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SURFACES WITH DEFECTIVE TANGENTIAL VARIETIES

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Surfaces with h-defective tangential varieties are classified for any h. For such surfaces the corresponding defects are calculated. Smooth surfaces with this property not coinciding with $v_3(\mathbb{P}^2)$ are pointed out. It is shown that varieties of osculatings (of any order) to curves are not defective.

1. Introduction.

In this paper we work over complex numbers. Suppose that $X \subset \mathbb{P}^N$ is a smooth variety. It is a well-known fact that if $N > 2\dim X + 1$ then X can be projected isomorphically to $\mathbb{P}^{2\dim X+1}$. One can ask whether X can be projected isomorphically to $\mathbb{P}^m$ with $m < 2\dim X + 1$. In order to answer this question we notice that X can be projected isomorphically to $\mathbb{P}^m$ iff $\dim SX \leq m$, where $SX = \overline{\bigcup_{x,y\in X, x\neq y}\langle x, y\rangle}$ is a secant variety (by $\langle U\rangle$ we denote the linear span of a set U), because the projection $\pi_L : \mathbb{P}^N \dashrightarrow \mathbb{P}^m$ from a linear subspace L is an isomorphism on X iff $L \cap SX = \emptyset$. By simple dimension count one can see that $\dim SX$ does not exceed $2\dim X + 1$, and equality holds for "general" X. Because of that, in general X can be isomorphically projected only to $\mathbb{P}^{2\dim X+1}$. But if we are interested in isomorphic projection to $\mathbb{P}^m$, then we should find varieties for which $\dim SX \leq m < 2\dim X + 1$ and also

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$\dim SX \leq m < N$. Such varieties are called *1-defective*. Straightforward generalization leads us to varieties for which the dimension of the variety of $(h+1)$-secant h-dimensional subspaces $S^h X$ is less then the expected one, i. e. $\dim S^h X < \min\{N, (h+1)\cdot \dim X + h\}$. Such varieties are called *h-defective*. It is more or less clear that there are no h-defective curves.

The main tool in studying defective varieties is the following lemma.

Lemma. (Terracini). *For general point $x_0, \ldots, x_h \in X$ and general point $q \in \langle x_0, \ldots, x_h\rangle \subset S^h X$ holds*

$$T_q S^h X = \langle T_{x_0} X, \ldots, T_{x_h} X\rangle.$$

Defective surfaces were classified by many authors. Classically such surfaces were considered by Palatini [9] and [10] whose classification theorem has a serious gap. Then Terracini [13] completed Palatini's classification. Also Scorza [12] and Bronowski [1] worked on this topic. Both Palatini's and Terracini's papers are obscure and difficult to read. Chiantini and Ciliberto [3] classified weakly defective surfaces, of which defective surfaces form a special case. Their approach is easier and faster then the previous ones. The result is as follows:

Theorem. *A surface X is h-defective iff X is one of the following:*

(1) *A non-degenerate surface $X \subset Cone_L C$, where C is a curve, L is a linear space of dimension $h-1$, $N \geq 3h+2$ such that for any linear subspace $l \subset L$ one has $\dim \pi_l(X) = 2$;*
(2) *$X = v_2(Y) \subset \mathbb{P}^{3h+2}$, where $Y \subset \mathbb{P}^{h+1}$ is a non-degenerate surface of minimal degree.*

1-defective threefolds were classified by Scorza [11]. In more recent times Zak [14], Fujita-Roberts [6] and Fujita [5] considered smooth defective threefolds. Ciliberto and Chiantini [4] reworked the Scorza classification in an easy and fast way.

For higher dimensional defective varieties only general properties are known, see Zak [14].

In [2] Bronowski first considered a surface whose tangential variety is defective and stated that [2, 3] *through a general point of 9-dimensional space there passes no 5-dimensional space containing 2 tangent planes to the Del Pezzo surface*. So, one can ask *whether $v_3(\mathbb{P}^2) \subset \mathbb{P}^9$ is the only surface in $\mathbb{P}^9$ with the following property: the $\mathbb{P}^5$'s spanned by two tangent planes to X do not fill up $\mathbb{P}^9$*? This question can be reformulated in the following way: *when* $\dim S(TX) < 9$ *(or TX is 1-defective)*?

Our main Theorem 3 gives an answer to this question, i.e. it describes all surfaces for which the tangential varieties are h-defective. It appears that $v_3(\mathbb{P}^2)$ is not the only such surface. For example, a general surface on a cone with vertex a plane over a curve has 1-defective tangential variety when the dimension of the ambient space is big enough.

This paper is organized in the following way: in Sections 2 and 3 we gather the preliminaries; the varieties of osculatings to curves are discussed in Section 4; in Section 5 the main property of surfaces with defective tangential varieties is pointed out; Section 6 is devoted to describing examples, including smooth ones, and calculating the corresponding defects; in Section 7 we state our main Theorem 3. In the subsequent Sections we give a proof of Theorem 3: in Section 8 we prove Theorem 3 in the case $\dim TX = 3$; Section 9 contains a proof in the case $\dim TX = 4$ and $h = 1$, in Subsection 9.1 we consider the case when the surface is mapped to a curve under the projection from the osculating space of order 2 at a general point, in Subsection 9.2 the case when the image of this projection is a surface is described; in Section 8 we prove Theorem 3 for $h > 1$. Finally, in Section 11 Corollary 5 is discussed.

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2. h-secant varieties and their defects.

Definition. The variety $S^h X = \overline{\bigcup_{x_0,\dots,x_h \in X,\ \dim\langle x_0,\dots x_h\rangle = h} \langle x_0, \dots, x_h \rangle}$ is called *h-secant variety of the variety X.*

By counting dimensions one can see that if N is big enough then the expected dimension of $S^h X$ is $\dim X \cdot (h + 1) + h$. Hence for an arbitrary N the expected dimension of $S^h X$ is $\min\{N, \dim X \cdot (h + 1) + h\}$.

Definition. The number $d_h(X) = \dim X(h + 1) + h - \dim S^h X$ is called the *cumulative h-defect* of the variety $X \subset \mathbb{P}^N$. The number $\delta_h(X) = \min N, \dim X(h + 1) + h - \dim S^h X$ is called the *h-th defect* of the variety $X \subset \mathbb{P}^N$.

Remark 1.

1. $d_h(X)$ is called *cumulative* defect because of the following. It is clear that $S^hX = S(X, S^{h-1}X)$, where

$$S(Y, Z) = \overline{\bigcup_{y\in Y, z\in Z, y\neq z} \langle y, z\rangle}.$$

For a general point $s \in S^hX$ the variety

$$\Sigma_s = \overline{\{x | x \in X,\ \exists t \in S^{h-1}X,\ t \neq x : s \in \langle x, t\rangle\}}$$

is called the *entry locus for* s. Put $\dim \Sigma_s = \sigma_h(X)$. One has $\dim S^hX = \dim S(X, S^{h-1}X) = (\dim X + \dim S^{h-1}X + 1) - \sigma_h(X)$. So, $d_h(X) - d_{h-1}(X) = \dim X + 1 - \dim S^hX + \dim S^{h-1}X = (\dim X + 1 + \dim S^{h-1}X) - \dim S^hX = \sigma_h(X)$, and $d_h(X) = \sigma_1(X) + \ldots + \sigma_h(X)$.

2. $\delta_h(X) = \min\{N - \dim S^hX, d_h(X)\} = \min\{N - (\dim X \cdot (h+1) + h - d_h(X)), d_h(X)\} = \min\{d_h(X) + (N - \dim X \cdot (h+1) - h), d_h(X)\}$; $\delta_h(X) < d_h(X)$ iff $N < \dim X(h+1) + h$.

3. As we saw above, $d_{h+1}(X) = d_h(X) + \sigma_{h+1}(X)$, and so $d_{h+1}(X) \geq d_h(X)$. Moreover, for $N \geq \dim X(h+2) + h + 1$ ($h \leq \frac{N-1-2\dim X}{\dim X+1} = \frac{N-\dim X}{\dim X+1} - 1$) also $\delta_{h+1}(X) \geq \delta_h(X)$. For $h > \frac{N-\dim X}{\dim X+1}$, $\delta_{h+1}(X) = N - \dim S^{h+1}X \leq N - S^hX = \delta_h(X)$.

4. $\forall\ i$, $\delta_i(X) \geq 0$, $d_i(X) \geq 0$. At the end, if for some h, $\delta_h(X) = 0$ and $\delta_{h+1}(X) > 0$, then $h \leq \frac{N-\dim X}{\dim X+1}$ ($N \geq \dim X(h+1) + h$) and $\forall i < h$, $\delta_i(X) = 0$, because $\delta_i(X) \leq \delta_h(X) = 0$. Also since $N \geq \dim X(h+1) + h$, $\forall i \leq h$, $d_h(X) = 0$, $d_{h+1}(X) \geq \delta_{h+1}(X) > 0$.

Definition. A variety $X \subset \mathbb{P}^N$ is called *h-defective*, if $\delta_h(X) > 0$ and $\delta_{h-1}(X) = 0$.

Remark 2. If X is h-defective then $d_h(X) > 0$ and $d_{h-1}(X) = 0$. If $d_h(X) > 0$ and $d_{h-1}(X) = 0$ then $S^hX = \mathbb{P}^N$ or X is h-defective.

Proposition 1. *Suppose that a variety $X \subset \mathbb{P}^N$ is non-degenerate, $q_0 \in S^{m-1}X$ is a general point and π is the projection with the center at $T_{q_0}S^{m-1}X$. Then* $\dim \pi(X) = \dim X - d_m(X) + d_{m-1}(X)$ *and for any $k \geq 1$ one has* $d_k(\pi(X)) = d_{k+m}(X) - d_m(X) - k(d_m(X) - d_{m-1}(X))$. *If $d_m(X) = 0$ then* $\dim \pi(X) = \dim X$ *and for any $k \geq 1$,* $\delta_k(\pi(X)) = \delta_{k+m}(X)$.

Proof. Choose general points $y_0, y_1, \ldots \in X$ and general points $q_1, q_2, \ldots \in \mathbb{P}^N$ such that $\forall i \geq 1$, $q_i \in \langle y_0, \ldots, y_{m-1+i}\rangle$, i. e. $q_i \in S^{m-1+i}X$. Suppose in addition that $q_0 \in \langle y_0, \ldots, y_{m-1}\rangle$. Then by Terracini's lemma for $k \geq 1$ one has $T_{q_k}S^{m-1+k}X = \langle T_{y_0}X, \ldots, T_{y_{m-1+k}}X\rangle = \langle T_{q_0}S^{m-1}X, T_{y_m}X, \ldots, T_{y_{m-1+k}}X\rangle = \langle T_{q_0}S^{m-1}X, \langle T_{y_m}X, \ldots, T_{y_{m-1+k}}X\rangle\rangle$. We have $\dim\pi(X) = \dim T_{\pi(y_m)}\pi(X) = \dim\pi(T_{y_m}X) = \dim T_{q_1}S^mX - \dim T_{q_0}S^{m-1}X - 1 = (\dim X\cdot(m+1)+m-d_m(X)) - (\dim X\cdot m+m-1-d_{m-1}(X)) - 1 = \dim X - d_m(X) + d_{m-1}(X)$. Hence $d_k(\pi(X)) = \dim\pi(X)\cdot(k+1)+k-\dim S^k(\pi(X)) = (\dim X-d_m(X)+d_{m-1}(X))\cdot(k+1)+k-\dim T_{\pi(q_{k+1})}S^k(\pi(X)) = (\dim X-d_m(X)+d_{m-1}(X))\cdot(k+1)+k-\langle T_{\pi(y_m)}\pi(X), \ldots, T_{\pi(y_{m+k})}\pi(X)\rangle = (\dim X-d_m(X)+d_{m-1}(X))\cdot(k+1)+k-\langle\pi(T_{y_m}X), \ldots, \pi(T_{y_{m+k}}X)\rangle = (\dim X-d_m(X)+d_{m-1}(X))\cdot(k+1)+k-(\dim T_{q_k}S^{m+k}X-\dim T_{q_0}S^{m-1}X-1) = (\dim X-d_m(X)+d_{m-1}(X))\cdot(k+1)+k-((\dim X\cdot(m+k+1)+m+k-d_{m+k}(X))-(\dim X\cdot m+m-1-d_{m-1}(X))-1) = (d_{m+k}(X)-d_m(X))-k(d_m(X)-d_{m-1}(X))$.

