LE MATEMATICHE
Vol. LXXVII (2022) – Issue I, pp. 95–105
doi: 10.4418/2022.77.1.5

BOUNDARY VALUE PROBLEMS IN GENERAL RELATIVITY

G. CIMATTI

Certain theorems of existence, non-existence and uniqueness for boun dary value problems modeling axial symmetric problems in general relativity are presented using the Weyl's metric. A solution related to the classical Poiseuille solution of non-relativistic fluid mechanics is also presented.

1. Introduction

The theory of newtonian potential is essentially a theory of elliptic boundary value problems. On the other hand, in general relativity the coefficients of the metric play the role of unknown potentials [7], but the corresponding boundary value problems seems to have received little attention. Even in simple case of absence of matter and electromagnetic fields, the Einstein equations do not fit immediatly in any of the type in which the partial differential equations are classified. They are nonlinear and form an over-determined system. If one consider a bounded domain, another difficulty is to find the additional conditions needed to single out a specific solution in order to have a well-posed problem. This question was well present to A. Einstein see [2].

In this paper we consider, in Section 2, the simple case in which the energy-stress tensor vanishes everywhere and we give a theorem of existence and uniqueness for the corresponding boundary value problem stated in a bounded axial

Received on July 2, 2021

AMS 2010 Subject Classification: 83C10, 83C05

Keywords: Axial symmetric gravitational fields, existence and uniqueness of solutions, Poiseuille solution

symmetric domain when the values of the coefficients of the metric are suitably prescribed. In Section 3 we treat the case of the energy-stress tensor corresponding to a continuous distribution of fluid in absence of body forces and fluid motion. We obtain a result of existence and non-existence of solutions. Finally in Section 4 we find an analogue of the Poiseuille solution of classical fluid mechanics starting from the Einstein's equations. We always consider axial symmetric situations and use the Weyl's metric [10]

$$ds^2 = e^{2\psi}dt^2 - e^{2\gamma-2\psi}d\rho^2 - e^{2\gamma-2\psi}dz^2 - e^{-2\psi}\rho^2 d\varphi^2, \tag{1}$$

where the unknown "potentials" ψ and γ are assumed to be functions of ρ and z only, i.e. $\psi = \psi(\rho,z)$, $\gamma = \gamma(\rho,z)$. We recall that the non-vanishing components of the Einstein's tensor G_{ik} corresponding to the metric (1) are [8]

$$G_{11} = e^{4\psi+2\gamma}\Big[-2\big(\psi_{\rho\rho} + \frac{\psi_\rho}{\rho} + \psi_{zz}\big) + \psi_\rho^2 + \psi_z^2 + \gamma_{\rho\rho} + \gamma_{zz}\Big] \tag{2}$$

$$G_{22} = \psi_\rho^2 - \frac{\gamma_\rho}{\rho} - \psi_z^2, \quad G_{23} = 2\psi_\rho\psi_z - \frac{\gamma_z}{\rho}, \quad G_{33} = -\psi_\rho^2 + \frac{\gamma_\rho}{\rho} + \psi_z^2 \tag{3}$$

$$G_{44} = -e^{-2\gamma}\rho^2\big(\psi_\rho^2 + \psi_z^2 + \gamma_{\rho\rho} + \gamma_{zz}\big). \tag{4}$$

2. The case $T_{ik} = 0$

Let us consider an axial symmetric bounded subset Ω of $\mathbf{R}^3$ with a regular boundary Γ. We suppose Ω free of gravitational masses and of electric charges. Outside Ω there exists a distribution of mass, also axial symmetric, which determines on Γ the values of ψ and γ in a way not depending on angular variable φ. We assume the energy-stress tensor T_{ik} to vanish in Ω. The Einstein's equations

$$G_{ik} = K\,T_{ik}, \quad K = 8\pi\frac{G}{c^4}, \quad \text{G gravitational constant} \tag{5}$$

determine the "potentials" $\psi(\rho,z)$ and $\gamma(\rho,z)$ via the following overdetermined system of partial differential equations [1], [9]

