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TWO PROPERTIES OF NORMS IN ORLICZ SPACES

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A characterization of the inclusion between L^p-spaces is well-known (see for instance [7], [3]). Here we present an analogous characterization for Orlicz Spaces. To this aim we use some definitions of the Orlicz and Luxemburg norms that are a little bit general than usual. Also this allows us to extend to Orlicz spaces the well-known property that in a finite measure space the L^p-norm tends to the L^∞-norm as $p \to +\infty$.

1. Preliminaries.

1.1. *Young Functions.* Throughout this paper, the term Young function will have a little more restrictive meaning than the usual one (see [5]). In fact we assume that the function M in the definition below is left continuous. This assumption assures the uniqueness of the integral representation of M and, on the other hand, does not imply any restriction on the associated Orlicz space L_M (see Remark 4).

Definition 1. *By a Young function M we mean a function $M : \overline{\mathbb{R}} \longrightarrow \overline{\mathbb{R}}$ satisfying the following conditions:*

a) *M is convex on $\mathbb{R}$;*
b) *M is even on $\overline{\mathbb{R}}$, $M(0) = 0$ and $M(\pm\infty) = +\infty$;*
c) *M is such that $\lim_{x \to c^-} M(x) = M(c)$ where $c = \sup\{x \in \overline{\mathbb{R}} : M(x) < +\infty\}$.*

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The following characterization of Young functions is easy checked to hold true.

Proposition 1. *A function $M : \overline{\mathbb{R}} \longrightarrow \overline{\mathbb{R}}$ is a Young function if and only if it admits an integral representation of the form*

$$M(x) = \int_0^{|x|} p(t)\,dt \quad \forall x \in \overline{\mathbb{R}},$$

where $p : [0, +\infty[\longrightarrow [0, +\infty]$ is a function that satisfies the following conditions:

i) p is increasing,
ii) p is right continuous,
iii) p is different from the constant functions $\mathbf{0}$ and ∞;

moreover, such a function p is unique.

Starting from a function p that satisfies the above conditions *i), ii)* and *iii),* we can define the *right inverse function* $q : [0, +\infty[\longrightarrow [0, +\infty]$ of p by the position:

$$q(s) = \begin{cases} 0 & \text{if} \quad \{x \in [0, +\infty[: p(x) \leq s\} = \emptyset \\ \sup\{x \in [0, +\infty[: p(x) \leq s\} & \text{if} \quad \{x \in [0, +\infty[: p(x) \leq s\} \neq \emptyset \end{cases}.$$

From this definition it follows that q is increasing and that the following inequalities hold:

- $q(p(t)) \geq t \quad \forall t \in [0, +\infty[$,
- $p(q(s)) \geq s \quad \forall s \in [0, +\infty[$,
- $q(p(t) - \epsilon) \leq t \quad \forall t \in \{t \in [0, +\infty[: 0 < p(x) < +\infty\}, \forall \epsilon \in]0, p(t)]$,
- $p(q(s) - \epsilon) \leq s \quad \forall t \in \{s \in [0, +\infty[: 0 < q(x) < +\infty\}, \forall \epsilon \in]0, q(s)]$.

Moreover the set $\{s \in [0, +\infty[: q(s) \geq \alpha\}$ is closed in $[0, +\infty[$ for each $\alpha \in \mathbb{R}$, in fact it results

$$\{s \in [0, +\infty[: q(s) \geq \alpha\} = \begin{cases} [0, +\infty[& \text{if} \quad \alpha \leq 0 \\ [\lim_{t \to \alpha^-} p(t), +\infty[& \text{if} \quad \alpha > 0 \end{cases},$$

whereas $\{s \in [0, +\infty[: q(s) < +\infty\}$ is open in $[0, +\infty[$. The above facts imply that q is like p and that the right inverse function of q is just p.

Two Young functions M and N are called *complementary Young functions* provided that their integral representations

$$M(x) = \int_0^{|x|} p(t)\,dt \quad \forall x \in \overline{\mathbb{R}} \quad \text{and} \quad N(y) = \int_0^{|y|} q(s)\,ds \quad \forall y \in \overline{\mathbb{R}},$$

hold for some functions p and q right inverse to each other.
The following theorem is well known (see for example [5]):

Theorem (Young Inequality). *Let M and N be complementary Young functions. Then*

$$xy \leq M(x) + N(y) \quad \forall x, y \in [0, +\infty]$$

and equality occurs if and only if at least one of the two equalities, $y = p(x)$ and $x = q(y)$ holds.

In the next sections the following simple classification of the Young functions will be useful.

