4-HARMONIC FUNCTIONS AND BEYOND

A. GRECU - M. MIHĂILESCU

The family of partial differential equations $-\Delta_4 u - \varepsilon \Delta_\infty u = 0$ ($\varepsilon > 0$) is studied in a bounded domain $\Omega$ for given boundary data. We show that for each $\varepsilon > 0$ the problem has a unique viscosity solution which is exactly the $(4 + \varepsilon)$-harmonic map with the given boundary data. We also explore the connections between the solutions of these problems and infinity harmonic and 4-harmonic maps by studying the limiting behavior of the solutions as $\varepsilon \to \infty$ and $\varepsilon \to 0^+$, respectively.

1. Introduction

1.1. Statement of the problem

Let $D \geq 1$ be an integer and let $\Omega \subset \mathbb{R}^D$ be a bounded domain with smooth boundary $\partial \Omega$. Next, let $g \in C^1(\Omega) \cap C(\overline{\Omega})$ be a given function. The main goal of this paper is to analyse the problem

$$\begin{cases}
-\Delta_4 u - \varepsilon \Delta_\infty u = 0 & \text{in } \Omega \\
u = g & \text{on } \partial \Omega,
\end{cases}$$

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where $\varepsilon > 0$ is a real parameter, $\Delta_4 u := \text{div}( |\nabla u|^2 \nabla u)$ is the 4-Laplace operator and $\Delta_\infty u := \sum_{i,j=1}^{\mathcal{D}} \partial_x u \partial_x j \partial_x u \partial_x i \partial_x j$ stands for the $\infty$-Laplace operator.

### 1.2. Motivation

Under the same assumptions as in the previous section the following problem was investigated

$$
\begin{align*}
-\Delta u - \varepsilon \Delta_\infty u &= 0 \quad \text{in} \quad \Omega \\
u &= g \quad \text{on} \quad \partial \Omega,
\end{align*}
$$

(2)

see, e.g. [6], [12] or [1]. By [12] we know that problem (2) has a unique classical solution, say $u_\varepsilon$ (which is, actually, an analytical function). Moreover, since problem (2) can be investigated in connection with the equation

$$
\begin{align*}
-\text{div}(e^{\varepsilon |\nabla u|^2} \nabla u) &= 0 \quad \text{in} \quad \Omega \\
u &= g \quad \text{on} \quad \partial \Omega,
\end{align*}
$$

(3)

the solution $u_\varepsilon$ minimizes the Euler-Lagrange functional associated to the problem (3), namely

$$
I(u) := \int_\Omega e^{\varepsilon |\nabla u|^2} dx,
$$

over a closed and convex subset of an Orlicz-Sobolev space defined with the aid of the $N$-function $\Phi(t) := e^{\varepsilon t^2} - 1$ and taking into account the boundary condition (see [1] for more details). Furthermore, it was proved that $u_\varepsilon$ converges uniformly over $\Omega$, as $\varepsilon \to \infty$, to the unique $\infty$-harmonic function with $g$ boundary data (see [7, Section 4.1] or [1, Proposition 4]), while in the case where $\varepsilon \to 0^+$, $u_\varepsilon$ converges in $W^{1,4}(\Omega)$ to the unique harmonic function in $\Omega$ with $g$ boundary value data (see [1, Theorem 3]).

Motivated by the studies on problem (2) here we analyse a similar problem, that is problem (1), where we replace the Laplace operator with the 4-Laplace operator. At this point a natural question which can be considered is investigating problem (1) when the 4-Laplace operator is replaced by the $p$-Laplace operator with $p \in (1, \infty) \setminus \{2, 4\}$. Unfortunately, the arguments provided for the cases $p \in \{2, 4\}$ seem to not hold true when $p \not\in \{2, 4\}$. More precisely, for $p \in \{2, 4\}$ we can prove that the viscosity solutions of problems (2) and (1) are actually variational solutions of some equivalent equations, which can be obtained as minimizers of their corresponding Euler-Lagrange functionals. That fact is crucial in the proofs of Lemmas 4.1 and 4.5 from this paper (and also in the proofs from [1] and [7] and [11] for similar results). In the case when $p \not\in \{2, 4\}$ the lack of a variational characterization for the solutions should ask for a different treatment of the problem.
1.3. Main results

The main results of this paper are formulated below in the following two theorems.

