LE MATEMATICHE
Vol. LXXVII (2022) – Issue I, pp. 107–118
doi: 10.4418/2022.77.1.6

4-HARMONIC FUNCTIONS AND BEYOND

A. GRECU - M. MIHĂILESCU

The family of partial differential equations $-\Delta_4 u - \varepsilon\Delta_\infty u = 0$ ($\varepsilon > 0$) is studied in a bounded domain Ω for given boundary data. We show that for each $\varepsilon > 0$ the problem has a unique viscosity solution which is exactly the $(4+\varepsilon)$-harmonic map with the given boundary data. We also explore the connections between the solutions of these problems and infinity harmonic and 4-harmonic maps by studying the limiting behavior of the solutions as $\varepsilon \to \infty$ and $\varepsilon \to 0^+$, respectively.

1. Introduction

1.1. Statement of the problem

Let $D \geq 1$ be an integer and let $\Omega \subset \mathbb{R}^D$ be a bounded domain with smooth boundary $\partial\Omega$. Next, let $g \in C^1(\Omega) \cap C(\overline{\Omega})$ be a given function. The main goal of this paper is to analyse the problem

$$\begin{cases} -\Delta_4 u - \varepsilon\Delta_\infty u = 0 & \text{in} \quad \Omega \\ u = g & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

Received on July 5, 2021

AMS 2010 Subject Classification: 35B40, 35D30, 35D40, 49J27, 49J45.

Keywords: Γ-convergence, 4-harmonic maps, infinity harmonic maps, viscosity solutions.

Andrei Grecu has been partially supported by CNCS-UEFISCDI Grant No. PN-III-P1-1.1-TE-2019-0456.

The authors would like to thank the anonymous referee for her/his careful reading of the original manuscript and for a number of relevant comments that led to improvements in the exposition in this paper.

where $\varepsilon > 0$ is a real parameter, $\Delta_4 u := \operatorname{div}(|\nabla u|_D^2 \nabla u)$ is the 4-Laplace operator and $\Delta_\infty u := \sum_{i,j=1}^D \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i x_j}$ stands for the ∞-Laplace operator.

1.2. Motivation

Under the same assumptions as in the previous section the following problem was investigated

$$\begin{cases} -\Delta u - \varepsilon \Delta_\infty u = 0 & \text{in} \quad \Omega \\ u = g & \text{on} \quad \partial\Omega, \end{cases} \tag{2}$$

see, e.g. [6], [12] or [1]. By [12] we know that problem (2) has a unique classical solution, say u_ε (which is, actually, an analytical function). Moreover, since problem (2) can be investigated in connection with the equation

$$\begin{cases} -\operatorname{div}(e^{\varepsilon|\nabla u|_D^2}\nabla u) = 0 & \text{in} \quad \Omega \\ u = g & \text{on} \quad \partial\Omega,, \end{cases} \tag{3}$$

the solution u_ε minimizes the Euler-Lagrange functional associated to the problem (3), namely

$$I(u) := \int_\Omega e^{\varepsilon|\nabla u|_D^2}\,dx,$$

over a closed and convex subset of an Orlicz-Sobolev space defined with the aid of the N-function $\Phi(t) := e^{\varepsilon t^2} - 1$ and taking into account the boundary condition (see [1] for more details). Furthermore, it was proved that u_ε converges uniformly over $\overline{\Omega}$, as $\varepsilon \to \infty$, to the unique ∞-harmonic function with g boundary data (see [7, Section 4.1] or [1, Proposition 4]), while in the case where $\varepsilon \to 0^+$, u_ε converges in $W^{1,4}(\Omega)$ to the unique harmonic function in Ω with g boundary value data (see [1, Theorem 3]).