$$-2\big(\psi_{\rho\rho} + \frac{\psi_\rho}{\rho} + \psi_{zz}\big) + \psi_\rho^2 + \psi_z^2 + \gamma_{\rho\rho} + \gamma_{zz} = 0 \tag{6}$$

$$\psi_\rho^2 - \frac{\gamma_\rho}{\rho} - \psi_z^2 = 0 \tag{7}$$

$$2\psi_\rho\psi_z - \frac{\gamma_z}{\rho} = 0 \tag{8}$$

$$\psi_\rho^2 + \psi_z^2 + \gamma_{\rho\rho} + \gamma_{zz} = 0. \tag{9}$$

In this paper we prove that a specific solution of the system (6)-(9) is determined in a unique way if ψ is prescribed on the boundary of the domain Ω under consideration by a given function ψ_Γ (under suitable assumptions on Ω and ψ_Γ) and γ is prescribed in a single point. This is not surprising since, after inserting (9) in (6), ψ intervenes with first and second order partial derivatives whereas γ intervenes only with first order derivatives. To prove this result use will be made of the following

Lemma 2.1. *Let Ω be an axial symmetric bounded subset of $\mathbf{R}^3$ referred to coordinates (ρ, z, ϕ) (z the axis of symmetry) with a regular boundary $\Gamma \in C^\alpha$. Let ψ_Γ be a function of class $C^{0,\alpha}(\Gamma)$ not depending on the angular variable ϕ. Then the problem*

$$\psi_{\rho\rho} + \frac{\psi_\rho}{\rho} + \psi_{zz} = 0 \ \ in\ \Omega, \quad \psi = \psi_\Gamma \ on\ \Gamma \tag{10}$$

has one and only one solution $\psi(\rho, z) \in C^{2,\alpha}(\bar{\Omega})$.

Proof. The problem (10) is a "piece" of the problem for the laplacian

$$\Delta\tilde{\psi} = 0 \quad \text{in } \Omega, \quad \tilde{\psi} = \psi_\Gamma \quad \text{on } \Gamma \tag{11}$$

which is simply the Dirichlet's problem. By standard results, see e.g. [3], problem (11) has one and only one solution $\tilde{\psi} \in C^{2,\alpha}(\bar{\Omega})$. On the other hand, $\tilde{\psi}(\rho, z, \phi + K)$ is also a solution of problem (11) in view of the axial symmetry of Ω and ψ_Γ. Hence by uniqueness $\tilde{\psi}(\rho, z, \phi + K) = \tilde{\psi}(\rho, z, \phi)$. Therefore $\tilde{\psi}$ does not depend on ϕ and thus it is the unique $C^{2,\alpha}(\bar{\Omega})$-solution of problem (10). □

Using Lemma 2.1 we can prove that problem (6)-(9) with the stated additional conditions on ψ and γ has one and only one solution.

We prove in fact in the next Lemma a more general result.

Lemma 2.2. *Let Γ be of class C^α. Assume the sections of Ω with an arbitrary half-plane containing the z-axis be simply connected. Let the function $\rho h(\rho, z) + ig(\rho, z)$ of the complex variable $\rho + iz$ be analytic. Assume $\psi_\Gamma \in C^\alpha(\Gamma)$ and $\gamma_0 \in \mathbf{R}^1$. Then the overdetermined system of P.D.E.*

$$-2\left(\psi_{\rho\rho} + \frac{\psi_\rho}{\rho} + \psi_{zz}\right) + \psi_\rho^2 + \psi_z^2 + \gamma_{\rho\rho} + \gamma_{zz} = 0 \tag{12}$$

$$\psi_\rho^2 - \frac{\gamma_\rho}{\rho} - \psi_z^2 = g \tag{13}$$

$$2\psi_\rho \psi_z - \frac{\gamma_z}{\rho} = h \tag{14}$$

$$\psi_\rho^2 + \psi_z^2 + \gamma_{\rho\rho} + \gamma_{zz} = 0 \tag{15}$$

with the conditions

$$\psi = \psi_\Gamma \quad \textit{on } \Gamma \tag{16}$$

$$\gamma = \gamma_0 \qquad \textit{in an arbitrary point of } \Omega \tag{17}$$

has one and only one solution $\psi(\rho,z) \in C^{2,\alpha}(\bar{\Omega})$ *and* $\gamma(\rho,z) \in C^{2,\alpha}(\bar{\Omega})$.