**Theorem 1.1.** For each \( g \in C^1(\Omega) \cap C(\overline{\Omega}) \) and each \( \varepsilon > 0 \) problem (1) has a unique viscosity solution which is exactly the \((4 + \varepsilon)\)-harmonic map with boundary values \( g \).

**Theorem 1.2.** Let \( g \in C^1(\Omega) \cap C(\overline{\Omega}) \) and for each \( \varepsilon > 0 \) let \( u_\varepsilon \) be the unique viscosity solution of problem (1). Then \( u_\varepsilon \) converges uniformly over \( \Omega \), as \( \varepsilon \to \infty \), to the unique \( \infty \)-harmonic function with \( g \) boundary data. Moreover, letting \( u_0 \) be the 4-harmonic map with \( g \) boundary data then \( u_\varepsilon \) converges weakly to \( u_0 \) in \( W^{1,4}(\Omega) \), as \( \varepsilon \to 0^+ \), and

\[
\lim_{\varepsilon \to 0^+} \int_{\Omega} |\nabla u_\varepsilon|^4_D \, dx = \int_{\Omega} |\nabla u_0|^4_D \, dx.
\]

The rest of the paper is structured as follows: in Section 2 we present some well-known results on \( p \)-harmonic functions and on \( \infty \)-harmonic functions which will be useful in our subsequent analysis; in Section 3 we give the proof of Theorem 1.1; in Section 4 we present the proof of Theorem 1.2.

2. A quick overview on \( p \)-harmonic functions and \( \infty \)-harmonic functions

2.1. \( p \)-harmonic functions

Let \( \Omega \subset \mathbb{R}^D \) be a bounded domain with smooth boundary \( \partial \Omega \), and let \( g \in C^1(\Omega) \cap C(\overline{\Omega}) \) be given. For each real number \( p \in (1, \infty) \) we consider the problem

\[
\begin{align*}
-\Delta_p u &= 0 & \text{in} & & \Omega \\
        u &= g & \text{on} & & \partial \Omega,
\end{align*}
\]

where \( \Delta_p u := \text{div}(|\nabla u|^{p-2}_{D} \nabla u) \) is the \( p \)-Laplace operator. The solutions of problem (4) are sought in the set

\[
W_{g}^{1,p}(\Omega) := \{ u \in W^{1,p}(\Omega) : u = g \text{ on } \partial \Omega \},
\]

which is a closed and convex subset of the Sobolev space \( W^{1,p}(\Omega) \), and are understood in a variational sense. More precisely, \( u_p \) is a week solution of problem (4) if \( u_p \in W_{g}^{1,p}(\Omega) \) and satisfies

\[
\int_{\Omega} |\nabla u_p|^{p-2}_{D} \nabla u_p \nabla \phi \, dx = 0, \quad \forall \phi \in W^{1,p}_0(\Omega).
\]
Such a function is also called a $p$-harmonic function with $g$ boundary data (see also [10, Definition 2.1] for the definition of a $p$-harmonic function). It is well-known (see, e.g. [13, Theorem 2.4] or [14, Theorem 2.16]) that problem (4) has a unique weak solution which turns out to be the unique minimizer on $W^{1,p}_g(\Omega)$ of the Euler-Lagrange functional associated with equation (4), namely, $I_p : W^{1,p}(\Omega) \to \mathbb{R}$ given by

$$ I_p(u) := \int_{\Omega} |\nabla u|^p_D \, dx. \quad (6) $$

Note also that when $\Omega$ is bounded and connected with the boundary $\partial \Omega$ of class $C^{1,\alpha}$ and $g \in W^{2,p}(\Omega)$ then $u_p \in C^{1,\alpha}_{\text{loc}}(\Omega)$ (see, e.g. [11, Section 2] or [14, Theorem 2.19]).