Motivated by the studies on problem (2) here we analyse a similar problem, that is problem (1), where we replace the Laplace operator with the 4-Laplace operator. At this point a natural question which can be considered is investigating problem (1) when the 4-Laplace operator is replaced by the p-Laplace operator with $p \in (1,\infty)\setminus\{2,4\}$. Unfortunately, the arguments provided for the cases $p \in \{2,4\}$ seem to not hold true when $p \notin \{2,4\}$. More precisely, for $p \in \{2,4\}$ we can prove that the viscosity solutions of problems (2) and (1) are actually variational solutions of some equivalent equations, which can be obtained as minimizers of their corresponding Euler-Lagrange functionals. That fact is crucial in the proofs of Lemmas 4.1 and 4.5 from this paper (and also in the proofs from [1] and [7] and [11] for similar results). In the case when $p \notin \{2,4\}$ the lack of a variational characterization for the solutions should ask for a diferent treatment of the problem.

1.3. Main results

The main results of this paper are formulated below in the following two theorems.

Theorem 1.1. *For each* $g \in C^1(\Omega) \cap C(\overline{\Omega})$ *and each* $\varepsilon > 0$ *problem (1) has a unique viscosity solution which is exactly the* $(4+\varepsilon)$*-harmonic map with boundary values g.*

Theorem 1.2. *Let* $g \in C^1(\Omega) \cap C(\overline{\Omega})$ *and for each* $\varepsilon > 0$ *let* u_ε *be the unique viscosity solution of problem (1). Then* u_ε *converges uniformly over* $\overline{\Omega}$*, as* $\varepsilon \to \infty$*, to the unique* ∞*-harmonic function with g boundary data. Moreover, letting* u_0 *be the* 4*-harmonic map with g boundary data then* u_ε *converges weakly to* u_0 *in* $W^{1,4}(\Omega)$*, as* $\varepsilon \to 0^+$*, and*

$$\lim_{\varepsilon \to 0^+} \int_\Omega |\nabla u_\varepsilon|_D^4 \, dx = \int_\Omega |\nabla u_0|_D^4 \, dx .$$

The rest of the paper is structured as follows: in Section 2 we present some well-known results on p-harmonic functions and on ∞-harmonic functions which will be useful in our subsequent analysis; in Section 3 we give the proof of Theorem 1.1; in Section 4 we present the proof of Theorem 1.2.

2. A quick overview on p-harmonic functions and ∞-harmonic functions

2.1. p-harmonic functions

Let $\Omega \subset \mathbb{R}^D$ be a bounded domain with smooth boundary $\partial\Omega$, and let $g \in C^1(\Omega) \cap C(\overline{\Omega})$ be given. For each real number $p \in (1, \infty)$ we consider the problem

$$\begin{cases} -\Delta_p u = 0 & \text{in} \quad \Omega \\ u = g & \text{on} \quad \partial\Omega , \end{cases} \tag{4}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|_D^{p-2} \nabla u)$ is the p-Laplace operator. The solutions of problem (4) are sought in the set

$$W_g^{1,p}(\Omega) := \{ u \in W^{1,p}(\Omega) \; : \; u = g \text{ on } \partial\Omega \} ,$$

which is a closed and convex subset of the Sobolev space $W^{1,p}(\Omega)$, and are understood in a variational sense. More precisely, u_p is a week solution of problem (4) if $u_p \in W_g^{1,p}(\Omega)$ and satisfies

$$\int_\Omega |\nabla u_p|_D^{p-2} \nabla u_p \nabla \phi \, dx = 0, \quad \forall \, \phi \in W_0^{1,p}(\Omega) . \tag{5}$$

Such a function is also called a p-harmonic function with g boundary data (see also [10, Definition 2.1] for the definition of a p-harmonic function). It is well-known (see, e.g. [13, Theorem 2.4] or [14, Theorem 2.16]) that problem (4) has a unique weak solution which turns out to be the unique minimizer on $W_g^{1,p}(\Omega)$ of the Euler-Lagrange functional associated with equation (4), namely, $I_p : W^{1,p}(\Omega) \to \mathbb{R}$ given by

$$I_p(u) := \int_\Omega |\nabla u|_D^p \, dx. \tag{6}$$

Note also that when Ω is bounded and connected with the boundary $\partial\Omega$ of class $C^{1,\alpha}$ and $g \in W^{2,p}(\Omega)$ then $u_p \in C^{1,\alpha}_{\text{loc}}(\Omega)$ (see, e.g. [11, Section 2] or [14, Theorem 2.19]).