If $d_m = 0$ then $d_{m-1} = 0$ and $\dim\pi(X) = \dim X - d_m(X) + d_{m-1}(X)\} = \dim X$. More, $N_1 = \dim\pi(\mathbb{P}^N) = N - \dim T_{q_0}S^{m-1}X - 1 = N - (\dim X\cdot m+m-1-d_{m-1}(X)) - 1 = N - \dim X\cdot m - m$. At the end, $\delta_k(\pi(X)) = \min\{d_k(\pi(X)) + (N_1 - \dim\pi(X)\cdot(k+1)-k), d_k(\pi(X))\} = \min\{((d_{m+k}(X)-d_m(X))-k(d_m(X)-d_{m-1}(X)))+((N-\dim X\cdot m-m)-\dim X\cdot(k+1)-k), (d_{m+k}(X)-d_m(X))-k(d_m(X)\text{ - }d_{m-1}(X))\} = \min\{d_{m+k}(X)+(N-\dim X\cdot(m+k+1)-(m+k)), d_{m+k}(X)\} = \delta_{m+k}(X)$. □

The following fact was also proved in [14 Chapter V, Proposition 1.7].

Corollary 1. *For any* $k > 1$, $d_{k+1}(X) - d_k(X) \geq d_k(X) - d_{k-1}(X)$, *i.e.* $\sigma_{k+1}(X) \geq \sigma_k(X)$ *in the notation of Remark 1.*

Remark 3. Really, if X is smooth, then much more stronger statement is true:

Theorem. ([14] Chapter V, Theorem 1.8) *For any* $k > 1$, $S^k(X) \neq \mathbb{P}^N$, *one has* $\sigma_{k+1}(X) \geq \sigma_k(X) + \sigma_1(X)$, *i.e.* $d_{k+1}(X) - d_k(X) \geq (d_k(X) - d_{k-1}(X)) + d_1(X)$.

3. Osculating and tangent spaces.

By $G(n, N)$ we denote the grassmannian of linear subspaces of dimension n in $\mathbb{P}^N$. For any point $\alpha \in G(n, N)$ denote the corresponding linear subspace by $\mathbb{P}^n_\alpha$.

Let $f : X \dashrightarrow G(n, N)$ be a family of $\mathbb{P}^n$'s in $\mathbb{P}^N$ and put $U_f = \overline{\bigcup_{x\in X}\mathbb{P}^n_{f(x)}}$.

Definition. A family $g : X - - \twoheadrightarrow G(m, N)$ is *generally tangent to the family* f if for a general point $x \in X$ and a general point $u \in \mathbb{P}^n_{f(x)}$ one has $\mathbb{P}^m_{g(x)} \supset T_u U_f$.

This definition could be reformulated in terms of Grassmannian as follows:

Definition. A family $g : X - - \twoheadrightarrow G(m, N)$ is *generally tangent to the family* f if for a general point $x \in X$ one has $T_{f(x)} f(X) \subset T_{f(x)} G_{g(x)}(n, m) \subset T_{f(x)} G(n, N)$, where $G_\alpha(n, m)$ is the Grassmannian of $\mathbb{P}^n$'s in $\mathbb{P}^m_\alpha \subset \mathbb{P}^N$ for $\alpha \in G(n, N)$.

Definition. A family $g : X - - \twoheadrightarrow G(m, N)$ generally tangent to f is called *tangent to* f if m is minimal possible. Such m is called *the dimension of the family* f and is denoted by $\dim f$.

Example 1.
1. If $f : \mathbb{P}^1 - - \twoheadrightarrow G(n, 2n + 1)$ is the family of fibers of the Segre variety $\mathbb{P}^1 \times \mathbb{P}^n$ then $\dim f = 2n + 1$.
2. If $f : \mathbb{P}^1 - - \twoheadrightarrow G(n, N)$ is the family of osculating spaces of order n to $v_N(\mathbb{P}^1) \subset \mathbb{P}^N$, then $\dim f = n + 1$.

Remark 4. Having a family $f : X - - \twoheadrightarrow G(n, N)$ and a linear susbspace $L \subset \mathbb{P}^N$ one can construct the new (projected) family f_L in the following way: suppose that for a general point $x \in X$, $\dim \mathbb{P}^n_{f(x)} \cap L = l_f$. Then a family $f_L : X - - \twoheadrightarrow G(n - l_f - 1, N - \dim L - 1)$ is such that for a general point $x \in X$, $\pi_L(\mathbb{P}^n_{f(x)}) = \mathbb{P}^{n-l_f-1}_{f_L(x)}$, where π_L is the projection from L. It is easy to see that if a family $g : X - - \twoheadrightarrow G(m, N)$ is generally tangent to the family f, then the family g_L is generally tangent to the family f_L.

Suppose now that $X \subset \mathbb{P}^N$ is a projective variety.

Definition. $T^0 : X - - \twoheadrightarrow G(0, N)$ is a family of points of X, $T^0(x) = x$ for any $x \in X$. For any $k \geq 1$ the family $T^k : X - - \twoheadrightarrow G(n_k, N)$ is tangent to the family T^{k-1}, $n_k = \dim T^{k-1}$. Consider the following diagram:

$$\begin{array}{ccccc} \Gamma \subset X \times G(n_k, N) & & I \subset G(n_k, N) \times \mathbb{P}^N & & \\ q_1 \swarrow & \searrow q_2 & q_3 \swarrow & \searrow q_4 & \\ X & & G(n_k, N) & & \mathbb{P}^N, \end{array}$$

where $\Gamma = \overline{\{(x, \alpha) | \alpha = f(x)\}}$ is the closure of the graphic of f, $I = \{(\alpha, p) | p \in \mathbb{P}^{n_k}_\alpha\}$, q_i, $1 \leq i \leq 4$, are the projections. The cone $T^k_x X =$

$q_4(q_3^{-1}(q_2(q_1^{-1}(x))))$ is called *the osculating cone of order k to the variety X at the point x*. The number $n_k = \dim T^{k-1}$ is denoted by $\dim_k X$. $T^k X = q_4(q_3^{-1}(q_2(q_1^{-1}(X))))$, which is the union of all osculating cones, is called *the variety of osculatings of order k to X*.

Proposition 2. *Suppose that $x \in X$ is a general point,* $\dim X = n$ *and $F : U_0 \dashrightarrow X$, $F(0) = x$ is a local parametrization ($U_0 \subset \mathbb{C}^n$ is a neighborhood of 0, u_i, $1 \le i \le n$, are the coordin in $\mathbb{C}^n$). Then*

(1) *$T_x^k X$ is a linear subspace and for a general point $p \in T_x^{k-1} X$ one has $T_p T^{k-1} X \subset T_x^k X$;*

(2)
$$T_x^k X = \langle F(0), F_{u_1}(0), \dots, F_{u_n}(0), \dots, F_{\underbrace{u_1 \dots u_1 u_1}_{k}}(0), \dots$$
$$\dots, F_{\underbrace{u_1 \dots u_1 u_n}_{k}}(0), \dots, F_{\underbrace{u_n \dots u_n}_{k}}(0)\rangle;$$

(3) *a hyperplane $H \subset \mathbb{P}^N$ contains $T_x^k X$ iff the equation $H(F(u)) = 0$ has zero at 0 of order at least $k+1$ (here we put $H(*) = 0$ for the corresponding linear equation).*

Proof.

1. Since x is a general point, $q_1^{-1}(x) = \{(x, T^k(x)\}$ and $T_x^k X = \mathbb{P}_{T^k(x)}^{\dim_k X}$ is a liner subspace. By definition, the family T^k is tangent to the family T^{k-1}, i.e. $\mathbb{P}_{T^k(x)}^{\dim_k X} \supset T_p T^{k-1} X$ for a general point $p \in \mathbb{P}_{T^{k-1}(x)}^{\dim_{k-1} X}$. So, $T_x^k X \supset T_p T^{k-1} X$ for a general point $p \in T_x^{k-1} X$.

2. We prove this by induction on k. For $k = 0$ it is clear. Suppose that

$$T_x^k X = \langle F(0), F_{u_1}(0), \dots, F_{u_n}(0), \dots, F_{\underbrace{u_1 \dots u_1 u_1}_{k}}(0), \dots$$
$$\dots, F_{\underbrace{u_1 \dots u_1 u_n}_{k}}(0), \dots, F_{\underbrace{u_n \dots u_n}_{k}}(0)\rangle$$

for a general point $x \in X$. Then for any numbers $1 \le j_1, \dots, j_{k+1} \le n$ the variety $T^k X$ contains points $F_{u_{j_1} \dots u_{j_k}}(u(t))$, where $u_i(t) = 0$ for $i \ne j_{k+1}$ and $u_{j_{k+1}}(t) = t$, $t \in \mathbb{C}^1$ is a parameter from a small neighborhood of 0. Hence $T_x^{k+1} X$ should contain the point $F_{u_{j_1} \dots u_{j_{k+1}}}(0)$ as well as $T_x^k X$. Therefore

$$T_x^{k+1} X \supset \langle F(0), F_{u_1}(0), \dots, F_{u_n}(0), \dots, F_{\underbrace{u_1 \dots u_1 u_1}_{k+1}}(0), \dots$$

$$\dots, F_{\underbrace{u_1 \dots u_1 u_n}_{k+1}}(0), \dots, F_{\underbrace{u_n \dots u_n}_{k+1}}(0)\rangle.$$

On the other hand, it is clear that the space

$$\langle F(0), F_{u_1}(0), \dots, F_{u_n}(0), \dots, F_{\underbrace{u_1 \dots u_1 u_1}_{k+1}}(0), \dots$$

$$\dots, F_{\underbrace{u_1 \dots u_1 u_n}_{k+1}}(0), \dots, F_{\underbrace{u_n \dots u_n}_{k+1}}(0)\rangle$$

contains any tangent line to $T^k X$ at a general point $p \in T_x^k X$. Since, by definition, $T_x^{k+1} X$ should be minimal subspace containing $T_p T^k X$ for a general point $p \in T_x^k X$, one has

$$T_x^{k+1} X = \langle F(0), F_{u_1}(0), \dots, F_{u_n}(0), \dots, F_{\underbrace{u_1 \dots u_1 u_1}_{k+1}}(0), \dots$$

$$\dots, F_{\underbrace{u_1 \dots u_1 u_n}_{k+1}}(0), \dots, F_{\underbrace{u_n \dots u_n}_{k+1}}(0)\rangle.$$

3. Since $H(*)$ is a linear function, for any numbers $1 \le j_1, \dots, j_m \le n$, $m \le k$, one has $(H \circ F)_{u_{j_1} \dots u_{j_m}}(0) = (H \circ F_{u_{j_1} \dots u_{j_m}})(0)$. So, $(H \circ F)_{u_{j_1} \dots u_{j_m}}(0) = 0$ iff the hyperplane H contains the point $F_{u_{j_1} \dots u_{j_m}}(0)$. Moreover, $H(F(u)) = 0$ has zero at 0 of order at least $k+1$ iff for any number $0 \le m \le k$ and any numbers $1 \le j_1, \dots, j_m \le n$ one has $(H \circ F)_{u_{j_1} \dots u_{j_m}}(0) = 0$. Hence, $H(F(u)) = 0$ has zero at 0 of order at least $k+1$ iff the hyperplane H contains all points of type $F_{u_{j_1} \dots u_{j_m}}(0)$, i.e. $H \supset T_x^k X$. □

Corollary 2. $\dim T^{k-1} X \le \dim_k X \le \binom{n+k}{k} - 1.$

The following fact will be used throughout the paper because one of our principal methods is taking projections from various subspaces.

Proposition 3. *If $L \subset \mathbb{P}^N$ is a linear space, then for a general point $x \in X$ and any $k \ge 0$ one has $\pi_L(T_x^k X) = T^k_{\pi_L(x)} \pi_L(X)$ and $\pi_L(T^k X) = T^k(\pi_L(X))$.*

Proof. This fact essentially follows from Remark 4 and definitions. □

Proposition 4. *If $X \subset \mathbb{P}^N$ is a non-degenerate curve and $k \le N$, then* $\dim_k X = k$.

Proof. Assume the opposite. Since $\dim_0 X = 0$ and $\dim_{j+1} X \geq \dim_j X$ for any $j \geq 0$, one can find an index $l < k$ such that $l = \dim_l X = \dim_{l+1} X$. Hence, for a general point $x \in X$ one has $T_x^l X = T_x^{l+1} X$. By Proposition 2 for a general point $p \in T_x^l X$ it is true that $T_p T^l X \subset T_x^{l+1} X = T_x^l X$. Since $T_x^l X \subset T_p T^l X$ and $T_x^l X \subset T^l X$, $T^l X = T_x^l X$. Moreover, $X \subset T^l X$. Therefore $T_x^l X = \mathbb{P}^N$ and $l = \dim_l X = N$. Since $l < k \leq N$, we obtain a contradiction. □

For any point $x \in X$ we have three well-defined objects related to the notion of tangency:

(1) the tangent (Zariski) space $T_x X$;
(2) the tangent star $\Theta_x X$, which is formed by all lines being limits of $\langle y, z\rangle$, $y, z \in X$, $y \neq z$, while y and z tend to x;
(3) the tangent cone $T'_x X$, which is formed by all lines being limits of $\langle x, y\rangle$, $y \in X$, $y \neq x$, while y tends to x;

It is clear that $T'_x X \subset \Theta_x X \subset T_x X$. In this section we introduced the osculating cone of order 1 $T_x^1 X$.