Next, we point out the fact that the weak solutions of problem (4) are equivalent with the viscosity solutions of the same equation (see, e.g. [10, Corollary 2.8]). Let us recall the definition of a viscosity solution of an equation of type

$$ \begin{cases} 
F(\nabla u, D^2 u) = 0 & \text{in } \Omega, \\
\quad u = g & \text{on } \partial \Omega, 
\end{cases} \quad (7) $$

where $D^2 u$ stands for the Hessioan matrix of $u$ (see, e.g. [2]).

**Definition 2.1.** (i) An upper semicontinuous function $u : \Omega \to \mathbb{R}$ is called a viscosity subsolution of (7) if $u|_{\partial \Omega} \leq g$ and, whenever $x_0 \in \Omega$ and $\Psi \in C^2(\Omega)$ are such that $u(x_0) = \Psi(x_0)$ and $u(x) < \Psi(x)$ if $x \in B_r(x_0) \setminus \{x_0\}$ for some $r > 0$, then we have $F(\nabla \Psi(x_0), D^2 \Psi(x_0)) \leq 0$;

(ii) A lower semicontinuous function $u : \Omega \to \mathbb{R}$ is called a viscosity supersolution of (7) if $u|_{\partial \Omega} \geq g$ and, whenever $x_0 \in \Omega$ and $\Psi \in C^2(\Omega)$ are such that $u(x_0) = \Psi(x_0)$ and $u(x) > \Psi(x)$ if $x \in B_r(x_0) \setminus \{x_0\}$ for some $r > 0$, then $F(\nabla \Psi(x_0), D^2 \Psi(x_0)) \geq 0$;

(iii) A continuous function $u : \Omega \to \mathbb{R}$ is called a viscosity solution of (7) if it is both a viscosity subsolution and a viscosity supersolution of (7).

Note that problem (4) is an equation of type (7) since if we assume that $u : \Omega \to \mathbb{R}$ is a sufficiently smooth function then the $p$-Laplacian of $u$ becomes

$$ \Delta_p u = |\nabla u|^p_D - 4 \left| |\nabla u|^2_D \Delta u + (p-2)\Delta_{\infty} u \right|, \quad (8) $$

or, taking into account that $\Delta u = \text{Trace}(D^2 u)$ and $\Delta_{\infty} u = \langle D^2 u \nabla u, \nabla u \rangle$ we get that

$$ \Delta_p u = F_p(\nabla u, D^2 u), $$

where

$$ F_p(\xi, S) := |\xi|^p_D - 4 \left( |\xi|^2_D \text{Trace}(S) + (p-2)\langle S\xi, \xi \rangle \right), $$
when $\xi \in \mathbb{R}^D$ and $S \in M_{sym}^{D \times D}(\mathbb{R})$. If $p \in (1, 2)$ the function $F_p$ is not defined at $\xi = 0$ (or $\nabla u = 0$). Consequently, this case requires special attention in relation with the definition of a viscosity solution (that is Definition 2.1 above). More precisely, in order to fix this problem we have to add the requirement $\nabla u(x_0) \neq 0$ in the definition of a viscosity solution. Consequently, there is no condition to be verified at the critical points in the definition of a viscosity solution. Note also that when $p \in [2, \infty)$ the above problem does not appear since $\Delta_p u(x_0) = 0$ in that case (see, e.g., [14, p. 78] for more details).

2.2. $\infty$-harmonic functions

Under the same assumptions as in the previous section we consider the equation

$$\begin{cases} -\Delta_{\infty} u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$

(9)

We call a viscosity solution of problem (9) an $\infty$-harmonic function with $g$ boundary data. Let $u_\infty$ be the unique (see [8]) viscosity solution of problem (9). It can be shown that the family of $p$-harmonic functions with $g$ boundary data, $u_p$, converges uniformly over $\Omega$, as $p \to \infty$, to $u_\infty$ (see, e.g., [11, Section 2] for a simple and quick explanation).