Next, we point out the fact that the weak solutions of problem (4) are equivalent with the viscosity solutions of the same equation (see, e.g. [10, Corollary 2.8]). Let us recall the definition of a viscosity solution of an equation of type

$$\begin{cases} F(\nabla u, D^2 u) = 0 & \text{in} \quad \Omega \\ u = g & \text{on} \quad \partial\Omega, \end{cases} \tag{7}$$

where $D^2 u$ stands for the Hessioan matrix of u (see, e.g. [2]).

Definition 2.1. (i) An upper semicontinuous function $u : \Omega \to \mathbb{R}$ is called a viscosity subsolution of (7) if $u|_{\partial\Omega} \le g$ and, whenever $x_0 \in \Omega$ and $\Psi \in C^2(\Omega)$ are such that $u(x_0) = \Psi(x_0)$ and $u(x) < \Psi(x)$ if $x \in B_r(x_0) \setminus \{x_0\}$ for some $r > 0$, then we have $F(\nabla\Psi(x_0), D^2\Psi(x_0)) \le 0$;
(ii) A lower semicontinuous function $u : \Omega \to \mathbb{R}$ is called a viscosity supersolution of (7) if $u|_{\partial\Omega} \ge g$ and, whenever $x_0 \in \Omega$ and $\Psi \in C^2(\Omega)$ are such that $u(x_0) = \Psi(x_0)$ and $u(x) > \Psi(x)$ if $x \in B_r(x_0) \setminus \{x_0\}$ for some $r > 0$, then $F(\nabla\Psi(x_0), D^2\Psi(x_0)) \ge 0$;
(iii) A continuous function $u : \Omega \to \mathbb{R}$ is called a viscosity solution of (7) if it is both a viscosity subsolution and a viscosity supersolution of (7).

Note that problem (4) is an equation of type (7) since if we assume that $u : \Omega \to \mathbb{R}$ is a sufficiently smooth function then the p-Laplacian of u becomes

$$\Delta_p u = |\nabla u|_D^{p-4}(|\nabla u|_D^2 \Delta u + (p-2)\Delta_\infty u), \tag{8}$$

or, taking into account that $\Delta u = \text{Trace}(D^2 u)$ and $\Delta_\infty u = \langle D^2 u \nabla u, \nabla u\rangle$ we get that

$$\Delta_p u = F_p(\nabla u, D^2 u),$$

where

$$F_p(\xi, S) := |\xi|_D^{p-4}(|\xi|_D^2 \text{Trace}(S) + (p-2)\langle S\xi, \xi\rangle),$$

when $\xi \in \mathbb{R}^D$ and $S \in \mathbb{M}^{D\times D}_{symm}(\mathbb{R})$. If $p \in (1,2)$ the function F_p is not defined at $\xi = 0$ (or $\nabla u = 0$). Consequently, this case requires special attention in relation with the definition of a viscosity solution (that is Definition 2.1 above). More precisely, in order to fix this problem we have to add the requirement $\nabla u(x_0) \neq 0$ in the definition of a viscosity solution. Consequently, there is no condition to be verified at the critical points in the definition of a viscosity solution. Note also that when $p \in [2,\infty)$ the above problem does not appear since $\Delta_p u(x_0) = 0$ in that case (see, e.g., [14, p. 78] for more details).

2.2. ∞-harmonic functions

Under the same assumptions as in the previous section we consider the equation

$$\begin{cases} -\Delta_\infty u = 0 & \text{in} \quad \Omega \\ u = g & \text{on} \quad \partial\Omega\,. \end{cases} \tag{9}$$

We call a viscosity solution of problem (9) an ∞-harmonic function with g boundary data. Let u_∞ be the unique (see [8]) viscosity solution of problem (9). It can be shown that the family of p-harmonic functions with g boundary data, u_p, converges uniformly over Ω, as $p \to \infty$, to u_∞ (see, e.g. [11, Section 2] for a simple and quick explanation).