Proof. Let $\psi(\rho,z)$ be the unique solution of problem (10) given by Lemma 2.1 and let us consider the first order system

$$\gamma_\rho = F(\rho,z), \quad \gamma_z = G(\rho,z) \tag{18}$$

with the condition

$$\gamma(\rho_0,z_0) = \gamma_0, \tag{19}$$

where $F(\rho,z) = \rho\big(\psi_\rho^2 - \psi_z^2 + g\big), \quad G(\rho,z) = 2\rho\big(\psi_\rho \psi_z + h\big)$. We have

$$G_\rho - F_z = 2\rho\psi_z\big(\psi_{\rho\rho} + \frac{\psi_\rho}{\rho} + \psi_{zz}\big) + (\rho h)_\rho - \big(\rho g\big)_z.$$

Since $\rho h + i\rho g$ is analytic we conclude that $G_\rho = F_z$ by (10). Therefore, the system (18) is integrable and with the condition (19) its solution is unique. It remains to prove that

$$\psi_\rho^2 + \psi_z^2 + \gamma_{\rho\rho} + \gamma_{zz} = 0. \tag{20}$$

From (18) we have

$$\gamma_{\rho\rho} + \gamma_{zz} = \psi_\rho^2 - \psi_z^2 + 2\rho\psi_\rho\psi_{\rho\rho} + 2\rho\psi_\rho\psi_{zz} + \big(g + \rho g_\rho + \rho h_z\big).$$

On the other hand, $\rho h + i\rho g$ is analytic, hence

$$\gamma_{\rho\rho} + \gamma_{zz} = \psi_\rho^2 - \psi_z^2 + 2\rho\psi_\rho\psi_{\rho\rho} + 2\rho\psi_\rho\psi_{zz}. \tag{21}$$

Substituting (21) in (20) we have, by (10),

$$\psi_\rho^2 + \psi_z^2 + \gamma_{\rho\rho} + \gamma_{zz} = 2\psi_\rho\rho\big(\psi_{\rho\rho} + \frac{\psi_\rho}{\rho} + \psi_{zz}\big) = 0.$$

Therefore, $(\psi(\rho,z),\gamma(\rho,z))$ is a solution of the problem (12)-(17). On the other hand, this solution is also unique. For, let (ψ^*,γ^*) be a second solution. By difference from (6) and (9) we obtain

$$\psi^*_{\rho\rho}+\frac{\psi^*_\rho}{\rho}+\psi^*_{zz}=0 \ \text{ in } \Omega, \quad \psi^*=\psi_\Gamma \text{ on } \Gamma.$$

Thus ψ and ψ^* are both solutions of a problem for which there is uniqueness. Hence $\psi=\psi^*$. Since $F(\rho,z)=F^*(\rho,z)$, $G(\rho,z)=G^*(\rho,z)$ and $\gamma(\rho_0,z_0)=\gamma^*(\rho_0,z_0)$, we also have $\gamma(\rho,z)=\gamma^*(\rho,z)$. We conclude that the problem (12)-(17) has one and only one solution. □

In Lemma 2.2 the physically relevant case of interest to us occurs when $g=0$ and $h=0$.

Remark 1. The above method may probably be applied to other static problems not only to axially symmetric ones.