3. Proof of Theorem 1.1

First, note that when $p = 4$ relation (8) implies that

$$\Delta_4 u = |\nabla u|_D^2 \Delta u + 2\Delta_{\infty} u.$$

Using that fact we observe that for each $p \in (4, \infty)$ we can rewrite relation (8) in the following way

$$\Delta_p u = |\nabla u|_D^{p-4} (\Delta_4 u + (p-4)\Delta_{\infty} u).$$

Thus, for each $\varepsilon > 0$ we have

$$\Delta_{4+\varepsilon} u = |\nabla u|_D^{\varepsilon} (\Delta_4 u + \varepsilon\Delta_{\infty} u).$$

(10)

Lemma 3.1. For each $\varepsilon > 0$ a function $u_\varepsilon$ is a viscosity solution of problem (1) if and only if it is a viscosity solution of problem (4) when $p = 4 + \varepsilon$.

Proof. Step I: Equivalence of the viscosity supersolutions.

• Let $u_\varepsilon$ be a viscosity supersolution of problem (1). We show that it is a viscosity supersolution of problem (4) when $p = 4 + \varepsilon$, too.
Indeed, first note that $u_{\varepsilon} = g$ on $\partial \Omega$. Next, let $x_0 \in \Omega$ and $\Psi \in C^2(\Omega)$ be such that $u_{\varepsilon}(x_0) = \Psi(x_0)$ and $u_{\varepsilon}(x) > \Psi(x)$ if $x \in B_r(x_0) \setminus \{x_0\}$ for some $r > 0$. Then

$$-\Delta_4 \Psi(x_0) - \varepsilon \Delta_\infty \Psi(x_0) \geq 0.$$ It follows that

$$-|\nabla \Psi(x_0)|_D^p (\Delta_4 \Psi(x_0) + \varepsilon \Delta_\infty \Psi(x_0)) \geq 0,$$ or, by (10) we deduce that

$$-\Delta_{4+\varepsilon} \Psi(x_0) \geq 0,$$

which shows that $u_{\varepsilon}$ is a viscosity supersolution of problem (4) when $p = 4 + \varepsilon$.

Let $u_{\varepsilon}$ be a viscosity supersolution of problem (4) when $p = 4 + \varepsilon$. We show that it is a viscosity supersolution of problem (1), too.

It is obvious that the boundary data condition is satisfied. Further, let $x_0 \in \Omega$ and $\Psi \in C^2(\Omega)$ be such that $u_{\varepsilon}(x_0) = \Psi(x_0)$ and $u_{\varepsilon}(x) > \Psi(x)$ if $x \in B_r(x_0) \setminus \{x_0\}$ for some $r > 0$. Then

$$-\Delta_4 \Psi(x_0) - \varepsilon \Delta_\infty \Psi(x_0) \geq 0,$$

or, by (10) we have

$$-|\nabla \Psi(x_0)|_D^p (\Delta_4 \Psi(x_0) + \varepsilon \Delta_\infty \Psi(x_0)) \geq 0.$$ If $|\nabla \Psi(x_0)|_D > 0$ then we get

$$-\Delta_4 \Psi(x_0) - \varepsilon \Delta_\infty \Psi(x_0) \geq 0,$$

which means that $u_{\varepsilon}$ is a viscosity supersolution of problem (1). Otherwise, if $|\nabla \Psi(x_0)|_D = 0$ then $\frac{\partial \Psi}{\partial x_i}(x_0) = 0$ for each integer $i \in \{1, \ldots, D\}$ and taking into account the definitions of the 4-Laplacian and the $\infty$-Laplacian we conclude that

$$-\Delta_4 \Psi(x_0) - \varepsilon \Delta_\infty \Psi(x_0) = 0,$$

which leads again to the conclusion that $u_{\varepsilon}$ is a viscosity supersolution of problem (1).

**Step II:** Equivalence of the viscosity subsolutions.

The analysis of this case can be done similarly with the one from the first part of this proof and consequently we omit it.