Definition 2. *We say that a Young function M is:*

- *Superlinear if and only if* $\lim_{x\to+\infty} \frac{M(x)}{x} = +\infty$;
- *Sublinear if and only if* $\lim_{x\to+\infty} \frac{M(x)}{x} \in [0, +\infty[$.

For the sake of convenience we call *positive* a Young function M, provided that $M(x) = 0$ if and only if $x = 0$, *finite* provided that $M(x) < +\infty \quad \forall x \in \mathbb{R}$, *null* provided that there exists $0 < c < +\infty$ such that

$$M(x) = \begin{cases} 0 & \text{if} \quad |x| \leq c \\ +\infty & \text{if} \quad |x| > c \end{cases}.$$

We will refer to the previous function as the null Young function *pointed at c*.

It is clear that whenever M and N are complementary Young functions, then M is not finite if and only if N is sublinear.

Finally, M^{-1} will denote the inverse function of the restriction of a finite and positive Young function M to $[0, +\infty]$.

1.2 L_M *Spaces.* Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. We will assume throughout that $\mathcal{A}$ contains a set of positive and finite measure. Moreover we identify two measurable functions provided that they are almost everywhere equal. As usual we denote by χ_A the characteristic function of a set A. We refer to [6] for concepts and basic results of functional analysis.

For a given function M, we will consider some subsets of the set of all measurable functions $f : \Omega \to \overline{\mathbb{R}}$, namely: $S_{M,a} = \{f : \int M(f)d\mu \leq a\}$ $\forall a \in]0, +\infty[$ and $S_M = \bigcup_{0<a<+\infty} S_{M,a} = \{f : \int M(f)d\mu < +\infty\}$. By the above assumption on $\mathcal{A}$, all sets $S_{M,a}$ and S_M are not empty. Moreover by the dominated convergence theorem we have that S_M is contained in the vector space generated by each $S_{M,a}$. Since $S_{M,a} \subseteq S_M$ for each $a \in]0, +\infty[$, it follows that all sets $S_{M,a}$ and S_M span the same vector space, which will be denoted by L_M.

The properties of M and the Jensen inequality imply that all $S_{M,a}$'s (and hence also S_M) are convex, balanced, absorbing sets in L_M.

In the following proposition we define the Luxemburg a-norms $\|\cdot\|_{(M),a}$ on L_M, that are slight generalizations of the Luxemburg norm, and for sake of entirety, we furnish a new proof of the completeness of the space.

Proposition 2. *Let M be a Young function. Then the position*

$$\|f\|_{(M),a} = \inf\left\{\rho > 0 : \int M\left(\frac{f}{\rho}\right) d\mu \le a\right\} \quad \forall f \in L_M$$

defines a complete norm on L_M for each $a \in]0, +\infty[$. Moreover, all the norms $\|\cdot\|_{(M),a}$, $a \in]0, +\infty[$, are equivalent.

Proof. The functional $\|\cdot\|_{(M),a}$ is a seminorm on L_M because it is the Minkowski functional relative to the set $S_{M,a}$. To show that $\|\cdot\|_{(M),a}$ is a norm, fix $f \in L_M$, $f \neq 0$, and assume by contradiction that $\|f\|_{(M),a} = 0$, i.e. $\int M(\frac{f}{\rho})\, d\mu \le a\ \forall \rho > 0$. Letting $\rho \to 0$, by Fatou's lemma we get the contradiction. Now, observe that $S_{M,a}$ is equal to the closed unit ball $B = \{f \in L_M : \|f\|_{(M),a} \le 1\}$. Indeed it is obvious that $S_{M,a} \subseteq B$. On the other hand, $S_{M,a}$ contains the open unit ball $\{f \in L_M : \|f\|_{(M),a} < 1\}$ and this fact, by the Monotone Convergence theorem, implies that $B \subseteq S_{M,a}$. Let us prove that the norm $\|\cdot\|_{(M),a}$ is complete. We must show that for each sequence $\{f_n\} \subseteq S_{M,a}$, and each sequence $\{\lambda_n\} \subseteq [0,1]$ such that $\sum_{n=1}^{+\infty} \lambda_n = 1$, then $\sum_{n=1}^{+\infty} \lambda_n f_n \in S_{M,a}$. In fact we have $\int M\big(\sum_{i=1}^{n} \lambda_i |f_i|\big)\, d\mu \le \sum_{i=1}^{n} \lambda_i \int M(f_i) \le a \quad \forall n \in \mathbb{N}$, and hence, by the Monotone Convergence theorem, $\int M\big(\sum_{n=1}^{+\infty} \lambda_n |f_n|\big)\, d\mu \le a$. Thus, the properties of M imply that the function $\sum_{n=1}^{+\infty} \lambda_n f_n$ is defined a.e. on Ω and that it belongs to $S_{M,a}$.