3. The boundary value problem in presence of a fluid

In this Section we assume again to be in the axial symmetric situation of Section 2 with the metric (1). The domain Ω is supposed to be filled with an incompressible viscous fluid. The energy-stress tensor reads ([4] page 512)

$$T_{ik}=-pg_{ik}+(p+\varepsilon)u_iu_k-c\eta\left(u_{i;k}+u_{k;i}-u_iu^lu_{k;l}-u_ku^lu_{i;k}\right), \qquad (22)$$

where u_i is the covariant four-velocity, p denotes the pressure, η is the viscosity and ε the energy density which is assumed here to be a given constant. We consider first an hydrostatic case in which the physical velocity $\mathbf{v}=(v_\rho,v_z,v_\varphi)$ is assumed to vanish. Correspondingly the covariant four-velocity is given, recalling the metric (1), by

$$u_i=\left(e^\psi,0,0,0\right). \qquad (23)$$

We want to determine ψ, γ and p assuming the additional conditions used for the system (6)-(9). The non-vanishing components of the covariant derivative $u_{i;j}$ are

$$u_{2;1}=-\psi_\rho e^\psi, \quad u_{3;1}=-\psi_z e^\psi$$

and those of the energy-stress tensor T_{ik} are given by

$$T_{11}=-pe^{2\psi}+(\varepsilon+p)e^{2\psi}, \quad T_{22}=pe^{2\gamma-2\psi}, \quad T_{33}=pe^{2\gamma-2\psi}, \quad T_{44}=pe^{-2\psi}\rho^2.$$

The Einstein's equations (5) become, by (2)-(4),

$$e^{4\psi-2\gamma}\big[\psi_\rho^2+\psi_z^2+\gamma_{\rho\rho}+\gamma_{zz}-2\big(\psi_{\rho\rho}+\frac{\psi_\rho}{\rho}+\psi_{zz}\big)\big]=K\big[\big(\varepsilon+p\big)e^{2\psi}-pe^{2\psi}\big] \tag{24}$$

$$\psi_\rho^2-\psi_z^2-\frac{\gamma_\rho}{\rho}=Kpe^{2\gamma-2\psi} \tag{25}$$

$$\frac{\gamma_\rho}{\rho}-\psi_\rho^2+\psi_z^2=Kpe^{2\gamma-2\psi} \tag{26}$$

$$2\psi_r\psi_z-\frac{\gamma_z}{\rho}=0 \tag{27}$$

$$-e^{-2\gamma}\rho^2\big(\psi_\rho^2+\psi_z^2+\gamma_{\rho\rho}+\gamma_{zz}\big)=Kpe^{-2\psi}\rho^2. \tag{28}$$

By adding (25) and (26) we infer

$$p=0.$$

Hence the system (24)-(28) can be rewritten as

$$\psi_\rho^2+\psi_z^2+\gamma_{\rho\rho}+\gamma_{zz}-2\big(\psi_{\rho\rho}+\frac{\psi_\rho}{\rho}+\psi_{zz}\big)=K\varepsilon e^{2\gamma-2\psi} \tag{29}$$

$$\gamma_\rho=\rho\big(\psi_\rho^2-\psi_z^2\big) \tag{30}$$

$$\gamma_z=2\rho\,\psi_\rho\psi_z \tag{31}$$

$$\psi_\rho^2+\psi_z^2+\gamma_{\rho\rho}+\gamma_{zz}=0. \tag{32}$$

From (29) and (32) we obtain

$$-2\big(\psi_{\rho\rho}+\frac{\psi_\rho}{\rho}+\psi_{zz}\big)=K\varepsilon\; e^{2\gamma-2\psi}. \tag{33}$$

Let $F(\rho,z)=\rho\big(\psi_\rho^2-\psi_z^2\big),\quad G(\rho,z)=2\rho\,\psi_\rho\psi_z$. From (30) and (31) we have

$$F_z-G_\rho=-2\rho\,\psi_z\big(\psi_{\rho\rho}+\frac{\psi_\rho}{\rho}+\psi_{zz}\big).$$

Hence, by (33) we obtain

$$F_z-G_\rho=2K\varepsilon e^{2\gamma-2\psi}\rho\,\psi_z.$$

Therefore, the system (30), (31) i.e.