**Proof of Theorem 1.1 (concluded).** Let $\varepsilon > 0$ be arbitrary but fixed. By [13, Theorem 2.4] or [14, Theorem 2.16] the problem (4) with $p = 4 + \varepsilon$ has a unique weak solution, say $u_{\varepsilon}$, which is a $(4 + \varepsilon)$-harmonic map (with boundary values $g$). Moreover, $u_{\varepsilon}$ is a continuous function. Then by [10, Corollary 2.8] we infer that $u_{\varepsilon}$ is a viscosity solution of problem (4) with $p = 4 + \varepsilon$. Combining that fact with Lemma 3.1 it follows that $u_{\varepsilon}$ is the unique viscosity solution of problem (1).
4. Proof of Theorem 1.2

By Theorem 1.1 we deduce that for each \( \epsilon > 0 \) the unique viscosity solution of problem (1), \( u_\epsilon \), is a \((4 + \epsilon)\)-harmonic function with boundary values \( g \) (or the unique weak solution of problem (4) with \( p = 4 + \epsilon \)). Then it is well-known that \( u_\epsilon \) converges uniformly, as \( \epsilon \to \infty \) to the unique infinite-harmonic function with boundary values \( g \) (see, e.g. [11, Section 2]).

Next, we analyse the convergence of \( u_\epsilon \) as \( \epsilon \to 0^+ \).

We start by showing the following result:

**Lemma 4.1.** The sequence \( \{u_\epsilon\}_{\epsilon > 0} \) is bounded in \( W^{1,4}(\Omega) \).

**Proof.** For each \( \epsilon > 0 \), \( u_\epsilon \in W^{1,4+\epsilon}_g(\Omega) \subset W^{1,4}_g(\Omega) \). Let \( u_0 \) be the 4-harmonic map with boundary values \( g \). By [13, Theorem 2.4] or [14, Theorem 2.16], \( u_0 \) is a minimizer of \( I_4 \) given by relation (6) when \( p = 4 \). Consequently, we have

\[
\int_\Omega |\nabla u_0|^4_D \, dx \leq \int_\Omega |\nabla u_\epsilon|^4_D \, dx, \quad \forall \epsilon > 0. \tag{11}
\]

Similar arguments can be used to point out that \( u_\epsilon \) is a minimizer of \( I_{4+\epsilon} \) given by relation (6) when \( p = 4 + \epsilon \). Consequently, we deduce that

\[
\int_\Omega |\nabla u_\epsilon|^{4+\epsilon}_D \, dx \leq \int_\Omega |\nabla g|^{4+\epsilon}_D \, dx, \quad \forall \epsilon > 0. \tag{12}
\]

On the other hand, Hölder’s inequality yields

\[
\int_\Omega |\nabla u_\epsilon|^4_D \, dx \leq \left( \int_\Omega |\nabla u_\epsilon|^{4+\epsilon}_D \, dx \right)^{4/(4+\epsilon)} m(\Omega)^{\epsilon/(4+\epsilon)}, \quad \forall \epsilon > 0. \tag{13}
\]

Combining (12) and (13) we find

\[
\left\| |\nabla u_\epsilon|^4_D \right\|_{L^4(\Omega)} \leq \left\| |\nabla u_\epsilon|^{4+\epsilon}_D \right\|_{L^{4+\epsilon}(\Omega)} \left( m(\Omega) + 1 \right)^{1/4} \leq \left\| |\nabla g|^4_D \right\|_{L^{4+\epsilon}(\Omega)} \left( m(\Omega) + 1 \right)^{1/4}, \quad \forall \epsilon > 0. \tag{14}
\]

Finally, using Poincaré’s inequality and relation (14) we deduce the existence of a constant \( C > 0 \) for which we get

\[
\left\| u_\epsilon \right\|_{W^{1,4}(\Omega)} \leq \left\| u_\epsilon - g \right\|_{W^{1,4}(\Omega)} + \left\| g \right\|_{W^{1,4}(\Omega)} \leq C \left\| |\nabla u_\epsilon| - |\nabla g|^4_D \right\|_{L^4(\Omega)} + \left\| |\nabla g|^4_D \right\|_{L^1(\Omega)} + \left\| g \right\|_{W^{1,4}(\Omega)} \leq C \left( \left\| |\nabla g|_D \right\|_{L^\infty(\Omega)} \left( m(\Omega) + 1 \right)^{1/4} + \left\| |\nabla g|^4_D \right\|_{L^1(\Omega)} \right) + \left\| g \right\|_{W^{1,4}(\Omega)}, \quad \forall \epsilon > 0.
\]
Since the right hand side of the above estimate is constant (it does not depend on $\varepsilon > 0$) we deduce the conclusion of the lemma.