3. Proof of Theorem 1.1

First, note that when $p = 4$ relation (8) implies that

$$\Delta_4 u = |\nabla u|_D^2 \Delta u + 2\Delta_\infty u\,.$$

Using that fact we observe that for each $p \in (4,\infty)$ we can rewrite relation (8) in the following way

$$\Delta_p u = |\nabla u|_D^{p-4}(\Delta_4 u + (p-4)\Delta_\infty u)\,.$$

Thus, for each $\varepsilon > 0$ we have

$$\Delta_{4+\varepsilon} u = |\nabla u|_D^{\varepsilon}(\Delta_4 u + \varepsilon\Delta_\infty u)\,. \tag{10}$$

Lemma 3.1. *For each $\varepsilon > 0$ a function u_ε is a viscosity solution of problem (1) if and only if it is a viscosity solution of problem (4) when $p = 4+\varepsilon$.*

Proof. **Step I:** Equivalence of the viscosity supersolutions.

• Let u_ε be a viscosity supersolution of problem (1). We show that it is a viscosity supersolution of problem (4) when $p = 4+\varepsilon$, too.

Indeed, first note that $u_\varepsilon = g$ on $\partial\Omega$. Next, let $x_0 \in \Omega$ and $\Psi \in C^2(\Omega)$ be such that $u_\varepsilon(x_0) = \Psi(x_0)$ and $u_\varepsilon(x) > \Psi(x)$ if $x \in B_r(x_0) \setminus \{x_0\}$ for some $r > 0$. Then

$$-\Delta_4\Psi(x_0) - \varepsilon\Delta_\infty\Psi(x_0) \geq 0\,.$$

It follows that

$$-|\nabla\Psi(x_0)|_D^{\varepsilon}(\Delta_4\Psi(x_0) + \varepsilon\Delta_\infty\Psi(x_0)) \geq 0\,,$$

or, by (10) we deduce that

$$-\Delta_{4+\varepsilon}\Psi(x_0) \geq 0\,,$$

which shows that u_ε is a viscosity supersolution of problem (4) when $p = 4+\varepsilon$.

• Let u_ε be a viscosity supersolution of problem (4) when $p = 4+\varepsilon$. We show that it is a viscosity supersolution of problem (1), too.

It is obvious that the boundary data condition is satisfied. Further, let $x_0 \in \Omega$ and $\Psi \in C^2(\Omega)$ be such that $u_\varepsilon(x_0) = \Psi(x_0)$ and $u_\varepsilon(x) > \Psi(x)$ if $x \in B_r(x_0) \setminus \{x_0\}$ for some $r > 0$. Then

$$-\Delta_{4+\varepsilon}\Psi(x_0) \geq 0\,,$$

or, by (10) we have

$$-|\nabla\Psi(x_0)|_D^{\varepsilon}(\Delta_4\Psi(x_0) + \varepsilon\Delta_\infty\Psi(x_0)) \geq 0\,.$$

If $|\nabla\Psi(x_0)|_D > 0$ then we get

$$-\Delta_4\Psi(x_0) - \varepsilon\Delta_\infty\Psi(x_0) \geq 0\,,$$

which means that u_ε is a viscosity supersolution of problem (1). Otherwise, if $|\nabla\Psi(x_0)|_D = 0$ then $\frac{\partial\Psi}{\partial x_i}(x_0) = 0$ for each integer $i \in \{1,...,D\}$ and taking into account the definitions of the 4-Laplacian and the ∞-Laplacian we conclude that

$$-\Delta_4\Psi(x_0) - \varepsilon\Delta_\infty\Psi(x_0) = 0\,,$$

which leads again to the conclusion that u_ε is a viscosity supersolution of problem (1).

Step II: Equivalence of the viscosity subsolutions.