Example 2. If $X = Cone_p(C) \subset \mathbb{P}^N$ is non-degenerate then

(1) $T_p X = \mathbb{P}^N$;
(2) $\Theta_p X = Cone_p(S(C))$;
(3) $T'_p X = Cone_p(C)$;
(4) $T_p^1 X = Cone_p(T^1 C)$.

Proposition 5. *For any point $x \in X$ one has $T'_x X \subset T_x^1 X \subset \Theta_x X \subset T_x X$.*

Proof. It is clear, that if $x \in X$ is a smooth point, then all four objects coincide. If x is a singular point, then by definition, $T_x^1 X$ is obtained as various limits of $T_y^1 X$ while y tends to x, $y \in X$ is smooth. Since $T_y^1 X = T_y X$, $T_x^1 X$ consists of lines which are limits of lines, tangent to X at y while y tends to x. So, $T_x^1 X \subset \Theta_x X$.

On the other hand, for any line $l \subset T'_x X$, $l \ni x$, there exists a curve $K \subset X$ such that $x \in K$ and $T'_x K = l$ (e.g. one can take general intersection of X with a linear subspace of dimension codim $X - 1$ containing l). For this curve an easy local computation shows that $T_x^1 K = T'_x K$. Hence, $l = T'_x K = T_x^1 K \subset T_x^1 X$. □

From now on, we put $TX = T^1 X$. If for a point $x \in X$ the osculating cone $T_x^k X$ is a linear space of proper dimension ($\dim_k X$), then we will call $T_x^k X$ also *osculating space of order k at the point x.*

One can see that the expected dimension for $T^k X$ is $\dim X + \dim_k X$. If $\dim T^k X \leq 1 + \dim_k X$, then it is possible to give a full classification.

Proposition 6. *Suppose that $X \subset \mathbb{P}^N$ is a non-degenerate variety.*

(1) *If* $\dim T^k X = \dim_k X$ *then* $T^k X = \mathbb{P}^N$.
(2) *If* $\dim T^k X = 1 + \dim_k X$ *then one of the following conditions holds:*
   (a) $T^k X = \mathbb{P}^N$;
   (b) *one can find a (maybe empty) linear space $M \subset \mathbb{P}^N$ and a curve C such that $X \subset Cone_M(T^m C)$ and* $\dim M + m = \dim_k X - k - 1$.

Proof.
1. This is evident.

2. If $\dim T^k X = 1 + \dim_k X$ and $T^k X \neq \mathbb{P}^N$, then there exists only one-dimensional family of different linear spaces $T_x^k X$, i.e. $\dim T^k(X) = 1$. Put $K = T^k(X) \subset G(\dim_k X, N)$. For a general point $\alpha \in K$ put $Y_\alpha = (T^k)^{-1}(\alpha) \subset X$, $L_\alpha = \langle \bigcup_{x \in Y_\alpha} T_x^{k-1} X \rangle$. Then the codimension of L_α in $\mathbb{P}_\alpha^{\dim_k X}$ is at most 1. Really, take a general point $x \in Y_\alpha$ and use again the local parametrization of X in a small neighborhood of x as in Proposition 2, and suppose additionally that Y_α is (locally) defined by the equation $u_n = 0$. By Proposition 2, L_α contains the following points:

$$F(0), F_{u_1}(0), F_{u_n}(0), \dots, F_{\underbrace{u_1 \dots u_1 u_1}_{k-1}}(0), F_{\underbrace{u_1 \dots u_1 u_n}_{k-1}}(0), \dots, F_{\underbrace{u_n \dots u_n}_{k-1}}(0)$$

(because $L_\alpha \supset T_x^{k-1} X$) and points of type $F_{u_{j_1} \dots u_{j_{k-1}}}(u(t))$, where $u_i(t) = 0$ for $i \neq j_k$ and $u_{j_k}(t) = t$, $t \in \mathbb{C}^1$ is a parameter from a small neighborhood of 0, $1 \leq j_1, \dots, j_{k-1} \leq n$, $1 \leq j_k < n$ (because $L_\alpha \supset T_y^{k-1} X$, $y \in Y_\alpha$). Hence, L_α contains $F_{u_{j_1} \dots u_{j_k}}(0)$. Varying the numbers $j_1, \dots, j_k$ one can obtain that $\mathbb{P}_\alpha^{\dim_k X} = T_x^k X = \langle L_\alpha, F_{\underbrace{u_n \dots u_n}_{k}}(0) \rangle$.

Put $Z = \overline{\bigcup_{\alpha \in K} L_\alpha}$. By our construction, there exists r such that

$$L_\alpha \subset S^r(\bigcup_{x \in Y_\alpha} T_x^{k-1} X)$$

and for any $v < r$ one has $L_\alpha \neq S^v(\bigcup_{x \in Y_\alpha} T_x^{k-1} X)$. Therefore, for a general point $q \in L_\alpha$ there exist points $x_0, \dots, x_r \in Y_\alpha$ and points $p_0, \dots, p_r \in T^{k-1} X$, $p_i \in T_{x_i}^{k-1} X$, $0 \leq i \leq r$, such that $q \in \langle p_0, \dots, p_r \rangle$. Arguing as it is usually done in prooving Terracini lemma, one has $T_q Z \subset \langle T_{p_0} T^{k-1} X, \dots, T_{p_r} T^{k-1} X \rangle$.

Moreover, for any i, $0 \leq i \leq r$, $T_{p_i}T^{k-1}X \subset T^k_{x_i}X$. Since $x_i \in Y_\alpha$, $T^k_{x_i}X = \mathbb{P}^{\dim_k X}_\alpha$. So, $T_{p_i}T^{k-1}X \subset \mathbb{P}^{\dim_k X}_\alpha$ and $T_qZ \subset \mathbb{P}^{\dim_k X}_\alpha$. Hence, we have an one-dimensional family of linear spaces L_α such that for a general point $q \in L_\alpha$ one has T_qZ depends only on α ($Z = \overline{\bigcup_{\alpha\in K} L_\alpha}$). According to the classification of one-dimensional families of linear spaces (see e.g. [7,2.4]), there exist a linear space M, a curve C, a number l and a map $\varphi : K - - \succ C$ such that $L_\alpha = \langle M, T^l_{\varphi(\alpha)}C\rangle$ and $\dim L_\alpha = \dim M + l + 1$. Moreover, $\mathbb{P}^{\dim_k X}_\alpha = \langle M, T^{l+1}_{\varphi(\alpha)}C\rangle$ and $\dim_k X = \dim M + l + 1 + 1$. Since $T^kX = Cone_M(T^{l+1}C)$, $X \subset Cone_M(T^{l+1-k}C)$. Put $m = l + 1 - k$. Then $X \subset Cone_M(T^mC)$ and $\dim_k X = \dim M + l + 1 + 1 = \dim M + (m - 1 + k) + 1 + 1 = \dim M + m + k + 1$, i.e. $\dim M + m = \dim_k X - k - 1$. □

The following fact is well-known (e.g. [7], 5.37).

Corollary 3. *If $X \subset \mathbb{P}^N$ is a non-degenerate surface such that* $\dim TX = 3$, *then $N = 3$, $X = Cone_p(C)$ or $X = TK$, where $p \in \mathbb{P}^N$ is a point, C and K are curves.*

4. Curves.

Theorem 1. *Suppose that $C \subset \mathbb{P}^N$ is a non-degenerate curve. The variety T^kC is non-h-defective for every h and k, i.e.* $\dim S^h(T^kC) =$ *min $N, kh + k + 2h + 1$ for all $k \geq 0$, $h \geq 0$.*

Proof. We prove this fact by induction on h. For $h = 0$ by Proposition 4, $\dim T^kC = \min\{N, k + 1\}$. Suppose, that T^kC is non-h-defective. We claim that T^kC is non-$(h + 1)$-defective. Since $0 = \delta_h(T^kC) = \min\{N - \dim S^h(T^kC), d_h(T^kC)\}$, one has either $S^h(T^kC) = \mathbb{P}^N$ or $d_h(T^kC) = 0$. In the first case $S^{h+1}(T^kC) = \mathbb{P}^N$, and T^kC is non-$(h + 1)$-defective. In the second case, by Proposition 1 applied to $X = T^kC$ and $m = h + 1$, we have $\dim \pi(T^kC) = \dim T^kC - d_{h+1}(T^kC) + d_h(T^kC) = k + 1 - d_{h+1}(T^kC)$. But $\pi(T^kC) = T^k(\pi(C))$, and $\dim \pi(T^kC) = \dim T^k(\pi(C)) = \min\{N - \dim S^h(T^kC) - 1, k + 1\} = \min\{N - (kh + k + 2h + 1) - 1, k + 1\}$. Hence, $d_{h+1}(T^kC) = k + 1 - \dim \pi(T^kC) = k + 1 - \min\{N - (kh + k + 2h + 2), k + 1\} = \max\{kh + 2k + 2h + 3 - N, 0\}$, $\dim S^{h+1}(T^kC) = (k + 1)(h + 2) + h + 1 - d_{h+1}(T^kC) = kh + 2k + 2h + 3 - \max\{kh + 2k + 2h + 3 - N, 0\} = \min\{N, (k + 1)(h + 2) + h + 1\}$. □

5. General property of surfaces with defective tangential varieties.

Theorem 2. *Suppose that $\delta_h(TX) > 0$. Then for general points $x_0, \dots, x_h \in X$, $p_0, \dots, p_h \in TX$, $p_i \in T_{x_i}X$, $0 \le i \le h$, $q \in \langle p_0, \dots, p_h\rangle$ the linear space $T_q S^h(TX)$ osculates X at the points $x_0, \dots, x_h$ of order 2 (in other words, $\langle T_{p_0}TX, \dots, T_{p_h}TX\rangle = \langle T^2_{x_0}X, \dots, T^2_{x_h}TX\rangle$).*

Proof. By Proposition 2, $T^2_{x_i}X \supset T_{p_i}TX$ for $0 \le i \le h$. By the Terracini lemma $T_q S^h(TX) = \langle T_{p_0}TX, \dots, T_{p_h}TX\rangle$. Hence,

$$T_q S^h(TX) = \langle T_{p_0}TX, \dots, T_{p_h}TX\rangle \subset \langle T^2_{x_0}X, \dots, T^2_{x_h}TX\rangle.$$

So, it is enough to show that for any i, $0 \le i \le h$, one has $T^2_{x_i}X \subset \langle T_{p_0}TX, \dots, T_{p_h}TX\rangle = T_q S^h(TX)$.

If $\dim_2 X = 4$ or 3, then $\dim_2 X = \dim TX$ and $T^2_{x_i}X = T_{p_i}TX$. Really, if $\dim TX = 4$ then $\dim_2 X \ge 4$ and, hence, $\dim_2 X = 4$. If $\dim TX = 3$, then by Corollary 3 $X \subset \mathbb{P}^3$, $X = Cone_t(K)$ or $X = TK$ for a certain curve K and a point t. In all these cases one can see that $\dim_2 X = 3 = \dim TX$. Since $T^2_{x_i}X = T_{p_i}TX$, $T^2_{x_i}X \subset \langle T_{p_0}TX, \dots, T_{p_h}TX\rangle$.