Finally, the equivalence of the norms $\|\cdot\|_{(M),a}$, $a \in]0, +\infty[$, follows by a corollary to the Open Mapping theorem, from the fact that $(L_M, \|\cdot\|_{(M),a})$ is complete for each $a \in]0, +\infty[$ and from the inequalities $\|\cdot\|_{(M),b} \le \|\cdot\|_{(M),a}$ whenever $0 < a < b$.

Remark 1. Let M be a Young function. Then for $0 < a < b$ we have the inequality $a\|f\|_{(M),a} \le b\|f\|_{(M),b}\ \forall f \in L_M$. This fact can be easily checked estimating the Luxemburg a-norm of $\frac{f}{\gamma}$, where $\gamma = \frac{b}{a}$. In this way we obtain a direct proof of the equivalence of all norms $\|\cdot\|_{(M),a}$.

For $a = 1$ the norm $\|\cdot\|_{(M),a}$ reduces to the usual Luxemburg norm, that we denote by $\|\cdot\|_{(M)}$ following [4].

If we consider the Minkowski functional relative to S_M, we get no longer a norm in general. More precisely we have the following proposition whose proof, technically similar to the previous one, is omitted.

Proposition 3. *Let M be a Young function. Then the position*

$$p_M(f) = \inf\left\{\rho > 0 : \int M\left(\frac{f}{\rho}\right) d\mu < +\infty\right\}$$

defines a seminorm on L_M. This seminorm is related to the norms $\|\cdot\|_{(M),a}$ by the equality:

$$\lim_{a\to+\infty} \|f\|_{(M),a} = p_M(f) \quad \forall f \in L_M.$$

Moreover, p_M is a norm if and only if M is not finite. In this case we also have $L_M \subseteq L^\infty$.

The inclusion $L_M \subseteq L^\infty$ may hold also for a finite M, as the following proposition shows.

Proposition 4. *Let M be a Young function. Then:*

i) $L^\infty \subseteq L_M \iff$ *either M is not positive or* $\mu(\Omega) < +\infty$,

ii) $L_M \subseteq L^\infty \iff$ *either M is not finite or* $\inf\{\mu(A) : A \in \mathcal{A},\ \mu(A) > 0\} > 0$.

Proof. *i)* Assume $L^\infty \subseteq L_M$. Then every constant function is in L_M; thus fixed any $d \in]0, +\infty[$ we have $M(\frac{d}{\rho})\mu(\Omega) = \int M(\frac{d}{\rho})\, d\mu < +\infty$ for some $\rho > 0$. Consequently if M is positive then $\mu(\Omega) < +\infty$. This proves the implication " $\Longrightarrow$ ". Conversely, let $f \in L^\infty$ and denote by k the norm $\|f\|_\infty$. If M is not positive and $d > 0$ is such that $M(x) = 0$ for $x \in [-d, d]$, then choose $\rho > 0$ such that $\frac{k}{\rho} < d$; if M is positive, then choose $\rho > 0$ such that $M(\frac{k}{\rho}) < +\infty$. In any case we obtain $\int M(\frac{|f|}{\rho})\, d\mu \le M(\frac{k}{\rho})\mu(\Omega) < +\infty$. This proves the reverse implication " $\Longleftarrow$ ".

ii) Assume $L_M \subseteq L^\infty$. If M is finite we proceed by contradiction supposing that $\inf\{\mu(A) : A \in \mathcal{A},\ \mu(A) > 0\} = 0$. Choose a real sequence $\{a_n\}$ such that $M(a_n) \to +\infty$ as $n \to \infty$ and another sequence $\{b_n\} \subset]0, +\infty[$ such that the series $\sum_{n=1}^{+\infty} M(a_n)b_n$ converges. Then construct a sequence of sets $\{A_n\} \subseteq \mathcal{A}$, pairwise disjoint, such that $0 < \mu(A_n) < b_n\ \forall n \in \mathbb{N}$. This is possible. Indeed, by our assumption, we can find a sequence $\{B_n\}$ in $\mathcal{A}$ such that $0 < \mu(B_n) \le \frac{1}{2^{n+1}}$ for each $n \in \mathbb{N}$. If we let $C_n = B_n \cup B_{n+1} \cup \ldots,\ n \in \mathbb{N}$, it is possible to extract a subsequence $\{C_{n_k}\}$ such that $\mu(C_{n_1}) < b_1$ and, inductively, $\mu(C_{n_{k+1}}) \le \min\ \{b_{k+1}\ ,\ \frac{1}{2}\mu(C_{n_k})\}$; it is clear that the sequence defined by $A_k = C_{n_k} \setminus C_{n_{k+1}}$ has the required properties. Now, if we consider the function f defined by the position $f = \sum_{n=1}^{\infty} a_n \chi_{A_n}$, it results $f \notin L^\infty$. On the other side $\int M(f)\, d\mu = \sum_{n=1}^{\infty} M(a_n)\mu(A_n) < +\infty$. This contradiction concludes the implication " $\Longrightarrow$ ". Conversely, the inclusion $L_M \subseteq L^\infty$ holds when M is not finite by Proposition 3. So, assume that M is finite. Fixed $f \in L_M$ define