$$\gamma_\rho = F(\rho,z), \quad \gamma_z = G(\rho,z)$$

is not integrable if $\varepsilon \neq 0$ and the system (24)-(28) cannot have solutions in this case[1]. On the other hand, if $\varepsilon = 0$ the system (29)-(32) becomes

$$\psi_\rho^2 + \psi_z^2 + \gamma_{\rho\rho} + \gamma_{zz} - 2\big(\psi_{\rho\rho} + \frac{\psi_\rho}{\rho} + \psi_{zz}\big) = 0$$

$$\gamma_\rho = \rho\left(\psi_\rho^2 - \psi_z^2\right)$$

$$\gamma_z = 2\rho\,\psi_\rho\,\psi_z$$

$$\psi_\rho^2 + \psi_z^2 + \gamma_{\rho\rho} + \gamma_{zz} = 0.$$

Thus we are formally in the situation of Section 2 and the corresponding results of existence and uniqueness apply.

4. A Poiseuille-like solution in general relativity

The space outside an indefinite cylinder of radius R is filled with an incompressible viscous fluid[2]. Body forces are absent and the cylinder is supposed to move with a constant velocity v_R in the z-direction. If we assume

$$v_\rho = 0, \quad v_z = v(\rho), \quad v_\varphi = 0 \tag{34}$$

the Navier-Stockes equations reduce to the single equation, see [5],

$$v'' + \frac{v'}{\rho} = 0. \tag{35}$$

The non-slip condition gives

$$v(R) = v_R. \tag{36}$$

In view of (36) we have, solving (35)

$$v(\rho) = C\log\rho + v_R - C\log R.$$

[1] This implies that the condition $\varepsilon \neq 0$ is incompatible with the assumption (23) of absence of fluid motion

[2] The axis of the cylinder is the z-axis of the cylindrical coordinates system

Assuming the velocity to remain bounded we obtain the exceedingly simple solution

$$v(\rho) = v_R.$$

We ask the following question: starting with the same assumptions (34), (36) which solution is given by the equations of general relativity? We use the Weyl's metric (2) assuming ψ and γ to be functions of ρ only. In view of the one-dimensional character of the problem, the non-vanishing components of the Einstein tensor take on the form

$$G_{11} = e^{4\psi+2\gamma}\Big[\psi'^2 + \gamma'' - 2\Big(\psi'' + \frac{\psi'}{\rho}\Big)\Big], \quad G_{22} = \psi'^2 - \frac{\gamma'}{\rho}$$

$$G_{33} = -\psi'^2 + \frac{\gamma'}{\rho}, \quad G_{44} = -e^{-2\gamma}\rho^2\big(\psi'^2 + \gamma''\big).$$

The contravariant four-velocity

$$u^i = \frac{dx_i}{ds} = \big(e^{-\Psi}, 0, v(\rho)e^{-\Psi}, 0\big) \tag{37}$$

in covariant form is

$$u_i = \big(e^{\Psi}, 0, -e^{2\gamma-3\Psi}, 0\big).$$

Turning to the non vanishing components of the energy-stress tensor we find, using (22),

$$T_{11} = \varepsilon e^{2\Psi}, \quad T_{12} = c\eta v^2 e^{2\gamma-3\psi}\big(\gamma' - \psi'\big), \quad T_{13} = -\big(\varepsilon + p\big)v e^{2\gamma-2\psi}$$

$$T_{22} = p e^{2\gamma-2\Psi}, \quad T_{23} = c\eta v' e^{2\gamma-3\psi}$$

$$T_{33} = p e^{2\gamma-2\Psi} + (\varepsilon + p)e^{4\gamma-6\Psi}v^2, \quad T_{44} = p\rho e^{-2\Psi}.$$