Consider the case $\dim_2 X = 5$. Since $d_h(TX) > 0$, $\exists k \le h$: $d_k(TX) > 0$, $d_{k-1}(TX) = 0$. $\dim S^k(TX) - \dim S^{k-1}(TX) = (5k+4-d_k(TX)) - (5(k-1)+4-d_{k-1}(TX)) = 5-d_k(TX) < 5$. Take a general point $q_1 \in \langle p_0, \dots, p_{k-1}\rangle$ and a general point $q_2 \in \langle p_0, \dots, p_k\rangle$. Then by the Terracini lemma $T_{q_1}S^{k-1}(TX) = \langle T_{p_0}TX, \dots, T_{p_{k-1}}TX\rangle$, $T_{q_2}S^k(TX) = \langle T_{p_0}TX, \dots, T_{p_k}TX\rangle$.
Therefore, $\dim\langle T_{q_1}S^{k-1}(TX), T_{p_k}TX\rangle - \dim T_{q_1}S^{k-1}(TX) = \dim T_{q_2}S^k(TX) - \dim T_{q_1}S^{k-1}(TX) = \dim S^k(TX) - \dim S^{k-1}(TX) = 5 - d_k(TX)$. Hence, $\dim T_{q_1}S^{k-1}(TX) \cap T_{p_k}TX = 4-(5-d_k(TX)) = d_k(TX)-1 \ge 0$. Let us vary $p_k \in T_{x_k}X$. Then all tangent spaces $T_{p_k}TX$ under the projection from $T_{x_k}X$ will be mapped to tangent lines to some conic in the plane $\pi_{T_{x_k}X}(T^2_{x_k}X)$. So, in order for $T_{q_1}S^{k-1}(TX)$ to intersect $T_{p_k}TX$ in a linear space of dimension $d_k(TX)-1$ for general $p_k \in T_{x_k}X$ we need either $\dim T_{q_1}S^{k-1}(TX) \cap T_{x_k}X = d_k(TX)-1$ or $\dim T_{q_1}S^{k-1}(TX) \cap T^2_{x_k}X = d_k(TX)$. In the first case $d_k(TX) - 1 \le \dim T_{x_k}X = 2$, $d_k(TX) \le 3$. After the projection π from $T_{q_1}S^{k-1}(TX)$ we have $\dim T_{\pi(x_k)}\pi(X) = 2 - (d_k(TX) - 1) - 1 = 2 - d_k(TX)$. Since $x_k \in X$ is general, $\dim \pi(X) = 2 - d_k(TX)$. If $d_k(TX) = 3$ or 2 then $\pi(X)$ is an empty set or a point respectively, and $T^2_{\pi(x_k)}\pi(X) = \pi(X) = T_{\pi(x_k)}\pi(X)$. Hence, $\pi(T^2_{x_k}X) = \pi(T_{x_k}X)$, and $T^2_{x_k}X \subset \langle T_{q_1}S^{k-1}(TX), T_{x_k}X\rangle = T_{q_2}S^k(TX)$. If $d_k(TX) = 1$ then $\pi(X)$ is a curve, $\dim T^2_{\pi(x_k)}\pi(X) \le 2$. Therefore, $\dim\langle T_{q_1}S^{k-1}(TX), T^2_{x_k}X\rangle \le \dim T_{q_1}S^{k-1}(TX) + 3$, and $\dim T_{q_1}S^{k-1}(TX) \cap T^2_{x_k}X \ge 2$. But in this case for general $p_k \in T_{x_k}X$, $\dim T_{q_1}S^{k-1}(TX) \cap T_{p_k}TX \ge$

1 because $T_{p_k}TX$ is a hyperplane in $T^2_{x_k}X$. Since $\dim T_{q_1}S^{k-1}(TX) \cap T_{p_k}TX = d_k(TX) - 1$, $d_k(TX) \geq 2$, which is not the case.

If $\dim T_{q_1}S^{k-1}(TX) \cap T^2_{x_k}X = d_k(TX)$, then

$$\dim\langle T_{q_1}S^{k-1}(TX), T^2_{x_k}X\rangle = \dim T_{q_1}S^{k-1}(TX) + 5 - d_k(TX) =$$

$$(5k+4) + 5 - d_k(TX) = \dim T_{q_2}S^k(TX).$$

Since $\langle T_{q_1}S^{k-1}(TX), T^2_{x_k}X\rangle \supset \langle T_{q_1}S^{k-1}(TX)T_{p_k}TX\rangle = T_{q_2}S^k(TX)$, $\langle T_{q_1}S^{k-1}(TX), T^2_{x_k}X\rangle = T_{q_2}S^k(TX)$, and $T^2_{x_k}X \subset T_{q_2}S^k(TX)$.

Using the symmetry and that $T_qS^h(TX) \supset T_{q_2}S^k(TX)$, one can obtain that $T^2_{x_i}X \subset T_qS^h(TX)$ for any i, $0 \leq i \leq h$ and, hence, $T_qS^h(TX)$ osculates of order 2 to X at the points $x_0, \dots, x_h$. □

Corollary 4. $\delta_h(TX) > 0$ *iff* $\min\{5h+4, N\} - \dim\langle T^2_{x_0}X, \dots, T^2_{x_h}X\rangle > 0$. *Moreover, in this case* $\delta_h(TX) = \min\{5h+4, N\} - \dim\langle T^2_{x_0}X, \dots, T^2_{x_h}X\rangle$.

Proof. By the Terracini lemma, one has for general points $p_0, p_1, \dots, p_h \in TX$ and $q \in \langle p_0, \dots, p_h\rangle$, that $T_qS^h(TX) = \langle T_{p_0}TX, \dots, T_{p_h}TX\rangle$ and $\dim S^h(TX) = \dim\langle T_{p_0}TX, \dots, T_{p_h}TX\rangle$. Suppose that $x_0, \dots, x_h \in X$ are points such that $\forall i$, $0 \leq i \leq h$, $p_i \in T_{x_i}X$. Then by Proposition 2 $\forall i$, $T_{p_i}TX \subset T^2_{x_i}$. Hence, $\delta_h(TX) = \min\{(h+1)\dim TX + h, N\} - \dim\langle T_{p_0}TX, \dots, T_{p_h}TX\rangle \geq \min\{5h+4, N\} - \dim\langle T^2_{x_0}X, \dots, T^2_{x_h}X\rangle$. So, if $\min\{5h+4, N\} - \dim\langle T^2_{x_0}X, \dots, T^2_{x_h}X\rangle > 0$, then $\delta_h(TX) > 0$.

By Theorem 2, in the case when $\delta_h(TX) > 0$ one has

$$\langle T_{p_0}TX, \dots, T_{p_h}TX\rangle = \langle T^2_{x_0}X, \dots, T^2_{x_h}X\rangle$$

and, hence, $\delta_h(TX) = \min\{5h+4, N\} - \dim\langle T^2_{x_0}X, \dots, T^2_{x_h}X\rangle > 0$. □

6. Examples of surfaces with defective tangential varieties.

General examples 6.1.

Proposition 7.

(1) *If* $X = v_3(\mathbb{P}^2) \subset \mathbb{P}^9$, *then* TX *is 1-defective and* $\delta_1(TX) = 1$;
(2) *If* $X \subset Cone_L(TC)$, *where* $L \subset \mathbb{P}^N$ *is a linear subspace,* $0 \leq \dim L \leq h-1$, *C is a curve,* $N \geq \dim L + 4h + 5$, $\pi_L(X) = TC$, $X \neq TK$ *for any curve* K, $h \geq 1$, *then* TX *is* h*-defective and* $\delta_h(TX) = \min\{h -$

$\dim L_{min}, N - \dim L_{min} - 4h - 4\}$, *where* L_{min} *is such a linear space of minimal dimension*;

(3) *If* $X \subset Cone_L(C)$, *where* $L \subset \mathbb{P}^N$ *is a linear subspace*, $0 < \dim L \leq 2h$, C *is a curve*, $N \geq \dim L + 3h + 4$, $X \neq Cone_p(K)$ *for any point* p *and any curve* K, $h \geq 1$, *then* TX *is* h*-defective and* $\delta_h(TX) = \min\{2h + 1 - \dim L_{min}, N - \dim L_{min} - 3h - 3\}$, *where* L_{min} *is such a linear space of minimal dimension*;

(4) $X = Cone_p(C)$, *where* $p \in \mathbb{P}^N$ *is a point*, C *is a curve*, $N > 3h + 3$, *then* TX *is* h*-defective and* $\delta_h(TX) = \min\{N - 3h - 3, h\}$.

Proof. 1. $X = v_3(\mathbb{P}^2) \subset \mathbb{P}^9$, $h = 1$. Let us show that for general points $x_0, x_1 \in X$ one has $\dim\langle T_x^2 X, T_y^2 X\rangle = 8$. For any hyperplane $H \subset \mathbb{P}^9$, $v_3^{-1}(X \cap H)$ is a cubic. Moreover, $H \supset T_x^2 X$ iff the cubic $v_3^{-1}(X \cap H)$ has multiplicity 3 at the point $v_3^{-1}(x) \in \mathbb{P}^2$. Hence, $H \supset \langle T_{x_0}^2 X, T_{x_1}^2 X\rangle$ iff the cubic $v_3^{-1}(X \cap H)$ has multiplicity at least 3 at the points $v_3^{-1}(x_0)$ and $v_3^{-1}(x_1)$, i. e. $v_3^{-1}(X \cap H)$ is a triple line $\langle v_3^{-1}(x_0), v_3^{-1}(x_1)\rangle$. Since the map $v_3 : \mathbb{P}^2 \dashrightarrow \mathbb{P}^9$ is defined by the complete linear system of cubics in $\mathbb{P}^2$, there exists a unique hyperplane H with this property. So, $\dim\langle T_{x_0}^2 X, T_{x_1}^2 X\rangle = 8$. Therefore, by Corollary 4, $\delta_1(TX) = \min\{5 \cdot 1 + 4, N\} - \dim\langle T_{x_0}^2 X, T_{x_1}^2 X\rangle = \min\{9, 9\} - 8 = 9 - 8 = 1$.

2. If $X \subset Cone_L(TC)$, $\pi_L(X) = TC$ and $X \neq TK$ for any curve K, then $\dim TX = 4$. Really, if $\dim TX = 3$ then by Corollary 3 X is a surface in $\mathbb{P}^3$, a cone $Cone_p(K)$, or $X = TK$ for a certain curve K. The first is not the case because $N \geq \dim L + 4h + 5 \geq 5$. The second is not the case because for any linear space L, $\pi_L(X)$ could be a curve or a cone over a curve, but not TC. The last is not the case by the hypothesis.

Since $X \subset Cone_L(TC)$ and $\pi_L(X) = TC$, for a general point $p \in TX$ one has $\pi_L(T_pTX) = T_{\pi_L(p)}T\pi_L(X) = T_{\pi_L(p)}T^2C = T_y^3C$, where $y \in C$ is a point such that $\pi_L(p) \in T_y^2C$. Hence $\dim \pi_L(T_pTX) = \dim T_y^3C = 3$, and, since $\dim T_pTX = \dim TX = 4$, $\dim L \cap T_pTX = 0$. Suppose that L has minimal possible dimension. Since $\dim L \leq h - 1$, for general points $p_0, \dots, p_h \in TX$ one has $\dim\langle L \cap T_{p_0}TX, \dots, L \cap T_{p_h}TX\rangle \leq h - 1$, and $\langle L \cap T_{p_0}TX, \dots, L \cap T_{p_h}TX\rangle = \langle L \cap T_{p_0}TX, \dots, L \cap T_{p_m}TX\rangle$ for some $m < h$; so, for general $p_{m+1} \in TX$ one has $L \cap T_{p_{m+1}}TX \in \langle L \cap T_{p_0}TX, \dots, L \cap T_{p_m}TX\rangle$. Put $M = \langle L \cap T_{p_0}TX, \dots, L \cap T_{p_m}TX\rangle$. Then, for a general point $p \in TX$, $\dim M \cap T_pTX = 0$ and $\dim \pi_M(T_pTX) = 3$. So, $\dim T\pi_M(X) = \dim \pi_M(TX) = 3$, and by Corollary 3, $\pi_M(X)$ is a surface in $\mathbb{P}^3$, a cone over a curve, or TK for some curve K. But $M \subseteq L$, $N \geq \dim L + 4h + 5$ and $\pi_L(X) = \pi_{\pi_M(L)}(\pi_M(X))$ is TC, so, $\pi_M(X)$ can only be TK for some curve K. Hence, $X \subset Cone_M(TK)$. Since L has minimal dimension and $M \subseteq L$, one has $M = L$. Therefore $L = \langle L \cap T_{p_0}TX, \dots, L \cap$

$T_{p_m}TX\rangle = \langle L \cap T_{p_0}TX, \dots, L \cap T_{p_h}TX\rangle$ and $L \subset \langle T_{p_0}TX, \dots, T_{p_h}TX\rangle$. So, $\dim\langle T_{p_0}TX, \dots, T_{p_h}TX\rangle = \dim L + 1 + \dim\langle T^3_{y_0}C, \dots, T^3_{y_h}C\rangle$, where $y_0, \dots, y_h \in C$ are points such that $\pi_L(p_i) \in T^2_{y_i}C$ $(0 \le i \le h)$. Since $N \ge \dim L + 4h + 5$, $\dim\langle C\rangle = N - \dim L - 1 \ge (4h+5) - 1 = 4h+4$. So, by Theorem 1, $\dim\langle T^3_{y_0}C, \dots, T^3_{y_h}C\rangle = \min\{N - \dim L - 1, 4h+3\} = 4h+3$. Therefore $\delta_h(TX) = \min\{5h+4, N\} - \dim\langle T_{p_0}TX, \dots, T_{p_h}TX\rangle = \min\{5h+4, N\} - (\dim L + 1 + (4h+3)) = \min\{5h+4 - \dim L - 1 - (4h+3), (N - \dim L) - 1 - (4h+3)\} = \min\{h - \dim L, N - \dim L - 4h - 4\}$.