$A_\alpha = \{x \in \Omega : |f(x)| > \alpha\} \quad \forall \alpha > 0$. By the Chebyshev-Markov inequality we obtain $M(\frac{\alpha}{\rho})\mu(A_\alpha) \leq \int M(\frac{|f|}{\rho})\,d\mu \leq 1$ for some $\rho > 0$, hence, by the hypothesis on the sets of positive measure, there must be some $\overline{\alpha} > 0$ such that $\mu(A_\alpha) = 0 \;\; \forall \alpha \geq \overline{\alpha}$. This shows the reverse implication " $\Longleftarrow$ ". □

Remark 2. Let M be the null Young function pointed at c. Then by the previous proposition we have $L_M \doteq L^\infty$. Moreover it is immediate that $p_M(f) = \|f\|_{(M),a} = \frac{\|f\|_\infty}{c} \; \forall f \in L_M, \forall a > 0$.

Remark 3. Let M be a Young function and let $0 < c < +\infty$. Define a new Young function M_c by the position

$$M_c(x) = \begin{cases} M(x) & \text{if} \quad |x| \leq c \\ +\infty & \text{if} \quad |x| > c \end{cases}.$$

Then it is clear that $L_M \cap L^\infty = L_{M_c}$.

Remark 4. We already mentioned that requisite *c)* in Definition 1 is not restrictive. In fact if $\widetilde{M}$ is a standard Young function, i.e. $\widetilde{M}$ satisfies requisites *a)* and *b)*, but not necessarily *c)*, then the same arguments in the present section allow us to define in a completely analogous way the Banach space $L_{\widetilde{M}}$, the Luxemburg complete norms $\|\cdot\|_{(\widetilde{M}),a}$ and the seminorm $p_{\widetilde{M}}$. Moreover if we consider the Young function M, according to Definition 1, that is obtained from $\widetilde{M}$ by changing only the value at the point $c = \sup\{x \in \overline{\mathbb{R}} : \widetilde{M}(x) < +\infty\}$ if necessary, it is easy to verify that $L_{\widetilde{M}} = L_M$ when $c = +\infty$ or $\widetilde{M}(c) \in [M(c), +\infty[$, while $L_{\widetilde{M}} \subseteq L_M$ when $M(c) < +\infty = \widetilde{M}(c)$; here, of course, all set-theoretic inclusions also hold from the topological point of view. Concerning the last case we observe that if in addition it results $M(c) = 0$, then we actually have $L_{\widetilde{M}} = L_M$; moreover the equalities $p_{\widetilde{M}}(f) = \|f\|_{(\widetilde{M}),a} = \frac{\|f\|_\infty}{c} \; \forall f \in L_{\widetilde{M}}, \; \forall a > 0$, still hold.

Remark 5. Let M be a finite positive Young function. If $a > 0$ and $A \in \mathcal{A}$, $0 < \mu(A) < +\infty$, then it is easily checked that $\|\chi_A\|_{(M),a} = \frac{1}{M^{-1}(\frac{a}{\mu(A)})}$.

The following propositions display some facts concerning the Luxemburg a-norms, which will be useful in the sequel. Some of the proofs are omitted.