In order to go further, we recall the definition of $\Gamma$-convergence (introduced in [4], [5]) in metric spaces. The reader is referred to [3] for a comprehensive introduction to the subject.

**Definition 4.2.** Let $Y$ be a metric space. A sequence $\{F_n\}$ of functionals $F_n : Y \to \mathbb{R} := \mathbb{R} \cup \{\infty\}$ is said to $\Gamma(Y)$-converge to $F : Y \to \mathbb{R}$, and we write $\Gamma(Y) - \lim_{n \to \infty} F_n = F_\infty$, if the following hold:

(i) for every $u \in Y$ and $\{u_n\} \subset Y$ such that $u_n \to u$ in $Y$, we have

$$F(u) \leq \liminf_{n \to \infty} F_n(u_n);$$

(ii) for every $u \in Y$ there exists a sequence $\{u_n\} \subset Y$ (called a recovery sequence) such that $u_n \to u$ in $Y$ and

$$F(u) \geq \limsup_{n \to \infty} F_n(u_n).$$

The following two results are well-known and can be found, e.g., in [9, Lemma 6.1.1] and [9, Corollary 6.1.1].

**Proposition 4.3.** Let $Y$ be a topological space that satisfies the first axiom of countability, and assume that $\{u_n\}$ is a sequence such that $u_n \to u$ in $Y$ as $n \to \infty$,

$$\limsup_{n \to \infty} F(u_n) \leq F(u),$$

and such that for every $m \in \mathbb{N}$ there exists a sequence $\{u_{m,n}\}$, $u_{m,n} \to u_m$ as $n \to \infty$, with

$$\limsup_{n \to \infty} F_n(u_{m,n}) \leq F(u_m).$$

Then there exists a recovering sequence for $u$ in the sense of (ii) of Definition 4.2.

**Proposition 4.4.** Let $Y$ be a topological space satisfying the first axiom of countability, and assume that the sequence $\{F_n\}$ of functionals $F_n : Y \to \mathbb{R}$ $\Gamma$-converge to $F : Y \to \mathbb{R}$. Let $z_n$ be a minimizer for $F_n$. If $z_n \to z$ in $X$, then $z$ is a minimizer of $F$, and $F(z) = \liminf_{n \to \infty} F_n(z_n)$.

For each $\varepsilon \geq 0$ define $J_\varepsilon : L^1(\Omega) \to \mathbb{R}$ ($n \geq 2$) and $J_\infty : L^1(\Omega) \to \mathbb{R}$ by

$$J_\varepsilon(u) := \begin{cases} 
I_{4+\varepsilon}(u) + \infty & \text{if } u \in W_g^{1,4+\varepsilon}(\Omega) \\
\infty & \text{if } u \in L^1(\Omega) \setminus W_g^{1,4+\varepsilon}(\Omega).
\end{cases}$$
Lemma 4.5. $\Gamma(L^1(\Omega)) - \lim_{\varepsilon \to 0^+} J_\varepsilon = J_0$.

Proof. Let $v_\varepsilon \to v$ in $L^1(\Omega)$. If we have $\liminf_{\varepsilon \to 0^+} J_\varepsilon(v_\varepsilon) = +\infty$, there is nothing to prove. Thus, we may assume, without loss of generality, that $v_\varepsilon \in W^{1,4+\varepsilon}_g(\Omega)$ and, after eventually extracting a subsequence,

$$
\liminf_{\varepsilon \to 0^+} J_\varepsilon(v_\varepsilon) = \lim_{\varepsilon \to 0^+} J_\varepsilon(v_\varepsilon) =: L < +\infty. \tag{15}
$$