3. If $X \subset Cone_L(C)$ and $X \neq Cone_p(K)$ for any curve K and any point p, then $\dim TX = 4$. Really, if $\dim TX = 3$ then by Corollary 3 X is a surface in $\mathbb{P}^3$, a cone $Cone_p(K)$, or $X = TK$ for a certain curve K. The first is not the case because $N \ge \dim L + 3h + 4 \ge 4$. The second is not the case because of the hypothesis. The last is not the case because if $X = TK \subset Cone_L C$, then for a general point $x \in K$, $\pi_L(T_xK) \subset \pi_L(X) = C$ is a point or a line. Since C is not a line ($\dim\langle C\rangle = N - \dim L - 1 \ge 3h+3 \ge 3$), $\pi_L(T_xK)$ is a point. So, $\pi_L(K)$ is a point and X is degenerate, which is not the case.

If $X \subset Cone_L(C)$ then for a general point $p \in TX$ one has $\pi_L(T_pTX) = T_{\pi_L(p)}T\pi_L(X) = T_{\pi_L(p)}TC = T^2_yC$, where $y \in C$ is a point such that $\pi_L(p) \in T_yC$. Hence $\dim \pi_L(T_pTX) = \dim T^2_yC = 2$, and, since $\dim T_pTX = \dim TX = 4$, $\dim L \cap T_pTX = 1$. Suppose that L has minimal possible dimension. Since $\dim L \le 2h$, for general points $p_0, \dots, p_h \in TX$ one has $\dim\langle L \cap T_{p_0}TX, \dots, L \cap T_{p_h}TX\rangle \le 2h$, and $\langle L \cap T_{p_0}TX, \dots, L \cap T_{p_{h-1}}TX\rangle \cap (L \cap T_{p_h}TX) \neq \emptyset$. If $\dim(L \cap T_{p_h}TX) \cap \langle L \cap T_{p_0}TX, \dots, L \cap T_{p_{h-1}}TX\rangle = 1$, then $L \cap T_{p_h}TX \subset \langle L \cap T_{p_0}TX, \dots, L \cap T_{p_{h-1}}TX\rangle$ and $\dim T_{\pi(p)}T\pi(X) = \dim \pi(T_pTX) = 4 - 1 - 1 = 2$, where π is the projection from $\langle L \cap T_{p_0}TX, \dots, L \cap T_{p_{h-1}}TX\rangle \subset L$. Since $p_h \in TX$ is a general point, $\pi(X)$ is a curve. Since L has minimal dimension, $\dim L \le \dim\langle L \cap T_{p_0}TX, \dots, L \cap T_{p_{h-1}}TX\rangle$. So, $L = \langle L \cap T_{p_0}TX, \dots, L \cap T_{p_{h-1}}TX\rangle = \langle L \cap T_{p_0}TX, \dots, L \cap T_{p_h}TX\rangle$.

If $\dim(L \cap T_{p_h}TX) \cap \langle L \cap T_{p_0}TX, \dots, L \cap T_{p_{h-1}}TX\rangle = 0$, one has $\dim T_{\pi(p_h)}T\pi(X) = \dim \pi(T_{p_h}TX) = 3$, where π is the projection from $\langle L \cap T_{p_0}TX, \dots, L \cap T_{p_{h-1}}TX\rangle \subset L$. Since $p_h \in TX$ is a general point, $\dim \pi(TX) = 3$. So by Corollary 3, $\pi(X)$ is a surface in $\mathbb{P}^3$, $Cone_qK$, or TK for some curves C and K and a point q. But since $X \subset Cone_L(C)$ and $N \ge \dim L + 3h + 4$, $\pi(X) \subset Cone_{\pi(L)}(C)$, $\pi(X)$ can only be of the type $Cone_q(K)$, and $\pi(L) = q$ because L has minimal dimension. So, $\pi(L) = q \in T_{\pi(p_h)}T\pi(X) = \pi(T_{p_h}TX)$, and $L \subset \langle T_{p_h}TX, \langle L \cap T_{p_0}TX, \dots, L \cap T_{p_{h-1}}TX\rangle\rangle = \langle L \cap T_{p_0}TX, \dots, L \cap T_{p_h}TX\rangle$.

So, $L = \langle L \cap T_{p_0}TX, \dots, L \cap T_{p_h}TX\rangle$. Hence,$\dim\langle T_{p_0}TX, \dots, T_{p_h}TX\rangle =$

$\dim L + 1 + \dim\langle T^2_{y_0}C, \ldots, T^2_{y_h}C\rangle$, where $y_0, \ldots, y_h \in C$ are points such that $\pi_L(p_i) \in T_{y_i}C$ $(0 \le i \le h)$. Since $N \ge \dim L + 3h + 4$, $\dim\langle C\rangle = N - \dim L - 1 \ge 3h + 3$. So, by Theorem 1, $\dim\langle T^2_{\pi_L(y_0)}C, \ldots, T^2_{\pi_L(y_h)}C\rangle = \min\{N - \dim L - 1, 3h+2\} = 3h+2$. Therefore $\delta_h(TX) = \min\{5h+4, N\} - \dim\langle T_{p_0}TX, \ldots, T_{p_h}TX\rangle = \min\{5h + 4, N\} - (\dim L + 1 + (3h + 2)) = \min\{5h+4-\dim L-3h-3, N-\dim L-3h-3\} = \min\{2h+1-\dim L, N-\dim L - 3h - 3\}$.

4. If $X = Cone_p(C)$, then $TX = Cone_p(TC)$ and $S^h(TX) = Cone_p(S^h(TC))$. By Theorem 1, the dimension of $S^h(TC)$ is always equal to the expected one, therefore, since $C \subset \mathbb{P}^{N-1}$ is a non-degenerate curve, $\dim S^h(TX) = 1+\dim S^h(TC) = 1+\min\{N-1, 3h+2\}$. So, $\dim S^h(TX) = \min\{N, 3h + 3\} = 3h + 3$. Hence $\delta_h(TX) = \min\{N, 3(h + 1) + h\} - \dim S^h(TX) = \min\{N, 4h + 3\} - 3h - 3 = \min\{N - 3h - 3, h\}$.

Proposition 8. *The different classes of surfaces described in Proposition 7 have empty intersection with each other even for different values of h.*

Proof. It is clear, that the surfaces described in items 1 and 4 cannot belong to other classes.

Assume that there exists a surface $X \subset \mathbb{P}^N$ such that the following conditions hold:

2. $X \subset Cone_{L_1}(TC_1)$, where $L_1 \subset \mathbb{P}^N$ is a linear subspace, $0 \le \dim L_1 \le h_1 - 1$, C_1 is a curve, $N \ge \dim L_1 + 4h_1 + 5$, $\pi_{L_1}(X) = TC_1$, $X \ne TK$ for any curve K, $h_1 \ge 1$;
3. $X \subset Cone_{L_2}(C_2)$, where $L_2 \subset \mathbb{P}^N$ is a linear subspace, $0 < \dim L_2 \le 2h_2$, C_2 is a curve, $N \ge \dim L_2 + 3h_2 + 4$, $X \ne Cone_p(K)$ for any point q and any curve K, $h_2 \ge 1$.

Then, as we saw above, for a general point $x \in X$, $p \in TX$, $\dim L_1 \cap T_pTX = 0$, $\dim L_2 \cap T_pTX = 1$.

If $L_1 \cap T_pTX \subset L_2$, then for $L = L_1 \cap L_2$ one has $\dim L \cap T_pTX = 0$ and $3 = \dim \pi_L(T_pTX) = \dim T_{\pi_L(p)}T\pi_L(X) = \dim T\pi_L(X)$, where π_L is the projection from L. Hence, by Corollary 3, $\pi_L(X)$ is a surface in $\mathbb{P}^3$, TC or $Cone_r(C)$ for a certain curve C and a point r. Since $\dim \pi_L(\mathbb{P}^N) = N - \dim L - 1 \ge N - \dim L_1 - 1 \ge 4h_1 + 5 \ge 9$, $\pi_L(X)$ cannot be a surface in $\mathbb{P}^3$. Moreover, since $L \subset L_1$, after taking the projection from $\pi_L(L_1)$, one has $\pi_{\pi_L(L_1)}\pi_L(X) = \pi_{L_1}(X) = TC_1$. So, $X = TC$. On the other hand, since $L \subset L_2$, after taking the projection from $\pi_L(L_2)$, one has $\pi_{\pi_L(L_2)}\pi_L(X) = \pi_{L_2}(X) = C_2$, i.e. $\pi_{\pi_L(L_2)}TC = C_2$. Hence, for a general point $y \in C$, $T_yC \cap \pi_L(L_2) \ne \emptyset$. So, $\pi_{\pi_L(L_2)}C$ is a point, but not a curve, which is impossible.

Therefore, $L_1 \cap T_pTX \not\subset L_2$. Hence, after the projection from L_2 one has $\pi_{L_2}(L_1) \cap T_{\pi_{L_2}(p)}T\pi_{L_2}(X) = \pi_{L_2}(L_1 \cap T_pTX) \neq \emptyset$. Since $\pi_{L_2}(X) = C_2$, $T_{\pi_{L_2}(p)}T\pi_{L_2}X = T_y^2C_2$ for a certain point $y \in C_2$. Since $p \in TX$ is a general point, $\pi_{L_2}(L_1) \cap T_y^2C_2 \neq \emptyset$ for a general point $y \in C_2$. So, after taking the projection from $\pi_{L_2}(L_1)$ one has $\dim_2 \pi_{\pi_{L_2}(L_1)}(C_2) = \dim \pi_{\pi_{L_2}(L_1)}(T_y^2C_2) \leq 2 - 1 = 1$. By Proposition 4, $\langle \pi_{\pi_{L_2}(L_1)}(C_2) \rangle \leq 1$. Hence, the codimension of $\pi_{L_2}(L_1)$ in $\pi_{L_2}(\mathbb{P}^N)$ is not more than 2, which is equivalent to $\dim\langle L_1, L_2\rangle \geq N - 2$.

But on the other hand from the restrictions for L_1 one has $N \geq \dim L_1 + 4h_1 + 5 \geq \dim L_1 + 4(\dim L_1 + 1) + 5 = 5\dim L_1 + 9$ or $\dim L_1 \leq \frac{N-9}{5}$. From the restrictions for L_2 one has $N \geq \dim L_2 + 3h_2 + 4 \geq \dim L_2 + 3\frac{\dim L_2}{2} + 4 = \frac{5}{2}\dim L_2 + 4$ or $\dim L_2 \leq \frac{2(N-4)}{5}$. So, $\dim L_1 + \dim L_2 \leq \frac{N-9}{5} + \frac{2(N-4)}{5} = \frac{3N-17}{5} < \frac{5N-15}{5} = N-3$ and $\dim\langle L_1, L_2\rangle \leq \dim L_1 + \dim L_2 + 1 < N-3+1 = N-2$. This contradiction proves our Proposition. □

6.2. Smooth examples There exist smooth varieties among the surfaces with defective tangent surfaces described in Proposition 7.