Proposition 5. *Let M be a Young function. If $\{f_n\} \subseteq L_M$ is such that $|f_n| \uparrow f$ for some measurable f, then $\lim_n \|f_n\|_{(M),a} = \|f\|_{(M),a}$ if $f \in L_M$, $\lim_n \|f_n\|_{(M),a} = +\infty$ if $f \notin L_M$.*

Proposition 6. *Let M be a Young function. If $f_1, \dots, f_n \in L_M$ and at least one of these functions is different from zero, then $\int M\left(\frac{|f_1|+\dots+|f_n|}{\|f_1\|_{(M),a}+\dots+\|f_n\|_{(M),a}}\right) d\mu \leq a$.*

Proposition 7. *Let M be a Young function. If $f_n \to f$ in L_M then there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ μ-a.e..*

Proof. Obviously we can suppose that $\{f_n\}$ possesses no constant subsequence. Let $\{f_{n_k}\}$ be a subsequence of $\{f_n\}$ such that $\|f_{n_{k+1}} - f_{n_k}\|_{(M),a} < \frac{1}{2^k}$ $\forall k \in \mathbb{N}$. Set $g_1 = |f_{n_1}|$, $g_{k+1} = |f_{n_{k+1}} - f_{n_k}|$ $\forall k \in \mathbb{N}$ and observe that $M\left(\frac{\sum_{k=1}^{+\infty} g_k}{\sum_{k=1}^{+\infty} \|g_k\|_{(M),a}}\right) \leq \liminf_n M\left(\frac{\sum_{k=1}^{n} g_k}{\sum_{k=1}^{n} \|g_k\|_{(M),a}}\right)$. Thus, by Proposition 6 and Fatou's lemma, it follows that there exists a measurable function, say $\overline{f}$, such that $f_{n_k} \to \overline{f}$ μ-a.e. . To complete the proof it is enough to verify that we also have $f_{n_k} \to \overline{f}$ in L_M. To show this, fix any $\epsilon > 0$ and select $\overline{k} \in \mathbb{N}$ such that $\sum_{i=k+1}^{+\infty} \|g_i\|_{(M),a} < \epsilon$ for $k \geq \overline{k}$. Arguing as above we have $\int M\left(\frac{|f_{n_k} - \overline{f}|}{\sum_{i=k+1}^{+\infty} \|g_i\|_{(M),a}}\right) \leq \int \liminf_n M\left(\frac{\sum_{i=k+1}^{n} g_i}{\sum_{i=k+1}^{n} \|g_i\|_{(M),a}}\right) \leq a$ $\forall k \in \mathbb{N}$, thus $\|f_{n_k} - \overline{f}\|_{(M),a} < \epsilon$ $\forall k \geq \overline{k}$. □

It is clear that the the above argument furnishes us with an alternative proof of the completeness of L_M.

In the proposition below the Orlicz a-norms $\|\cdot\|_{M,a}$ are showed (for $a = 1$ we have the usual Orlicz norm), jointly with the relationships between them and the Luxemburg a-norms. The proofs are classic, so we omit them (refer to [5] and [10]).

Recall that the measure space $(\Omega, \mathcal{A}, \mu)$ (or, simply, the measure μ) is said to have the *finite subset property* (shortly denoted by f.s.p.) or is said to be semi-finite provided that for any $A \in \mathcal{A}$, with $\mu(A) > 0$, there exists $B \in \mathcal{A}$, $B \subseteq A$, such that $0 < \mu(B) < +\infty$ (refer to [5] and [9]).

Proposition 8. *Let M and N be complementary Young functions, with M superlinear. Also let $a > 0$ and let $f : \Omega \longrightarrow \overline{\mathbb{R}}$ be a measurable function. Then the following two statements hold true.*

1) If M is positive the following facts are equivalent:

i) $f \in L_M$;
ii) $\{x \in \Omega : f(x) \neq 0\}$ *is σ-finite and* $\int |fg|\, d\mu < +\infty$ $\forall g \in S_N$;
iii) $\{x \in \Omega : f(x) \neq 0\}$ *is σ-finite and* $\sup_{g \in S_{N,a}} \int |fg|\, d\mu < +\infty$.

2) If μ has the f.s.p. the following facts are equivalent:

j) $f \in L_M$;

jj) $\int |fg|\,d\mu < +\infty \quad \forall g \in S_N$;
jjj) $\sup_{g \in S_{N,a}} \int |fg|\,d\mu < +\infty$.

In both cases the position

$$\|f\|_{M,a} = \sup_{g \in S_{N,a}} \int |fg|\,d\mu \quad \forall f \in L_M$$

defines a norm on L_M *and the inequalities*

$$a\|f\|_{(M),a} \leq \|f\|_{M,a} \leq 2a\|f\|_{(M),a} \quad \forall f \in L_M$$

hold. Moreover we have

$$|f\|_{M,a} = \sup_{g \in S_{N,a}} |\int fg d\mu| \quad \forall f \in L_M.$$

Remark 6. It is easy to find an example in which both conditions *jj)* and *jjj)* hold, although the set $\{x \in \Omega : f(x) \neq 0\}$ is not σ-finite (hence, $f \notin L_M$ and μ has not the f.s.p.): take $\Omega = \{0, 1\}$, $\mathcal{A} = \mathcal{P}(\Omega)$, μ defined by the positions $\mu(\{0\}) = 1$, $\mu(\{1\}) = +\infty$ and $f \equiv 1$.