Since for each $\varepsilon > 0$ we have $W^{1,4+\varepsilon}_g(\Omega) \subset W_g^{1,4}(\Omega)$ then Young’s inequality implies

$$
\int_{\Omega} |\nabla v_\varepsilon|^4_D \, dx \leq \frac{4}{4+\varepsilon} \int_{\Omega} |\nabla v_\varepsilon|^{4+\varepsilon}_D \, dx + \frac{\varepsilon}{4+\varepsilon} m(\Omega) \leq (L+1) + m(\Omega) =: M,
$$

for all $\varepsilon > 0$, where $m(\Omega)$ stands for the Lebesgue measure of $\Omega$ and $M$ is a positive constant. It follows that $\{|\nabla v_\varepsilon|_D\}_{\varepsilon}$ is bounded in $L^4(\Omega)$. Next, for each $\varepsilon > 0$ the use of Poincaré’s inequality implies the existence of a positive constant $C$ such that

$$
\|v_\varepsilon\|_{W^{1,4}(\Omega)} \leq \|v_\varepsilon - g\|_{W^{1,4}(\Omega)} + \|g\|_{W^{1,4}(\Omega)} \leq C\|\nabla v_\varepsilon - \nabla g\|_D \|\nabla g\|_{L^4(\Omega)} + \|g\|_{W^{1,4}(\Omega)} \leq C(M^{1/4} + \|\nabla g\|_D \|\nabla g\|_{L^4(\Omega)}) + \|g\|_{W^{1,4}(\Omega)}, \quad \forall \varepsilon > 0.
$$

We deduce that $\{v_\varepsilon\}$ is bounded in $W^{1,4}(\Omega)$, and, thus, after eventually extracting a subsequence (not relabeled), we have $v_\varepsilon \rightharpoonup v$ weakly in $W^{1,4}(\Omega)$. Moreover, standard arguments from trace theory show that $u = g$ on $\partial \Omega$. Taking into account the fact that the functional $J_0$ is sequentially weakly lower semicontinuous in $W^{1,4}(\Omega)$, we obtain

$$
J_0(v) = \int_{\Omega} |\nabla v|^4_D \, dx \leq \liminf_{\varepsilon \to 0^+} \int_{\Omega} |\nabla v_\varepsilon|^4_D \, dx \leq \liminf_{\varepsilon \to 0^+} \left[ \frac{4}{4+\varepsilon} \int_{\Omega} |\nabla v_\varepsilon|^{4+\varepsilon}_D \, dx + \frac{\varepsilon}{4+\varepsilon} m(\Omega) \right] = \lim_{\varepsilon \to 0^+} J_\varepsilon(v_\varepsilon).
$$

It remains to prove the existence of a recovery sequence for the $\Gamma$-limit. To this, let $v \in L^1(\Omega)$ be arbitrary, and note that if $v \not\in W_g^{1,4}(\Omega)$ there is nothing to prove, since $J_0(v) = +\infty$ in this case. Next, assume that $v \in W_g^{1,4}(\Omega)$, and let $\{v_\varepsilon\} \subset C_0^\infty(\Omega)$ be such that $v_\varepsilon \rightharpoonup v - g$ as $\varepsilon \to 0^+$ in $W^{1,4}_0(\Omega)$. Thus, $v_\varepsilon + g \rightharpoonup v$
as \( \varepsilon \to 0^+ \) in \( W^{1,4}(\Omega) \) and \( v_\varepsilon + g \in W^{1,4+\varepsilon}_g(\Omega) \) for each \( \varepsilon > 0 \). In particular, we have \( \lim_{\varepsilon \to 0^+} J_0(v_\varepsilon + g) = J_0(v) \).

**Claim.** For each \( w \in C^\infty_0(\Omega) \) we have

\[
\lim_{\varepsilon \to 0^+} I_4(\varepsilon w + g) = I_4(w + g).
\]

Indeed, since \( w \in C^\infty_0(\Omega) \) and \( g \in C^1(\Omega) \) it follows that \( |\nabla (w + g)|_D \in L^\infty(\Omega) \subset L^5(\Omega) \). Thus, for each \( \varepsilon \in (0, 1) \) we have

\[
|\nabla (w + g)(x)|_D^{4+\varepsilon} \leq |\nabla (w + g)(x)|_D^{5} + 1 \in L^1(\Omega), \quad \forall \, x \in \Omega.
\]

On the other hand, it is clear that

\[
\lim_{\varepsilon \to 0^+} |\nabla (w + g)(x)|_D^{4+\varepsilon} = |\nabla (w + g)(x)|_D^{4}, \quad \forall \, x \in \Omega.
\]

Thus, a simple application of Lebesgue’s dominated convergence theorem concludes the result of the claim.