The analysis of this case can be done similarly with the one from the first part of this proof and consequently we omit it. □

Proof of Theorem 1.1 (concluded). Let $\varepsilon > 0$ be arbitrary but fixed. By [13, Theorem 2.4] or [14, Theorem 2.16]) the problem (4) with $p = 4+\varepsilon$ has a unique weak solution, say u_ε, which is a $(4+\varepsilon)$-harmonic map (with boundary values g). Moreover, u_ε is a continuous function. Then by [10, Corollary 2.8] we infer that u_ε is a viscosity solution of problem (4) with $p = 4+\varepsilon$. Combining that fact with Lemma 3.1 it follows that u_ε is the unique viscosity solution of problem (1).

4. Proof of Theorem 1.2

By Theorem 1.1 we deduce that for each $\varepsilon > 0$ the unique viscosity solution of problem (1), u_ε, is a $(4+\varepsilon)$-harmonic function with boundary values g (or the unique weak solution of problem (4) with $p = 4+\varepsilon$). Then it is well-known that u_ε converges uniformly, as $\varepsilon \to \infty$ to the unique ∞-harmonic function with boundary values g (see, e.g. [11, Section 2]).

Next, we analyse the convergence of u_ε as $\varepsilon \to 0^+$.

We start by showing the following result:

Lemma 4.1. *The sequence $\{u_\varepsilon\}_{\varepsilon>0}$ is bounded in $W^{1,4}(\Omega)$.*

Proof. For each $\varepsilon > 0$, $u_\varepsilon \in W_g^{1,4+\varepsilon}(\Omega) \subset W_g^{1,4}(\Omega)$. Let u_0 be the 4-harmonic map with boundary values g. By [13, Theorem 2.4] or [14, Theorem 2.16], u_0 is a minimizer of I_4 given by relation (6) when $p = 4$. Consequently, we have

$$\int_\Omega |\nabla u_0|_D^4\, dx \le \int_\Omega |\nabla u_\varepsilon|_D^4\, dx, \quad \forall\, \varepsilon > 0. \tag{11}$$

Similar arguments can be used to point out that u_ε is a minimizer of $I_{4+\varepsilon}$ given by relation (6) when $p = 4+\varepsilon$. Consequently, we deduce that

$$\int_\Omega |\nabla u_\varepsilon|_D^{4+\varepsilon}\, dx \le \int_\Omega |\nabla g|_D^{4+\varepsilon}\, dx, \quad \forall\, \varepsilon > 0. \tag{12}$$

On the other hand, Hölder's inequality yields

$$\int_\Omega |\nabla u_\varepsilon|_D^4\, dx \le \left(\int_\Omega |\nabla u_\varepsilon|_D^{4+\varepsilon}\, dx \right)^{4/(4+\varepsilon)} m(\Omega)^{\varepsilon/(4+\varepsilon)}, \quad \forall\, \varepsilon > 0. \tag{13}$$

Combining (12) and (13) we find

$$\begin{array}{rcl} \|\, |\nabla u_\varepsilon|_D \,\|_{L^4(\Omega)} & \le & \|\, |\nabla u_\varepsilon|_D \,\|_{L^{4+\varepsilon}(\Omega)} (m(\Omega)+1)^{1/4} \\ & \le & \|\, |\nabla g|_D \,\|_{L^{4+\varepsilon}(\Omega)} (m(\Omega)+1)^{1/4} \\ & \le & \|\, |\nabla g|_D \,\|_{L^\infty(\Omega)} (m(\Omega)+1)^{1/4}, \quad \forall\, \varepsilon > 0. \end{array} \tag{14}$$

Finally, using Poincaré's inequality and relation (14) we deduce the existence of a constant $C > 0$ for which we get

$$\begin{array}{rcl} \|u_\varepsilon\|_{W^{1,4}(\Omega)} & \le & \|u_\varepsilon - g\|_{W^{1,4}(\Omega)} + \|g\|_{W^{1,4}(\Omega)} \\ & \le & C\|\, |\nabla u_\varepsilon - \nabla g|_D \,\|_{L^4(\Omega)} + \|g\|_{W^{1,4}(\Omega)} \\ & \le & C(\|\, |\nabla u_\varepsilon|_D \,\|_{L^4(\Omega)} + \|\, |\nabla g|_D \,\|_{L^4(\Omega)}) + \|g\|_{W^{1,4}(\Omega)} \\ & \le & C(\|\, |\nabla g|_D \,\|_{L^\infty(\Omega)} (m(\Omega)+1)^{1/4} + \|\, |\nabla g|_D \,\|_{L^4(\Omega)}) \\ & & +\|g\|_{W^{1,4}(\Omega)}, \quad \forall\, \varepsilon > 0. \end{array}$$

