\eta)}, \;\; n \geq 0. \tag{87}$$

From (85), we have

$$xu = \eta v. \tag{88}$$

It is clear that (21) and (88) give

$$x(x-1)u = \eta\lambda \mathcal{G}^{(\alpha,\mu-\frac{1}{2})}. \tag{89}$$

So,

$$h_{-1}(x(x-1)u) = x(x-1)u. \tag{90}$$

Multiplying (90) by $x\big(x+\frac{1+2\mu}{1-2\mu}\big)$, we obtain

$$x\Phi(x)u = h_{-1}(B(x)u),$$

with

$$\Phi(x) = x(x-1)\Big(x+\frac{1+2\mu}{1-2\mu}\Big), \;\;\; B(x) = x^2(x-1)\Big(x-\frac{1+2\mu}{1-2\mu}\Big).$$

Substituting (88) into (22), we get

$$\mathcal{T}_\mu\big(x(x^2-1)u\big) - \frac{1+2\mu}{\lambda+1}x(x-\lambda-1)u = 0.$$

Then, u is Dunkl-semiclassical form and verifying (30) with

$$\Psi(x) = -\frac{(1+2\mu)^2}{\xi(1-2\mu)(\lambda+1)}x(x-\lambda-1)$$

and

$$K=\frac{1+2\mu}{\xi},$$

where

$$\xi=(\beta_1+2\beta_0^2)\gamma_1+\beta_0^3+\frac{4\mu}{1-2\mu}(\gamma_1+\beta_0^2)-\frac{1+2\mu}{1-2\mu}\beta_0,$$

$$\beta_0=\eta,\ \ \beta_1=\frac{\eta^2-2(1+\lambda)\eta+1+\lambda}{1+\lambda-\eta},\ \ \gamma_1=-(1+\lambda-\eta)(1+\lambda-2\eta).$$

So, we should choose η so that $\xi\neq 0$.

Furthermore, condition (29) is written as

$$\frac{1+2\mu}{\xi(1-2\mu)}\Big(\frac{-(1+2\mu)+\big(2\mu[r-1]-(r-1)\big)(\lambda+1)}{\lambda+1}\Big)\neq 0. \tag{91}$$

Prove now that $\{\mathrm{P}_n^{[1]}(.,\mu)\}_{n\geq 0}$ is quasi-orthogonal of order 1 with respect to

$$w(\mu)=\frac{1+2\mu}{\xi(1-2\mu)}x(x^2-1)u.$$

We have

$$\begin{aligned}\mu_{n+1}\big\langle w(\mu),x^m\mathrm{P}_n^{[1]}\big\rangle &= -\big\langle \mathcal{T}_\mu(x^m w(\mu)),\mathrm{P}_{n+1}\big\rangle\\ &= -\big\langle x^m\mathcal{T}_\mu w(\mu)+mx^{m-1}w(\mu)+2\mu H_{-1}x^m h_{-1}w(\mu),\mathrm{P}_{n+1}\big\rangle\\ &= \big\langle u,x^m\Psi\mathrm{P}_{n+1}\big\rangle-\frac{1+2\mu}{\xi(1-2\mu)}\big\langle u,mx^m(x^2-1)\mathrm{P}_{n+1}\big\rangle\\ &\quad+\frac{2\mu(1+2\mu)}{\xi(1-2\mu)}\big\langle u,(H_{-1}x^m)x(x-1)^2\mathrm{P}_{n+1}\big\rangle.\end{aligned}$$

So, from the orthogonality of $\{\mathrm{P}_n\}_{n\geq 0}$ with respect to u and the fact that $\deg\Psi=2$, we get

$$\big\langle w(\mu),x^m\mathrm{P}_n^{[1]}\big\rangle=0,\quad \forall\, 0\leq m\leq n-2.$$

Suppose that, for all $r\geq 1$, $\big\langle w(\mu),x^{r-1}\mathrm{P}_r^{[1]}(.,\mu)\big\rangle=0$. Then,

$$-\frac{1}{\mu_{r+1}}\big\langle x^{r-1}\mathcal{T}_\mu w(\mu)+(x^{r-1})'w(\mu)+2\mu H_{-1}(x^{r-1})h_{-1}w(\mu),\mathrm{P}_{r+1}\big\rangle=0. \tag{92}$$

Using the orthogonality of $\{\mathrm{P}_n\}_{n\geq 0}$ with respect to u, we obtain

$$\big\langle x^{r-1}\mathcal{T}_\mu w(\mu),\mathrm{P}_{r+1}\big\rangle=-\big\langle u,x^{r-1}\Psi,\mathrm{P}_{r+1}\big\rangle=\frac{(1+2\mu)^2}{\xi(1-2\mu)(\lambda+1)}\big\langle u,\mathrm{P}_{r+1}^2\big\rangle,$$

$$\begin{aligned}\big\langle (x^{r-1})'w(\mu),\mathrm{P}_{r+1}\big\rangle &= \frac{1}{\lambda\eta}\big\langle u,(r-1)x^{r-1}(x^2-1)\mathrm{P}_{r+1}\big\rangle\\ &= \frac{(1+2\mu)(r-1)}{\xi(1-2\mu)}\big\langle u,\mathrm{P}_{r+1}^2\big\rangle\end{aligned}$$

and

$$\begin{aligned}\big\langle H_{-1}(x^{r-1})h_{-1}w(\mu),\mathrm{P}_{r+1}\big\rangle &= -\frac{1}{\lambda\eta}\big\langle u,(H_{-1}x^{r-1})x(x-1)^2\mathrm{P}_{r+1}\big\rangle\\ &= -\frac{(1+2\mu)[r-1]}{\xi(1-2\mu)}\big\langle u,\mathrm{P}_{r+1}^2\big\rangle.\end{aligned}$$

So, (92) became

$$\frac{\big\langle u,\mathrm{P}_{r+1}^2\big\rangle}{\mu_{r+1}}\left(-\frac{1+2\mu}{\xi(1-2\mu)}\frac{-(1+2\mu)+\big(2\mu[r-1]-(r-1)\big)(\lambda+1)}{\lambda+1}\right)=0,$$

which contradicts (91). Thus $\{\mathrm{P}_n^{[1]}(.,\mu)\}_{n\geq 0}$ is quasi-orthogonal of order 1 with respect to $w(\mu)$.

Remark 3.1. In fact, for any $n\geq 1$, the polynomial $P_n^{[1]}(x,\mu)$ can be computed by the following algorithm. Using the relation (87) to compute a_{n-1}. Next, we can compute β_1 and γ_1 from (86) and finally, from the three-term recurrence relation, we get $P_{n+1}(x)$. Then compute $P_n^{[1]}(x,\mu)$ from (18). As an example, for $n=1$, $a_0=1+\lambda-\eta$. So

$$\beta_1=1+\lambda-\eta-\frac{\lambda(1+\lambda)}{1+\lambda-\eta},\quad \gamma_1=\eta(1+\lambda-\eta).$$

Then, we get

$$P_2(x)=\left(x-(1+\lambda-\eta)+\frac{\lambda(1+\lambda)}{1+\lambda-\eta}\right)(x-\eta)-\eta(1+\lambda-\eta).$$

Therefore, we obtain

$$\mathrm{P}_1^{[1]}(x,\mu)=x-\frac{1}{2}(1+2\mu)\left(1+\lambda-\frac{\lambda(1+\lambda)}{1+\lambda-\eta}\right).$$

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