Next, using the above claim we deduce that for each \( \delta > 0 \) small enough we have

\[
\lim_{\varepsilon \to 0^+} J_\varepsilon(v_\delta + g) = J_0(v_\delta + g).
\]

Finally, in view of Proposition 4.3 we conclude that

\[
J_0(v) \geq \limsup_{\varepsilon \to 0^+} J_\varepsilon(v_\varepsilon + g).
\]

The proof of Lemma 4.5 is complete. \( \square \)

Now, we are ready to discuss the convergence of \( u_\varepsilon \) as \( \varepsilon \to 0^+ \). First, note that by Lemma 4.1 we deduce that, passing eventually to a subsequence, we have that \( u_\varepsilon \) converges weakly in \( W^{1,4}(\Omega) \) and strongly in \( L^1(\Omega) \) to some \( u_0 \), as \( \varepsilon \to 0^+ \). Next, since Lemma 4.5 holds true we can apply Proposition 4.4 with \( X = L^1(\Omega) \), \( F_n = J_\varepsilon \), \( F = J_0 \), \( z_n = u_\varepsilon \) and taking into account the strong convergence of \( u_\varepsilon \) to \( u_0 \) we deduce the \( u_0 \) should be a minimizer of \( J_0 \) on \( L^1(\Omega) \) and consequently of \( I_4 \) on \( W^{1,4}_g(\Omega) \). Moreover,

\[
\liminf_{\varepsilon \to 0^+} \int_\Omega |\nabla u_\varepsilon|_D^{4+\varepsilon} \, dx = \int_\Omega |\nabla u_0|_D^4 \, dx.
\]

On the other hand, since by Young’s inequality we know that

\[
\int_\Omega |\nabla u_\varepsilon|_D^4 \, dx \leq \frac{4}{4+\varepsilon} \int_\Omega |\nabla u_\varepsilon|_D^{4+\varepsilon} \, dx + \frac{\varepsilon}{4+\varepsilon} m(\Omega), \quad \forall \, \varepsilon > 0,
\]

as \( \varepsilon \to 0^+ \) in \( W^{1,4}(\Omega) \) and \( v_\varepsilon + g \in W^{1,4+\varepsilon}_g(\Omega) \) for each \( \varepsilon > 0 \). In particular, we have \( \lim_{\varepsilon \to 0^+} J_0(v_\varepsilon + g) = J_0(v) \).

**Claim.** For each \( w \in C^\infty_0(\Omega) \) we have

\[
\lim_{\varepsilon \to 0^+} I_4(\varepsilon w + g) = I_4(w + g).
\]
letting $\varepsilon \to 0^+$ in the last inequality and taking into account the previous equality we get
\[
\limsup_{\varepsilon \to 0^+} \int_\Omega |\nabla u_\varepsilon|_D^4 \, dx \leq \int_\Omega |\nabla u_0|_D^4 \, dx.
\]
Next, since $u_\varepsilon$ converges weakly to $u_0$ in $W^{1,4}(\Omega)$ we have
\[
\int_\Omega |\nabla u_0|_D^4 \, dx \leq \liminf_{\varepsilon \to 0^+} \int_\Omega |\nabla u_\varepsilon|_D^4 \, dx.
\]
The last two inequalities lead to the conclusion that
\[
\lim_{\varepsilon \to 0^+} \int_\Omega |\nabla u_\varepsilon|_D^4 \, dx = \int_\Omega |\nabla u_0|_D^4 \, dx.
\]
The proof of Theorem 1.2 is complete.

REFERENCES


A. GRECU
Department of Mathematics
University of Craiova
and
Research group of the project PN-III-P1-1.1-TE- 2019-0456
Politehnica University of Bucharest
e-mail: andreigrecu.cv@gmail.com

M. MIHĂILESCU
Department of Mathematics
University of Craiova
and
"Gheorghe Mihoc - Caius Iacob“ Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy
e-mail: mmihales@yahoo.com