Since the right hand side of the above estimate is constant (it does not depend on $\varepsilon > 0$) we deduce the conclusion of the lemma. □

In order to go further, we recall the definition of Γ-convergence (introduced in [4], [5]) in metric spaces. The reader is referred to [3] for a comprehensive introduction to the subject.

Definition 4.2. Let Y be a metric space. A sequence $\{F_n\}$ of functionals $F_n : Y \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ is said to $\Gamma(Y)$-converge to $F : Y \to \overline{\mathbb{R}}$, and we write $\Gamma(Y) - \lim_{n\to\infty} F_n = F_\infty$, if the following hold:
(i) for every $u \in Y$ and $\{u_n\} \subset Y$ such that $u_n \to u$ in Y, we have

$$F(u) \leq \liminf_{n\to\infty} F_n(u_n);$$

(ii) for every $u \in Y$ there exists a sequence $\{u_n\} \subset Y$ (called a recovery sequence) such that $u_n \to u$ in Y and

$$F(u) \geq \limsup_{n\to\infty} F_n(u_n).$$

The following two results are well-known and can be found, e.g., in [9, Lemma 6.1.1] and [9, Corollary 6.1.1].

Proposition 4.3. *Let Y be a topological space that satisfies the first axiom of countability, and assume that $\{u_n\}$ is a sequence such that $u_n \to u$ in Y as $n \to \infty$,*

$$\limsup_{n\to\infty} F(u_n) \leq F(u),$$

and such that for every $m \in \mathbb{N}$ there exists a sequence $\{u_{m,n}\}_n$, $u_{m,n} \to u_m$ as $n \to \infty$, with

$$\limsup_{n\to\infty} F_n(u_{m,n}) \leq F(u_m).$$

Then there exists a recovering sequence for u in the sense of (ii) of Definition 4.2.

Proposition 4.4. *Let Y be a topological space satisfying the first axiom of countability, and assume that the sequence $\{F_n\}$ of functionals $F_n : Y \to \overline{\mathbb{R}}$ $\Gamma -$ converge to $F : Y \to \overline{\mathbb{R}}$. Let z_n be a minimizer for F_n. If $z_n \to z$ in X, then z is a minimizer of F, and $F(z) = \liminf_{n\to\infty} F_n(z_n)$.*

For each $\varepsilon \geq 0$ define $J_\varepsilon : L^1(\Omega) \to \overline{\mathbb{R}}$ $(n \geq 2)$ and $J_\infty : L^1(\Omega) \to \overline{\mathbb{R}}$ by

$$J_\varepsilon(u) := \begin{cases} I_{4+\varepsilon}(u) & \text{if } u \in W_g^{1,4+\varepsilon}(\Omega) \\ +\infty & \text{if } u \in L^1(\Omega) \setminus W_g^{1,4+\varepsilon}(\Omega)\,. \end{cases}$$

Lemma 4.5. $\Gamma(L^1(\Omega)) - \lim_{\varepsilon\to 0^+} J_\varepsilon = J_0$.

Proof. Let $v_\varepsilon \to v$ in $L^1(\Omega)$. If we have $\liminf_{\varepsilon\to 0^+} J_\varepsilon(v_\varepsilon) = +\infty$, there is nothing to prove. Thus, we may assume, without loss of generality, that $v_\varepsilon \in W_g^{1,4+\varepsilon}(\Omega)$ and, after eventually extracting a subsequence,

$$\liminf_{\varepsilon\to 0^+} J_\varepsilon(v_\varepsilon) = \lim_{\varepsilon\to 0^+} J_\varepsilon(v_\varepsilon) =: L < +\infty. \tag{15}$$