2. The inclusion $L_M \subseteq L_N$.

For all definitions in this section about atoms and so on we refer to [1].

Notation.

$$\mathcal{A}_0 = \{A \in \mathcal{A} : \mu(A) > 0\}, \qquad \mathcal{A}_\infty = \{A \in \mathcal{A} : \mu(A) < +\infty\},$$

$$\mathcal{T} = \{A \in \mathcal{A} : A \text{ is an atom}\},$$

$$l = \inf\{\mu(A) : A \in \mathcal{A}_0\}, \quad L = \sup\{\mu(A) : A \in \mathcal{A}_\infty\},$$

$$\mathbf{R} = \left\{\frac{1}{\mu(A)} : A \in \mathcal{A}_0 \cap \mathcal{A}_\infty\right\}.$$

It is clear that if the measure space $(\Omega, \mathcal{A}, \mu)$ possesses no atom of finite measure, then $\mathbf{R} = [\frac{1}{L}, +\infty[$.

Remark 7. It is easy to verify that the following two equivalences hold:

– $l > 0 \iff \mathcal{T} \neq \emptyset$, $\inf_{\mathcal{T}} \mu(T) > 0$ and for each $A \in \mathcal{A}_0 \cap \mathcal{A}_\infty$ there exist $T_1, \ldots, T_n \in \mathcal{T}$, pairwise disjoint, such that $A = \bigcup_{i=1}^n T_i$;

– $L < +\infty \iff$ there exist $S, T \in \mathcal{A}$ such that $S \cup T = \Omega$, $S \cap T = \emptyset$, $S \in \mathcal{A}_\infty$, and either $\mu(T) = 0$ or $T \in \mathcal{T}$ and $\mu(T) = +\infty$.

Moreover if $l > 0$ then $l = \inf_{\mathcal{T}} \mu(T)$.

Let M and N be two Young functions. Consider the following facts:

(0) $L_M \subseteq L_N$;
(1) $\exists\, k > 0 : \|u\|_{(N)} \leq k\, \|u\|_{(M)} \quad \forall u \in \mathrm{Span}(\{\chi_A : A \in \mathcal{A}_0 \cap \mathcal{A}_\infty\})$;
(2) $\exists\, k > 0 : \|\chi_A\|_{(N)} \leq k\, \|\chi_A\|_{(M)} \quad \forall A \in \mathcal{A}_0 \cap \mathcal{A}_\infty$;
(3) $\exists\, c > 0 : N(t) \leq M(ct) \quad \forall t \in [\frac{1}{L}, +\infty[$.

Remark 8. Easy calculations show that if M and N are two positive finite Young functions, then each of the following two facts is equivalent to *(2)*:

(2') $\exists\, k > 0 : \quad M^{-1}(r) \leq k N^{-1}(r) \quad \forall r \in \mathbf{R}$,
(2") $\exists\, k > 0 : \quad N(\frac{s}{k}) \leq M(s) \quad \forall s \in M^{-1}(\mathbf{R}) = \{M^{-1}(r) : r \in \mathbf{R}\}$.

Moreover, *(3)* is equivalent to:

(3') $\exists\, c > 0$, $\exists a \geq 0$ ($a = 0$ if $L = +\infty$) : $N(t) \leq M(ct) \quad \forall t \in [a, +\infty[$.

The following theorem characterizes, in finite atomless hypothesis, the inclusion *(0)* between two Orlicz spaces, in terms of the Luxemburg norms of characteristic functions (compare with Theorem 3 on page 155 of [5]).

Theorem 1. *Let M and N be two positive finite Young functions. Then:*

$$(3) \Longrightarrow (0) \iff (1) \Longrightarrow (2).$$

If, in addition, μ has no atom of finite measure, then all four facts are equivalent.

Proof. *(3)* $\Longrightarrow$ *(0)*. Let $f \in L_M$ and $\rho > 0$ such that $\frac{f}{\rho} \in S_M$. If $L = +\infty$ then it is apparent that $\frac{f}{c\rho} \in S_N$. Thus suppose $L < +\infty$. From Remark 7 it follows that $\mu(\{f(x) \neq 0\}) \leq L$. Integrating $N(\frac{|f|}{c\rho})$ over $\{f(x) \neq 0\} = \{0 < |f| < \frac{c\rho}{L}\} \cup \{|f| \geq \frac{c\rho}{L}\}$ one obtains $f \in L_N$.