(1) $X = v_3(\mathbb{P}^2)$ (this example was known to Bronowski, see [2,3]).
(2) If $X = Scroll_{2h-1,k}$, $k \geq 3(h+1)$, then $X \subset Cone_L(C)$ for $C = v_k(\mathbb{P}^1)$ and a certain $L \subset \mathbb{P}^N$, $\dim L = 2h-1$; $\dim_2 X = 4$. One has $\delta_{h+l}(TX) = \min\{2(h+l)+1-(2h-1), (k+(2h-1)+1)-(2h-1)-3(h+l)-3\} = \min\{2l+2, k-3h-3-3l+1\}$, and $\delta_{h+l}(TX) > 0$ iff $0 \leq l \leq \frac{k-3(h+1)}{3}$.
(3) If $X = Scroll_{2h,k}$, $k \geq 3(h+1)$, then $X \subset Cone_L(C)$ for $C = v_k(\mathbb{P}^1)$ and a certain $L \subset \mathbb{P}^N$, $\dim L = 2h$; $\dim_2 X = 4$. One has $\delta_{h+l}(TX) = \min\{2(h+l)+1-2h, (k+2h+1)-2h-3(h+l)-3\} = \min\{2l+1, k-3h-3-3l+1\}$, and $\delta_{h+l}(TX) > 0$ iff $0 \leq l \leq \frac{k-3(h+1)}{3}$.
(4) If $X = Scroll_{0,2h-1,k} \cap G$, where $k \geq 3(h+1)$, $G \subset \mathbb{P}^N$ is a non-degenerate general hypersurface, then $X \subset Cone_L(C)$, for $C = v_k(\mathbb{P}^1)$ and a certain $L \subset \mathbb{P}^N$, $\dim L = 2h$; $\dim_2 X = 5$. As we saw above, $\delta_{h+l}(TX) = \min\{2l+1, k-3h-3-3l+1\}$, and $\delta_{h+l}(TX) > 0$ iff $0 \leq l \leq \frac{k-3(h+1)}{3}$. Particularly, for $h = 1$ we obtain 1-defective surface X with $\delta_1(TX) = 1$, $X \subset Cone_L(C)$, where L is a plane.

7. Main theorem.

Let X be a non-degenerate surface in $\mathbb{P}^N$, $N > 2$.

Theorem 3. *The tangential variety TX to a surface $X \subset \mathbb{P}^N$ has positive h-th-defect iff the pair of X and h is one of the pairs described in Proposition*

7.

Corollary 5. *The tangential variety* $TX \subset \mathbb{P}^N$ *to a surface* $X \subset \mathbb{P}^N$ *is* h*-defective iff* X *is one of the following surfaces*:

(1) $v_3(\mathbb{P}^2) \subset \mathbb{P}^9$, $h = 1$; $\delta_1(TX) = 1$;
(2) $Cone_p(C)$, *where* $p \in \mathbb{P}^N$ *is a point,* C *is a curve,* $N \geq 7$, $h = 1$; $\delta_1(TX) = 1$;
(3) *a non-degenerate subvariety of* $Cone_L(TC)$, *where* $L \subset \mathbb{P}^N$ *is a linear subspace,* $\dim L = h-1$, C *is a curve,* $N \geq 5h+4$, *such that* $\pi_L(X) = TC$ *and for any (even empty) linear subspace* $l \subset L$, $\dim T(\pi_l(X)) > 3$; $\delta_h(TX) = 1$;
(4) *a non-degenerate subvariety of* $Cone_L(C)$, *where* $L \subset \mathbb{P}^N$ *is a linear subspace,* $2h-1 \leq \dim L \leq 2h$, C *is a curve,* $N \geq \dim L + 3h + 4$, *such that for any linear subspace* $l \subset L$ $\dim \pi_l(X) = 2$; $\delta_h(TX) = \min\{2h+1-\dim L, N - \dim L - 3h - 3\}$.

8. Proof of Theorem 3, the case $\dim TX = 3$.

In this case by Corollary 3, X is a surface in $\mathbb{P}^3$, a cone $Cone_p(C)$ over a curve C, or $X = TK$ for a certain curve K.

1. If $X \subset \mathbb{P}^3$ then $TX = \mathbb{P}^3 = S^h(TX)$, and the dimension of $S^h(TX)$ is equal to the expected one for any h.

2. If $X = Cone_p(C)$, then as we saw in the proof of Proposition 7, item 4, $\dim S^h(TX) = \min\{N, 3h+3\}$ and $\delta_h(TX) = \min\{N, 3(h+1)+h\} - \dim S^h(TX) = \min\{N, 4h+3\} - \min\{N, 3h+3\}$. Therefore $\delta_h(TX) > 0$ iff $N > 3h+3$ $(N \geq 3h+4)$.

3. If $X = TK$ then $TX = T^2K$, which is always non-h-defective by Theorem 1.

9. Proof of Theorem 3, the case $\dim TX = 4$, $h = 1$.

By Corollary 2 applied to $k = n = 2$, $4 \leq \dim_2 X \leq 5$.

Consider general points $x, y \in X$, $p \in T_xX$, $r \in T_yX$, $q \in \langle p, r\rangle$. Since $\delta_1(TX) > 0$, by Theorem 2 $T_qS(TX) = \langle T_x^2X, T_y^2X\rangle$, and $\dim\langle T_x^2X, T_y^2X\rangle < \min\{9, N\}$.
Hence, $\dim T_x^2X \cap T_y^2X = \dim T_x^2X + \dim T_y^2X - \dim\langle T_x^2X, T_y^2X\rangle \geq 2\dim_2 X - \min\{9, N\} + 1 = \max\{2\dim_2 X - 8, 2\dim_2 X - N + 1\}$. Fix $x \in X$ and consider the projection π from T_x^2X, $\pi : \mathbb{P}^N \dashrightarrow \mathbb{P}^{N-\dim_2 X-1}$. Then for

a general point $y \in X$, $T^2_{\pi(y)}\pi(X) = \pi(T^2_y X)$, so $\dim T^2_{\pi(y)}\pi(X) \leq \dim_2 X - \max\{2\dim_2 X - 8, 2\dim_2 X - N + 1\} - 1 = \min\{7 - \dim_2 X, N - \dim_2 X - 2\} \leq 7 - \dim_2 X \leq 7 - 4 = 3$.

If $\dim T^2_{\pi(y)}\pi(X) \leq 1$, then $\pi(X) = \langle\pi(X)\rangle = \dim T^2_{\pi(y)}\pi(X)$, and $X \subset \langle T^2_x X, T^2_y X\rangle$. Hence, $\dim\langle T^2_x X, T^2_y X\rangle = N$, which is impossible because $\dim\langle T^2_x X, T^2_y X\rangle < \min\{9, N\}$.

If $\dim T^2_{\pi(y)}\pi(X) = 2$, then $\pi(X)$ is a curve. Since $\dim T^2_{\pi(y)}\pi(X) \leq \min\{7 - \dim_2 X, N - \dim_2 X - 2\} \leq N - \dim_2 X - 2$, $N \geq \dim_2 X + 4$. More, $\dim T^2_x X \cap T^2_y X = \dim T^2_y X - \dim T^2_{\pi(y)}\pi(X) - 1 = \dim_2 X - 3$.

In the case $\dim T^2_{\pi(y)}\pi(X) = 3$, $\pi(X)$ is a surface. Since $\dim T^2_{\pi(y)}\pi(X) \leq \min\{7 - \dim_2 X, N - \dim_2 X - 2\}$, $7 - \dim_2 X \geq 3$ and $N - \dim_2 X - 2 \geq 3$, which is equivalent to $\dim_2 X \leq 4$ and $N \geq \dim_2 X + 5$. Since $\dim_2 X \geq 4$, $\dim_2 X = 4$ and $N \geq 9$. More, $\dim T^2_x X \cap T^2_y X = \dim T^2_y X - \dim T^2_{\pi(y)}\pi(X) - 1 = \dim_2 X - 3 - 1 = 0$.

9.1. $\dim \pi(X) = 1$. In this case $N \geq \dim_2 X + 4$, $\dim T^2_x X \cap T^2_y X = \dim_2 X - 3$.

Proposition 9. *If* $\dim \pi(X) = 1$, *then one of the following conditions holds*:

(1) $X \subset Cone_L(C)$, $\dim L = \dim_2 X - 3$, C *is a curve* ($N \geq \dim_2 X + 4 = \dim L + 7$);
(2) $\dim_2 X = 5$, $N = 9$, $X = v_3(\mathbb{P}^2)$.

Proof. Consider a general fiber $K = \overline{X \cap (\pi^{-1}(\pi(y)) \setminus T^2_x X)}$ for a general point $y \in X$. Take another general point $z \in K \subset X$. If $K \not\subset T^2_z X$ consider the projection $\pi' = \pi_{T^2_z X}$. Since $\dim T^2_x X \cap T^2_z X = \dim_2 X - 3$ and $z \in \pi^{-1}(\pi(y)) \supset T^2_x X$, $\dim T^2_z X \cap \pi^{-1}(\pi(y)) = \dim_2 X - 2$, and $\dim \pi'(\pi^{-1}(\pi(y))) \leq \dim \pi^{-1}(\pi(y)) - \dim(T^2_z X \cap \pi^{-1}(\pi(y))) - 1 = \dim_2 X + 1 - (\dim_2 X - 2) - 1 = 2$. So, $\pi'(K)$ lies in a plane, but it is also a subset of the curve $\pi'(X)$. Hence, $\pi'(K)$ is a subset of $\pi'(X) \cap \pi'(\pi^{-1}(\pi(y)))$, which is a finite number of points. So, for every component $K_i \subset K$ one has $\pi'(K_i)$ is a point and K_i is also a component of a fiber under the projection from $T^2_z X$. More, the linear span of such a component has codimension at least 2 in $\pi^{-1}(\pi(y))$, so its dimension is at most $\dim_2 X + 1 - 2 = \dim_2 X - 1$.

Now take all components of all fibers under all projections from $T^2_y X$ for a general $y \in X$. By taking the closure, we obtain the family $\mathcal{L}$ of curves on X.

Lemma 1. $\dim \mathcal{L} \leq 2$. *If* $\dim \mathcal{L} = 1$, *then* X *sits in a cone over a curve with the vertex of dimension* $\dim_2 X - 3$.

Proof. If for general component K of the fiber under the projection $\pi_{T_x^2X}$ one has $K \subset T_z^2X$ for $z \in K$, then K does not depend on $x \in X$. So, in this case $\dim \mathcal{L} = 1$. If $K \not\subset T_z^2X$, then primarily, after taking all components of all fibers under all projections from T_x^2X for a general $x \in X$, we can obtain a family of dimension no more than 3. Since every component of a general fiber is also a component under the projection from the osculating space related to a general point of this component, $\dim \mathcal{L} \leq 2$.

Suppose now, that $\dim \mathcal{L} = 1$. This means that through a general point $y \in X$ there passes only finite number of fibers. Take one such fiber K. Then for a general point $x \in X$, $K \subset \langle y, T_x^2X\rangle$. So, for general $z \in X$, $K \subset \langle y, T_x^2X\rangle \cap \langle y, T_z^2X\rangle$. We will show now that $\langle y, T_x^2X\rangle \cap \langle y, T_z^2X\rangle = \langle y, T_x^2X \cap T_z^2X\rangle$. Assume the opposite. Then there exists a line $l \ni y$, $l \cap T_x^2X \neq \emptyset$ and $l \cap T_z^2X \neq \emptyset$, but $l \cap T_x^2X \neq l \cap T_z^2X$. Hence, $T_z^2X \cap \langle y, T_x^2X\rangle \supset T_z^2X \cap T_x^2X$ and $\dim T_z^2X \cap \langle y, T_x^2X\rangle > \dim T_z^2X \cap T_x^2X = \dim_2 X - 3$. So, $\dim T_z^2X \cap \langle y, T_x^2X\rangle \geq \dim_2 X - 2$ and after the projection from $\langle y, T_x^2X\rangle$ (to $\mathbb{P}^{N-\dim_2 X-2}$) one can get that the osculating space to the image has dimension $\dim_2 X - (\dim_2 X - 2) - 1 = 1$; so the image is a line in $\mathbb{P}^{N-\dim_2 X-2}$ and $N = \dim_2 X + 2 + 1 = \dim_2 X + 3$, which is not the case.

So, $\langle y, T_x^2X\rangle \cap \langle y, T_z^2X\rangle = \langle y, T_x^2X \cap T_z^2X\rangle$ and the subspace $\langle y, T_x^2X \cap T_z^2X\rangle$ contains $\langle K\rangle$, i. e. $\pi_{T_x^2X \cap T_z^2X}(K)$ is a point. Varying $y \in X$ one can obtain that $\dim \pi_{T_x^2X \cap T_z^2X}(X) = 1$, and X sits in a cone with the vertex $T_x^2X \cap T_z^2X$ of dimension $\dim_2 X - 3$ over a certain curve C. If $X \subset Cone_M(C_1)$, $0 \leq \dim M < \dim_2 X - 3$, then $T_x^2X = Cone_M(T_j^2C)$, where $j \in C$, $x \in \langle M, j\rangle$, and $\dim_2 X = \dim T_x^2X = \dim M + 1 + 2 < \dim_2 X$, which is impossible.

Lemma 2. *If* $\dim \mathcal{L} = 2$, *then* $\dim_2 X = 5$, $N = 9$ *and* $X = v_3(\mathbb{P}^2)$.