Since for each $\varepsilon > 0$ we have $W_g^{1,4+\varepsilon}(\Omega) \subset W_g^{1,4}(\Omega)$ then Young's inequality implies

$$\int_\Omega |\nabla v_\varepsilon|_D^4\,dx \le \frac{4}{4+\varepsilon}\int_\Omega |\nabla v_\varepsilon|_D^{4+\varepsilon}\,dx + \frac{\varepsilon}{4+\varepsilon}m(\Omega) \le (L+1) + m(\Omega) =: M,$$

for all $\varepsilon > 0$, where $m(\Omega)$ stands for the Lebesgue measure of Ω and M is a positive constant. It follows that $\{|\nabla v_\varepsilon|_D\}_\varepsilon$ is bounded in $L^4(\Omega)$. Next, for each $\varepsilon > 0$ the use of Poincaré's inequality implies the existence of a positive constant C such that

$$\begin{aligned}
\|v_\varepsilon\|_{W^{1,4}(\Omega)} &\le \|v_\varepsilon - g\|_{W^{1,4}(\Omega)} + \|g\|_{W^{1,4}(\Omega)} \\
&\le C\|\,|\nabla v_\varepsilon - \nabla g|_D\,\|_{L^4(\Omega)} + \|g\|_{W^{1,4}(\Omega)} \\
&\le C(\|\,|\nabla v_\varepsilon|_D\,\|_{L^4(\Omega)} + \|\,|\nabla g|_D\,\|_{L^4(\Omega)}) + \|g\|_{W^{1,4}(\Omega)} \\
&\le C(M^{1/4} + \|\,|\nabla g|_D\,\|_{L^4(\Omega)}) + \|g\|_{W^{1,4}(\Omega)}, \quad \forall\,\varepsilon > 0.
\end{aligned}$$

We deduce that $\{v_\varepsilon\}$ is bounded in $W^{1,4}(\Omega)$, and, thus, after eventually extracting a subsequence (not relabeled), we have $v_\varepsilon \rightharpoonup v$ weakly in $W^{1,4}(\Omega)$. Moreover, standard arguments from trace theory show that $u = g$ on $\partial\Omega$. Taking into account the fact that the functional J_0 is sequentially weakly lower semicontinuous in $W^{1,4}(\Omega)$, we obtain

$$\begin{aligned}
J_0(v) = \int_\Omega |\nabla v|_D^4\,dx &\le \liminf_{\varepsilon\to 0^+}\int_\Omega |\nabla v_\varepsilon|_D^4\,dx \\
&\le \liminf_{\varepsilon\to 0^+}\left[\frac{4}{4+\varepsilon}\int_\Omega |\nabla v_\varepsilon|_D^{4+\varepsilon}\,dx + \frac{\varepsilon}{4+\varepsilon}m(\Omega)\right] \\
&= \lim_{\varepsilon\to 0^+} J_\varepsilon(v_\varepsilon).
\end{aligned}$$

It remains to prove the existence of a recovery sequence for the Γ-limit. To this, let $v \in L^1(\Omega)$ be arbitrary, and note that if $v \notin W_g^{1,4}(\Omega)$ there is nothing to prove, since $J_0(v) = +\infty$ in this case. Next, assume that $v \in W_g^{1,4}(\Omega)$, and let $\{v_\varepsilon\} \subset C_0^\infty(\Omega)$ be such that $v_\varepsilon \to v - g$ as $\varepsilon \to 0^+$ in $W_0^{1,4}(\Omega)$. Thus, $v_\varepsilon + g \to v$

as $\varepsilon \to 0^+$ in $W^{1,4}(\Omega)$ and $v_\varepsilon + g \in W_g^{1,4+\varepsilon}(\Omega)$ for each $\varepsilon > 0$. In particular, we have $\lim\limits_{\varepsilon\to 0^+} J_0(v_\varepsilon + g) = J_0(v)$.