(0) $\Longrightarrow$ *(1)*. It is enough to verify that the canonical inclusion $i : L_M \to L_N$ is continuous, i.e. its graphic is closed. So, let $\{f_n\}$ be a sequence in L_M such that $(f_n, f_n) \to (f, g)$ in $L_M \times L_N$. Proposition 7 implies the existence of a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ μ-a.e. and $f_{n_k} \to g$ μ-a.e.. Thus $f = g$ and the proof is complete.

(1) $\Longrightarrow$ *(0)*. Let $0 \neq f \in L_M$ and $\rho > 0$ such that $\frac{f}{\rho} \in S_M$. By the Chebyshev-Markov inequality we obtain that $\{x \in \Omega : f(x) \neq 0\} = \{x \in \Omega : M(\frac{|f|}{\rho}) \neq 0\}$ is a σ-finite set. Let $\{A_n\}$ be a sequence in $\mathcal{A}_\infty$ such that $A_n \uparrow \{f \neq 0\}$; also let $\{u_n\}$ be a sequence of simple functions such that $u_n \uparrow |f|$. Then $v_n = u_n \chi_{A_n}$ lies in $\mathrm{Span}(\{\chi_A : A \in \mathcal{A}_0 \cap \mathcal{A}_\infty\})$ for every $n \in \mathbb{N}$ and $v_n \uparrow |f|$. By our assumption we have $\|v_n\|_{(N)} \leq k\, \|v_n\|_{(M)}$ for every $n \in \mathbb{N}$ and the argument follows from Proposition 5 .

(1) $\Longrightarrow$ *(2)*. It is obvious.

Finally suppose that μ has no atom of finite measure and show that *(2)* $\Longrightarrow$ *(3)*. By Remark 8, *(2)* is equivalent to *(2")*. Setting $t = \frac{s}{k}$, by our assumption on atoms *(2")* becomes $\exists\, k > 0 : \ N(t) \leq M(kt)\ \forall t \in \left[\frac{1}{k} M^{-1}\left(\frac{1}{L}\right), +\infty\right[$, i.e. condition *(3')* holds. A further application of Remark 8 concludes the proof. □

The following corollary shows that when M and N satisfy some particular assumptions, the inclusion *(0)* can be characterized more simply in terms of conditions $l > 0$ and $L < +\infty$ already introduced at the beginning of this section. The proof, technically similar to the previous one, is omitted.

Corollary 1. *Let M and N be two positive finite Young functions.*

(i) If $\lim_{x\to+\infty} \frac{N^{-1}(x)}{M^{-1}(x)} = 0$ *and there exists* $\delta > 0$ *such that* $N(x) \leq M(x)$ *for* $|x| \leq \delta$*, then:*

$$(0) \Longleftrightarrow (1) \Longleftrightarrow (2) \Longleftrightarrow l > 0.$$

(ii) If $\lim_{x\to0^+} \frac{N^{-1}(x)}{M^{-1}(x)} = 0$ *and there exists* $\delta > 0$ *such that* $N(x) \leq M(x)$ *for* $|x| \geq \delta$*, then:*

$$(0) \Longleftrightarrow (1) \Longleftrightarrow (2) \Longleftrightarrow L < +\infty.$$

Example 1. If $1 \leq p < q < +\infty$ then $M(x) = |x|^p$, $N(x) = |x|^q$ is an example of a couple of positive Young functions verifying the hypotheses for Corollary 1, *(i)*. To get an analogous example for *(ii)* it is sufficient to interchange p and q.

Remark 9. Corollary 1 and Example 1, jointly with Remark 7 and Preposition 4, yields us in particular Theorems 1 and 2 of [7] (of course to allow p and q to range over all of $]0, +\infty]$ it suffices to consider that $L^p \subseteq L^q$ if and only if $L^{pt} \subseteq L^{qt}\ \forall t > 0$).

3. An extension of a continuity property of the L^p-norm.

When the measure μ is finite, it is well-known that if $f \in L^p \equiv L^p(\mu)$ for every $p \in [1, +\infty[$, then $\|f\|_p \to \|f\|_\infty$ as $p \to +\infty$, where $\|f\|_\infty$ is assumed to be $+\infty$ if $f \notin L^\infty$ (see [8], Theorem (8.1)). The following proposition generalizes this property.