For the non identically satisfied Einstein's equations (5) we find

$$e^{2\psi-2\gamma}\big[\psi'^2 + \gamma'' - 2\big(\psi'' + \frac{\psi'}{\rho}\big)\big] = \varepsilon K \tag{38}$$

$$c\eta v^2 e^{2\gamma-3\psi}\big(\gamma' - \psi'\big) = 0 \tag{39}$$

$$(\varepsilon + p)e^{2\gamma-2\Psi}v = 0 \tag{40}$$

$$\psi'^2 - \frac{\gamma'}{\rho} = Kpe^{2\gamma-2\psi} \tag{41}$$

$$c\eta v' e^{2\gamma-3\psi} = 0 \tag{42}$$

$$-\psi'^2 + \frac{\gamma'}{\rho} = K\left[pe^{2\gamma-2\psi} + (\varepsilon + p)e^{4\gamma-6\psi}v^2\right] \tag{43}$$

$$\psi'^2 + \gamma'' = -Kpe^{2\gamma-2\psi}. \tag{44}$$

From (42) we have $v(\rho) = v_R$, as in the Navier-Stockes case, if we assume the boundary condition $v(R) = v_R$. From (39) we obtain, if $v_R \neq 0$,

$$\gamma = \psi + C \tag{45}$$

C a constant to be determined. By (40) we have

$$p = -\varepsilon.$$

Thus (43) becomes

$$-\psi'^2 + \frac{\gamma'}{\rho} = Kpe^{2\gamma-2\psi}. \tag{46}$$

Adding (46) and (41) we obtain

$$2Kpe^{2\gamma-2\psi} = 0. \tag{47}$$

Hence

$$p = 0, \quad \varepsilon = 0. \tag{48}$$

We conclude that the problem is compatible only with a null energy density. Moreover, (44) becomes

$$\psi'^2 + \gamma'' = 0. \tag{49}$$

In addition (38) gives

$$\psi'' + \frac{\psi'}{\rho} = 0.$$

This means

$$\psi(\rho) = k_1 \log \rho + k_2.$$

On the other hand, by (45) we have

$$\psi' = \gamma' = \frac{k_1}{\rho}, \quad \gamma'' = -\frac{k_1}{\rho^2}.$$

By (49), it follows

$$\frac{k_1^2}{\rho^2} = \frac{k_1}{\rho^2}.$$

Hence, either $k_1 = 1$ or $k_1 = 0$. If $k_1 = 1$ we have

$$\psi(\rho) = \log\rho + k_2.$$

The constant k_2 is determined with a condition of the form $\psi(R) = \psi_R$ which gives $\psi(\rho) = \log\rho - \log R + \psi_R$ (compare for this solution [6]). Finally with a condition like $\gamma(R) = \gamma_R$ we determine the, still unknown, constant C entering in (45). We obtain $\gamma(\rho) = \log\rho - \log R + \gamma_R$. We conclude that the boundary conditions $\psi(R) = \psi_R$, $\gamma(R) = \gamma_R$ and $v(R) = v_R$ make the problem well-posed in agreement with the corresponding solution of the Navier-Stockes equations and determine completely also the metric (1).

Observe that the equations of motion $T^{ij}_{;j} = 0$ is in this case automatically satisfied.

Remark 2. If, in the contest of Newtonian hydrodynamics, we state the cognate problem of the viscous fluid motion between two coaxial cylinders of radii $R_2 > R_1 > 0$ moving respectively with given velocities v_2 and v_1 in the z direction, again in absence of body forces, the problem is well-posed and easily solved with the non-slip conditions $v(R_2) = v_2$, $v(R_1) = v_1$ and we find easily

$$v(\rho) = \frac{v_2 - v_1}{\log \frac{\rho}{R_1}} \log \frac{\rho}{R_1} + v_1.$$

However, in the context of general relativity there is no parallel situation since the equation (42) permits to impose only one boundary condition. This is inherent to the fact that the Einstein's equations are of the first order.

Acknowledgment.An interesting suggestion about possible generalizations made by the Referee is gratefully acknowledged.

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G. CIMATTI
Department of Mathematics
University of Pisa
e-mail: `cimatti@dm.unipi.it`