Proof. Through a general point $x \in X$ there passes an one-dimensional subfamily of $\mathcal{L}$. We saw, that all curves from this subfamily are components of fibers under the projection π from T_x^2X. Hence any fiber under this projection contains x. Take now a general hyperplane $H \supset T_x^2X$. Then $H \cap X$ contains $\deg \pi(X)$ fibers, passing through x. So, the multiplicity μ of $H \cap X$ at x is at least $\deg \pi(X)$. On the other hand, by Proposition 2, since $H \supset T_x^2X$, $\mu \geq 3$; since $H \not\supset T_x^3X$, $\mu < 4$. So, $\mu = 3$ and $\deg \pi(X) \leq \mu = 3$. Since $\pi(X)$ is irreducible and non-degenerate in $\mathbb{P}^{N-\dim_2 X-1}$, $N - \dim_2 X - 1 \geq 3$, $\deg \pi(X) = 3$, $N = \dim_2 X + 4$. Moreover, for general $K_1, K_2 \in \mathcal{L}$ one has $K_1 \cap K_2 \neq \emptyset$, and if $x \in K_1 \cap K_2$, x is general for X, then after the projection π from T_x^2X it appears that $\pi(K_1)$ and $\pi(K_2)$ are points on $v_3(\mathbb{P}^1)$. Hence, K_1 and K_2 are linearly equivalent. Moreover, if $H \subset \mathbb{P}^N$ is a hyperplane, $H \cap X$ is linearly equivalent to $3K$, $K \in \mathcal{L}$.

Now we prove the irreducibility of a general fiber of π. Through the point x there passes only one component of every fiber because of the same multiplicity counting. Now, take all components, passing through x. The closure of the variety swept out by them is 2-dimensional, so it is X. Hence, a general point of X lies on a component, containing x, of the corresponding fiber. So, such components are fibers themselves.

Show now that for a general point $z \in X$ the intersection $T_z^2 X \cap X$ consists of a finite number of points. Assume the opposite. Then $T_z^2 X \supset C_z$, C_z is a curve. One has $\pi(C_z) \subset \pi(T_z^2 X \cap X) = T^2_{\pi(z)}\pi(X) \cap \pi(X) = \pi(z)$, because $\pi(X) = v_3(\mathbb{P}^1)$. So, $C_z \subset \pi^{-1}(\pi(z))$. For a general point $y \in \pi^{-1}(\pi(z)) \cap X$ we also have $C_y \subset \pi^{-1}(\pi(y)) = \pi^{-1}(\pi(z))$. Since $X \cap \pi^{-1}(\pi(z))$ is a curve, $C_y = C_z$. Varying $x, z \in X$ one can obtain that for general points $z, y \in X$, $C_y = C_z$. Put $C = C_y$ for general $y \in X$. So, for general points $y, z \in X$, $C \subset T_y^2 X$, $C \subset T_z^2 X$ and $C \subset T_y^2 X \cap T_z^2 X$. Since $\dim T_y^2 X \cap T_z^2 X = \dim_2 X - 3$ and $\dim\langle C\rangle \geq 1$, after the projection $\pi_{\langle C\rangle}$ one has $\dim_2 X - 2 \geq \dim T^2_{\pi_{\langle C\rangle}(y)}\pi_{\langle C\rangle}(X) \geq \dim_2 X - (\dim_2 X - 3) - 1 = 2$. If $\dim_2 \pi_{\langle C\rangle}(X) = 2$, then $C' = \pi_{\langle C\rangle}(X)$ is a curve and $X \subset Cone_{\langle C\rangle}(C')$, $\dim\langle C\rangle \leq \dim_2 X - 3$. In this case $\dim \mathcal{L} = 1$, which is impossible. So, $\dim_2 \pi_{\langle C\rangle}(X) = \dim_2 X - 2 \neq 2$. Hence, $\dim_2 X = 5$ and $\dim_2 \pi_{\langle C\rangle}(X) = 3$. Moreover, since $\dim T_z^2 X \cap T_y^2 X = 2$ and $\dim\langle C\rangle = 1$, $\langle C\rangle \neq T_y^2 X \cap T_z^2 X$ and $T^2_{\pi_{\langle C\rangle}(y)}\pi_{\langle C\rangle}(X) \cap T^2_{\pi_{\langle C\rangle}(z)}\pi_{\langle C\rangle}(X) \neq \emptyset$. Since $\dim T\pi_{\langle C\rangle}(X) = 3$, by Corollary 3, $\pi_{\langle C\rangle}(X) = Cone_p(C')$ for a certain point p and a curve C'. Hence, $X \subset Cone_{\pi^{-1}_{\langle C\rangle}(p)}(C')$, $\dim \pi^{-1}_{\langle C\rangle}(p) = 1 + 1 = 2$. And again we obtain that $\dim \mathcal{L} = 1$, which is not the case. So, $T_z^2 Z X \cap X$ for a general point $z \in X$ does not contain curves.

The case $\dim_2 X = 4$. $X \subset \mathbb{P}^8$, for general points $x, y \in X$, $\dim T_x^2 X \cap T_y^2 X = 1$, fibers under all projections from $T_x^2 X$ for general $x \in X$ are irreducible space curves (in general) and make a 2-dimensional linear system $\mathcal{L}$ on X.

Consider a general curve $K \in \mathcal{L}$ and a general point $t \in X \setminus K$. Since K is not a fiber under $\pi_{T_t^2 X}$, $\pi_{T_t^2 X}(K) = v_3(\mathbb{P}^1)$. But $\dim\langle K\rangle \leq 3$, so, $K = v_3(\mathbb{P}^1)$.

Since $\pi(X)$ is a curve, $\dim T_x^2 X \cap T_y X = 0$ for a general point $y \in X$. Consider the projection π' from $T_x X$. So, for a general point $y \in X$, $\dim T_x X \cap T_y^2 X = 0$ and $3 = \dim \pi'(T_y^2 X) = \dim T^2_{\pi'(y)}\pi'(X) = \dim_2 \pi'(X) = \dim T\pi'(X)$. Hence, by Corollary 3, $\pi'(X)$ is a surface in $\mathbb{P}^3$, $Cone_p(C)$ or TC for a certain point p and a certain curve C. Since $\dim\langle \pi'(X)\rangle = 8 - 2 - 1 = 5 > 2$, $\pi'(X)$ cannot be a surface in $\mathbb{P}^3$. Moreover, for a general curve $K \in \mathcal{L}$, $K \ni x$, $K = v_3(\mathbb{P}^1)$ and $T_x K \subset T_x X$. So, $\pi'(K)$ is a line or a point. If $\pi'(K)$

is a point, then $\pi'(X)$ is a curve, $\deg \pi'(X) \geq 5 - 1 + 1 = 5$. Arguing as above, one can obtain that for any hyperplane $H \subset T_xX$ the multiplicity of $H \cap X$ at x is not less than 5, but by Proposition 2 it should be equal to 2. So, $\pi'(X)$ is a surface and $\pi'(K)$ is a line. Moreover, the line $\pi'(K)$ contains the point $\pi'(T_x^2K) \in \pi'(T_x^2X)$. So, $\pi'(X)$ is swept out by one-dimensional family of lines intersecting the line $\pi'(T_x^2X)$. If $\pi'(X) = TC$, then this family of lines has to be the family of tangent lines to C. So, all tangent lines to C intersect the line $\pi'(T_x^2X)$. Hence, C is a plane curve and $\pi'(X) = TC$ is a plane, which is impossible. Therefore, $\pi'(X) = Cone_p(C)$. Consider the 3-dimensional linear subspace $L_x = \pi'^{-1}(p)$. One has $\pi_{L_x}(X) = C$ is a curve. So, for any hyperplane $H \supset L_x$ the multiplicity of $H \cap X$ at x is at least $\deg C$. Since $\dim\langle C\rangle = 8 - 3 - 1 = 4$, $\deg C \geq 4$. By Proposition 2, $H \supset T_x^3X$. Therefore, $L_x \supset T_x^3X$ and $\dim_3 X \leq \dim L_x = 3 < 4 = \dim_2 X$, which is impossible.

So, $\dim_2 X = 4$ is not the case.

The case $\dim_2 X = 5$. $X \subset \mathbb{P}^9$, $\pi_{T_x^2X}(X)$ is a twisted cubic, fibers under the projections from T_x^2X for general $x \in X$ are irreducible curves in $\mathbb{P}^4$ (in general) and make a 2-dimensional linear system $\mathcal{L}$ on X.

Consider a general curve K and a general point $t \in X \setminus K$. Since K is not a fiber under $\pi_{T_t^2X}$, $\pi_{T_t^2X}(K) = v_3(\mathbb{P}^1)$. So $\dim\langle K\rangle \geq 3$, and $\dim\langle K\rangle$ is equal to 3 or 4. Consider these cases.

The case $\dim\langle K\rangle = 4$. Put $\langle K\rangle \cap T_t^2X = p_K$, p_K is a point. Then $\pi_{p_K}(K) = \pi_{T_t^2X}(K) = \pi_{T_t^2X}(X)$ is a twisted cubic. For a general point $y \in K$, $\pi_{p_K}^{-1}(\pi_{p_K}(y)) = \langle p_K, y\rangle$ and $\pi_{p_K}^{-1}(\pi_{p_K}(y)) = \langle K\rangle \cap \pi_{T_t^2X}^{-1}(\pi_{T_t^2X}(y)) = \langle K\rangle \cap \langle K_1, T_t^2X\rangle \supset \langle K\rangle \cap \langle K_1\rangle$, where $K_1 \in \mathcal{L}$, $K_1 \ni y$ is the corresponding fiber under the projection from T_t^2X. So, for two (general in $\mathcal{L}$) curves K and K_1 the linear subspaces $\langle K\rangle$ and $\langle K_1\rangle$ intersect each other at most by a line. Moreover, $K \cap \langle p_K, y\rangle \subset X \cap \langle p_K, y\rangle = X \cap (\langle K\rangle \cap \langle K_1, T_t^2X\rangle) \subset X \cap \langle K_1, T_t^2X\rangle = K_1 \cup (X \cap T_t^2X)$. So, $K_1 \supset (\langle p_K, y\rangle \cap K) \setminus T_t^2X$.

Assume that $p_K \in K$. So, $p_K \in X$ and $p_K \in T_t^2X \cap X$. As we saw above, $T_t^2X \cap X$ does not contain curves. Hence, $p_t = p_K$ does not depend on K and depends only on t. Since for a general point $t \in X$, $p_t \subset \langle K\rangle \cap \langle K_1\rangle$ for general curves $K, K_1 \in \mathcal{L}$, there exists a line $l \subset \mathbb{P}^9$ such that $l \ni p_t$ for a general point $t \in X$ or $p = p_t$ does not depend on $t \in X$. In the first case after the projection π_l from l one has for general $K, K_1 \in \mathcal{L}$ the intersection $\pi_l(K) \cap \pi_l(K_1)$ cannot be empty, and $\dim\langle K\rangle \cap \langle K_1\rangle = \dim Cone_l(\langle \pi_l(K)\rangle \cap \langle \pi_l(K_1)\rangle) \geq 1 + 1 + 0 = 2$, which is not the case. If $p = p_t$ does not depend on t, then $T_t^2X \ni p$ and after the projection $\pi_p : \mathbb{P}^9 \dashrightarrow \mathbb{P}^8$ one has $\dim_2 \pi_p(X) = \dim T^2_{\pi_p(t)}(\pi_p(X)) = \dim \pi_p(T_t^2X) = 4$ and $\pi_{T^2_{\pi_p(t)}(\pi_p(X))}(\pi_p(X)) = \pi_{T_t^2X}(X)$ is a twisted cubic. As we saw above in the case $\dim_2 X = 4$, this is impossible.