Claim. For each $w \in C_0^\infty(\Omega)$ we have

$$\lim_{\varepsilon\to 0^+} I_{4+\varepsilon}(w+g) = I_4(w+g)\,.$$

Indeed, since $w \in C_0^\infty(\Omega)$ and $g \in C^1(\Omega)$ it follows that $|\nabla(w+g)|_D \in L^\infty(\Omega) \subset L^5(\Omega)$. Thus, for each $\varepsilon \in (0,1)$ we have

$$|\nabla(w+g)(x)|_D^{4+\varepsilon} \le |\nabla(w+g)(x)|_D^5 + 1 \in L^1(\Omega), \quad \forall\, x \in \Omega\,.$$

On the other hand, it is clear that

$$\lim_{\varepsilon\to 0^+} |\nabla(w+g)(x)|_D^{4+\varepsilon} = |\nabla(w+g)(x)|_D^4, \quad \forall\, x \in \Omega\,.$$

Thus, a simple application of Lebesgue's dominated convergence theorem concludes the result of the claim.

Next, using the above claim we deduce that for each $\delta > 0$ small enough we have

$$\lim_{\varepsilon\to 0^+} J_\varepsilon(v_\delta + g) = J_0(v_\delta + g)\,.$$

Finally, in view of Proposition 4.3 we conclude that

$$J_0(v) \ge \limsup_{\varepsilon\to 0^+} J_\varepsilon(v_\varepsilon + g)\,.$$

The proof of Lemma 4.5 is complete. □

Now, we are ready to discuss the convergence of u_ε as $\varepsilon \to 0^+$. First, note that by Lemma 4.1 we deduce that, passing eventually to a subsequence, we have that u_ε converges weakly in $W^{1,4}(\Omega)$ and strongly in $L^1(\Omega)$ to some u_0, as $\varepsilon \to 0^+$. Next, since Lemma 4.5 holds true we can apply Proposition 4.4 with $X = L^1(\Omega)$, $F_n = J_\varepsilon$, $F = J_0$, $z_n = u_\varepsilon$ and taking into account the strong convergence of u_ε to u_0 we deduce the u_0 should be a minimizer of J_0 on $L^1(\Omega)$ and consequently of I_4 on $W_g^{1,4}(\Omega)$. Moreover,

$$\liminf_{\varepsilon\to 0^+} \int_\Omega |\nabla u_\varepsilon|_D^{4+\varepsilon}\, dx = \int_\Omega |\nabla u_0|_D^4\, dx\,.$$

On the other hand, since by Young's inequality we know that

$$\int_\Omega |\nabla u_\varepsilon|_D^4\, dx \le \frac{4}{4+\varepsilon}\int_\Omega |\nabla u_\varepsilon|_D^{4+\varepsilon}\, dx + \frac{\varepsilon}{4+\varepsilon} m(\Omega), \quad \forall\, \varepsilon > 0\,,$$

letting $\varepsilon \to 0^+$ in the last inequality and taking into acconut the previous equality we get

$$\limsup_{\varepsilon\to 0^+} \int_\Omega |\nabla u_\varepsilon|_D^4 \, dx \le \int_\Omega |\nabla u_0|_D^4 \, dx \, .$$

Next, since u_ε converges weakly to u_0 in $W^{1,4}(\Omega)$ we have

$$\int_\Omega |\nabla u_0|_D^4 \, dx \le \liminf_{\varepsilon\to 0^+} \int_\Omega |\nabla u_\varepsilon|_D^4 \, dx \, .$$

The last two inequalities lead to the conclusion that

$$\lim_{\varepsilon\to 0^+} \int_\Omega |\nabla u_\varepsilon|_D^4 \, dx = \int_\Omega |\nabla u_0|_D^4 \, dx \, .$$

The proof of Theorem 1.2 is complete.

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A. GRECU
Department of Mathematics
University of Craiova
and
Research group of the project PN-III-P1-1.1-TE- 2019-0456
Politehnica University of Bucharest
e-mail: `andreigrecu.cv@gmail.com`

M. MIHĂILESCU
Department of Mathematics
University of Craiova
and
"Gheorghe Mihoc - Caius Iacob" Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy
e-mail: `mmihailes@yahoo.com`