Theorem 2. *Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space. Moreover let $\{M_n\}$ be a sequence of superlinear positive Young functions which converges pointwise to the null Young function M pointed at c. Then for each $f \in \bigcap_n L_{M_n}$ and each $a > 0$ it results*

$$\lim_n \|f\|_{(M_n),a} = \lim_n \frac{\|f\|_{M_n,a}}{a} = \begin{cases} \frac{\|f\|_\infty}{c} & if \quad f \in L_M \\ +\infty & if \quad f \notin L_M \end{cases}.$$

(Recall that $L_M = L^\infty$ by Remark 2).

Proof. Let $f \in \bigcap_n L_{M_n}$, $f \neq 0$ and $\Lambda = \frac{\|f\|_\infty}{c}$ if $f \in L_M$, $\Lambda = +\infty$ if $f \notin L_M$. Fixed any λ such that $0 < \lambda < \Lambda$, consider the set $A = \{x \in \Omega : |f(x)| > \lambda c\}$ and observe that $\mu(A) > 0$ since $\lambda c < \|f\|_\infty$. Moreover, for every $n \in \mathbb{N}$ the inequality $M_n(\frac{\lambda c}{\rho})\chi_A \leq M_n(\frac{|f|}{\rho})$, $\rho > 0$, implies $\left\{\rho > 0 : \int M_n\left(\frac{|f|}{\rho}\right) d\mu \leq a\right\} \subseteq \left\{\rho > 0 : M_n\left(\frac{\lambda c}{\rho}\right)\mu(A) \leq a\right\}$, thus denoting by ρ_n the infimum of the right side and having in mind the definition of the Luxemburg norm we have $\liminf_n \rho_n \leq \liminf_n \|f\|_{(M_n),a}$. At this point fix any σ such that $0 < \sigma < \lambda$ and consider that $M_n(\frac{\lambda c}{\sigma}) \to +\infty$ as $n \to \infty$ so that $M_n(\frac{\lambda c}{\sigma}) > \frac{a}{\mu(A)}$ for n large enough. From this we deduce that for n large enough we have also $\sigma < \rho_n$ hence $\sigma \leq \liminf_n \rho_n$. Since σ is arbitrary we have also $\lambda \leq \liminf_n \|f\|_{(M_n),a}$. But λ is arbitrary too, so $\Lambda \leq \liminf_n \|f\|_{(M_n),a}$. If $\Lambda = +\infty$ the thesis follows from Proposition 8. Thus suppose $\Lambda < +\infty$ and for each $n \in \mathbb{N}$ denote by N_n the complementary Young function of M_n. Let $g : \Omega \longrightarrow \overline{\mathbb{R}}$ an arbitrary measurable function: from the inequalities $c|g| \leq M_n(c) + N_n(|g|)$ we have $c \int |g|\, d\mu \leq M_n(c)\mu(\Omega) + \int N_n(|g|)\, d\mu \quad \forall n \in \mathbb{N}$. It follows that

$$\|f\|_{M_n,a} \leq \Lambda \sup\left\{c \int |g|\, d\mu : \int N_n(|g|)\, d\mu \leq a\right\} \leq$$

$$\leq \Lambda M_n(c)\mu(\Omega) + \Lambda a \quad \forall n \in \mathbb{N}$$

and finally $\limsup_n \frac{\|f\|_{M_n,a}}{a} \leq \Lambda$. A further application of Proposition 8 concludes the proof. □

Example 2. It is well known that for the Young function Φ_p, $1 \leq p < +\infty$, defined by the position $\Phi_p(x) = \frac{|x|^p}{p}$ $\forall x \in \mathbb{R}$, it results $L_M = L^p$ and $\|\cdot\|_{(M)} = \left(\frac{1}{p}\right)^{(\frac{1}{p})} \|\cdot\|_p$. Now, assume that μ is a finite measure and that the function f belongs to L^p for every $p \in [1, +\infty[$. If $M_n = \Phi_{p_n}$, $n \in \mathbb{N}$, where $\{p_n\}$ is an arbitrary sequence in $[1, +\infty[$ such that $p_n \to +\infty$ and M is the null Young function pointed at $c = 1$, all hypotheses of Theorem 2 are verified. It follows that

$$\lim_n \|f\|_{p_n} = \lim_n \left(\frac{1}{p_n}\right)^{(\frac{1}{p_n})} \|f\|_{(M_n)} = \begin{cases} \|f\|_\infty & if \quad f \in L_M \\ +\infty & if \quad f \notin L_M \end{cases}.$$

Since $\{p_n\}$ is arbitrary we can conclude that

$$\lim_{p\to+\infty} \|f\|_p = \begin{cases} \|f\|_\infty & if \quad f \in L^\infty \\ +\infty & if \quad f \notin L^\infty \end{cases}.$$

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