So, $p_K \notin K$ and $K_1 \supset (\langle p_K, y\rangle \cap K) \setminus T_t^2 X = \langle p_K, y\rangle \cap K$. Also since $\pi_{p_K}(K) = v_3(\mathbb{P}^1)$, $\pi_{p_K}^{-1}(\pi_{p_K}(y)) \cap K = \langle p_K, y\rangle \cap K$ contains at least 2 different points for a general point $y \in K$. Hence, $\langle K\rangle \cap \langle K_1\rangle = \langle K \cap K_1\rangle$ is a line. Moreover, K_1 and K are fibers under the projection from $T_y^2 X$, because $y \in K_1$, $y \in K$. Two fibers can intersect only in the points of $T_y^2 X \cap X$, which does not contain curves, and if $K_2 \in \mathcal{L}$ is another fiber under this projection, then $K \cap K_1 \cap K_2 = K \cap K_1$, i.e. the set $K_1 \cap K$ depend only on $y \in K \cap K_1$. Hence if we vary $t \in X$, the fibers of the projection $\pi_{T_t^2 X} : K \dashrightarrow v_3(\mathbb{P}^1)$ remain the same and $T_t^2 X \ni p_K$. Take 4 general curves $K_1, K_2, K_3, K_4 \in \mathcal{L}$ and consider the lines $\langle K_1 \cap K_2\rangle$ and $\langle K_3 \cap K_4\rangle$. Take a curve $K_5 \in \mathcal{L}$, containing a point from $K_1 \cap K_2$ and a point from $K_3 \cap K_4$. So, $K_5 \supset K_1 \cap K_2$, $K_5 \supset K_3 \cap K_4$ and the lines $\langle K_1 \cap K_2\rangle$ and $\langle K_3 \cap K_4\rangle$ meet at p_{K_5}. Hence, we have a family of lines meeting each other. Since $X \not\subset \mathbb{P}^2$, the point p_K does not depend on $K \in \mathcal{L}$. Put $p = p_K$. Again after taking the projection π_p from that point p we obtain that $\dim_2 \pi_p(X) = \dim T^2_{\pi_p(t)} \pi_p(X) = \dim \pi_p(T_t^2 X) = 4$ and $\pi_{T^2_{\pi_p(t)}(\pi_p(X))}(\pi_p(X)) = \pi_{T_t^2 X}(X)$ is a twisted cubic. As we saw above in the case $\dim_2 X = 4$, this is impossible.

So, $\dim\langle K\rangle = 4$ is not the case.

If $\dim\langle K\rangle = 3$, then $K = v_3(\mathbb{P}^1)$. Also, $\pi_{T_t^2 X}|_K$ is an isomorphism, and $\# K \cap K_1 = \#\langle K\rangle \cap \langle K_1\rangle = 1$, where $K_1 \in \mathcal{L}$ is a general fiber of $\pi_{T_t^2 X}$, and the corresponding intersection is transverse. More, if we vary curves K and K_1, the point $K \cap K_1$ will also vary. Take two general curves $K_1, K_2 \in \mathcal{L}$ and put $M = \langle K_1, K_2\rangle$. Then $\dim M = \dim\langle K_1\rangle + \dim\langle K_2\rangle - \dim(\langle K_1\rangle \cap \langle K_2\rangle) = 3 + 3 - 0 = 6$. Consider the projection $\pi_M : \mathbb{P}^9 \dashrightarrow \mathbb{P}^2$ from M. For a general curve $K \in \mathcal{L}$, $\dim M \cap \langle K\rangle \geq 1$. So, $\pi_M(K)$ is a point or a line. In the first case also $\pi_M(X) = \pi_M(K)$ is a point, which is not the case. Hence, $\pi_M(K)$ is a line. Since $\pi_M(X)$ is non-degenerate, $\pi_M(X) = \mathbb{P}^2$. Moreover, since for a general hyperplane $H \subset \mathbb{P}^9$ the section $H \cap X$ is linearly equivalent to $3K$ and $\pi_M(K)$ is a line, π_M^{-1} is defined by a linear system of cubics on $\mathbb{P}^2$. But $\dim H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3)) = \binom{5}{2} = 10$ and we need $\langle \pi_M^{-1}(\mathbb{P}^2)\rangle = \mathbb{P}^9$. Therefore, π_M^{-1} is defined by the complete linear system of cubics on $\mathbb{P}^2$ and $X = v_3(\mathbb{P}^2)$. □

9.2. $\dim \pi(X) = 2$. In this case $\dim_2 X = 4$, $N \geq 9$, for general points $x, y \in X$, $\dim T_x^2 X \cap T_y^2 X = 0$, $\dim T^2_{\pi(y)} \pi(X) = 3$.

Proposition 10. *If* $\dim \pi(X) = 2$, *then one of the following conditions holds:*

(1) $X \subset Cone_L(C)$, *where* L *is a plane,* C *is a curve* ($N \geq \dim L + 7$);

(2) *$X \subset Cone_p(TK)$, where p is a point, K is a curve ($N \geq \dim p + 9$).*

Proof. Start our proof by the following lemma.

Lemma 3.

(1) *If $\exists p \in \mathbb{P}^N$, $T_x^2 X \ni p$ for general $x \in X$, then $X \subset Cone_p(TC)$ for a certain curve C.*
(2) *If $\exists L \subset \mathbb{P}^N$, L is a plane, $\dim T_x^2 X \cap L \geq 1$ for general $x \in X$, then $X \subset Cone_L(C)$ for a certain curve C.*
(3) *It is not the case that there exists a linear space $M \subset \mathbb{P}^N$, $\dim M \leq 6$, such that for general $x \in X$, $\dim M \cap T_x^2 X \geq 2$.*

Proof.

1. After the projection from p one has for a general point $x \in X$, $\dim T^2_{\pi_p(x)} \pi_p(X) = \dim \pi_p(T_x^2 X) = 3$. Hence by Corollary 3, $\pi_p(X)$ is a surface in $\mathbb{P}^3$, $Cone_q(C)$ for a certain point q and a certain curve C, or TC for a certain curve C. In the first case $N = 4$, which is not the case. In the second case $T_x^2 X$ always contains the line $\pi_p^{-1}(q)$, so for general $y \in X$, $\dim T_x^2 X \cap T_y^2 X \geq 1$, which is not the case. So, $\pi_p(X)$ is TC, and $X \subset Cone_p(TC)$.

2. After the projection from L one has for a general point $x \in X$, $\dim T^2_{\pi_L(x)} \pi_L(X) = \dim \pi_L(T_x^2 X) = 2$. Hence, $\pi_L(X)$ is a plane or a curve C. In the first case $N = 5$, which is not the case. So, $\pi_L(X)$ is C, and $X \subset Cone_L(C)$.

3. After the projection from M one has for a general point $x \in X$, $\dim T^2_{\pi_M(x)} \pi_M(X) = \dim \pi_M(T_x^2 X) = 1$. Hence, $\pi_M(X)$ is a line. In this case $N \leq 8$, which is not the case. □

By construction, $\pi(X)$ is a non-degenerate surface in $\mathbb{P}^{N - \dim_2 X - 1} = \mathbb{P}^{N-5}$, $N - 5 \geq 9 - 5 = 4$. So, $\dim T\pi(X) > 2$. But $3 = \dim T^2_{\pi(y)} \pi(X) \geq \dim T\pi(X)$. Therefore $\dim T\pi(X) = 3$. By Corollary 3, $\pi(X)$ is a surface in $\mathbb{P}^3$, $Cone_q(C)$ or TC. Since $\pi(X)$ is non-degenerate, $\pi(X)$ cannot be a surface in $\mathbb{P}^3$. So, $\pi(X)$ has the following property: it is swept out by an one-dimensional family of lines l_α, $\alpha \in R_x \subset G_x(1, N-5)$, $\dim R_x = 1$, and for a general point $z \in l_\alpha$ one has $T_z \pi(X) = T_\alpha^1$, $T_z^2 \pi(X) = T_\alpha^2$, where $T_\alpha^1, T_\alpha^2 \subset \mathbb{P}^{N-5}$ are certain spaces non dependent on the position of z inside l_α, $\dim T_\alpha^1 = 2$, $\dim T_\alpha^2 = 3$.

Take the preimage of one such line: $K = \overline{X \cap (\pi^{-1}(l_\alpha) \setminus T_x^2 X)}$ for a general point $\alpha \in R_x$, $K \subset \pi^{-1}(T_\alpha^2)$, $\dim \pi^{-1}(T_\alpha^2) = 8$. Take a general point $z \in K \subset X$. Then $\pi^{-1}(T_\alpha^2) = \langle T_x^2 X, T_z^2 X \rangle$. Consider the projection $\pi' = \pi_{T_z^2 X}$. Then $\pi'(\langle T_x^2 X, T_z^2 X \rangle) = \pi'(T_x^2 X) = T_\beta^2$ for some $\beta \in R_z \subset$

$G_z(1, N-5)$. So, if $K \not\subset T_z^2X$, then $\pi'(K) = l_\beta$. So, we need to consider two cases: $K \not\subset T_z^2X$ and $K \subset T_z^2X$.

The case $K \not\subset T_z^2X$. We have at most 2-dimensional family $\mathcal{L}$ of preimages of such lines under all projections from T_t^2X, t is a general point in X.

If $\dim \mathcal{L} = 2$, take a general point $y \in X$ and consider the preimage $K = \overline{X \cap (\pi^{-1}(l_\alpha) \setminus T_x^2X)}$ for $l_\alpha \ni \pi(y)$. Suppose $z \in K$ is a general point. Since $\pi(T_z^2X) = T_\alpha^2$ and codim l_α in T_α^2 is equal to 2, $\dim T_z^2X \cap \langle K \rangle \le \dim T_z^2X \cap \pi^{-1}(l_\alpha) = 2$. K is not a plane curve or a line because $K \not\subset T_z^2X$, and $\dim T_z^2K = 2$. Since $T_z^2X \cap \langle K \rangle \supset T_z^2K$, $\dim T_z^2X \cap \langle K \rangle = 2$ and $T_z^2X \cap \pi^{-1}(l_\alpha) = T_z^2K$. We saw that $\pi_{T_z^2X}(K)$ is a line. So, $\dim\langle K \rangle = 4$. Take now $T_K = \cup_{z \in K \cap Smooth(X)} T_zX \subset \pi^{-1}(T_\alpha^1)$. Since codim T_α^1 in T_α^2 is equal to 1, $\dim T_z^2X \cap \pi^{-1}(T_\alpha^1) = 3$. More, for a general point $q \in T_zX$ one has $T_z^2X \cap \pi^{-1}(T_\alpha^1) \subset T_qT_K$. Since $\dim T_z^2X \cap \pi^{-1}(T_\alpha^1) = 3 = \dim T_K$, $T_z^2X \cap \pi^{-1}(T_\alpha^1) = T_qT_K$. Consider now the projection $\pi'' = \pi_{T_y^2K}$. Since $\dim\langle K \rangle = 4$, $\dim T_y^2K \cap T_z^2K = 0$. Since $T_z^2K = T_z^2X \cap \pi^{-1}(l_\alpha) \subset T_z^2X \cap \pi^{-1}(T_\alpha^1) = T_qT_K$, $\dim T_y^2K \cap T_qT_K \ge 0$. Hence, $\dim \pi''(T_K) = \dim \pi''(T_qT_K) = \dim T_qT_K - 1 - \dim T_y^2K \cap T_qT_K \le 2$. Since $\dim \pi''(X) \ge \dim \pi_{T_y^2X}(X) = 2$ and y and z are general points in X (it is possible to make points $y, z \in X$ general, because $\dim \mathcal{L} = 2$), $\dim \pi''(X) = \dim \pi''(T_zX) = 2$ and $\dim \pi''(T_zX) = \dim \pi''(T_K)$. Hence, $\pi''(T_zX) = \pi''(T_K)$ and the subspace $M = \pi''^{-1}(\pi''(T_K))$ of dimension 5 has the following properties: $T_K \subset M$, i.e. for a general point $z \in K$ one has $T_zX \subset M$; $\dim T_z^2X \cap M = \dim T_qTX \cap M \ge \dim T_qT_K \cap M = \dim T_K = 3$. Hence, $\dim T_y^2X \cap M \ge 3$ too. Therefore, $\dim \pi_{T_y^2X}(M) \le 1$, and $\dim \pi_{T_y^2X}(T_zX) \le 1$, which is not the case because $y, z \in X$ are general.

So, $\dim \mathcal{L} = 1$, and for a general point $t \in X$ we have $T_t^2X \cap \langle K \rangle$ has codimension 2 in $\langle K \rangle$. Since for general $y \in X$, $\dim T_t^2X \cap T_y^2X = 0$, $\dim\langle K \rangle \le 4$. If $\dim\langle K \rangle = 4$, then $\dim T_t^2X \cap \langle K \rangle = 2$, and by Lemma 3 this is impossible. If $\dim\langle K \rangle = 3$, then we obtain a family of lines of type $T_t^2X \cap \langle K \rangle$. If for general $t, y \in X$ one has $T_t^2X \cap T_y^2X \notin \langle K \rangle$, then $\dim\langle K, T_t^2X \rangle \cap T_y^2X \ge 2$, which is impossible by Lemma 3. If for general t, y one has $T_t^2X \cap T_y^2X \in \langle K \rangle$, then all lines of type $T_t^2X \cap \langle K \rangle$ pass through a certain point $p \in \mathbb{P}^N$ or lie in a plane $L \subset \mathbb{P}^N$. By Lemma 3, $X \subset Cone_p(TC)$ or $X \subset Cone_L(C)$.

If $\dim \mathcal{L} = 1$ and $\dim\langle K \rangle \le 2$, then $K \subset T_z^2X$ for a general $z \in K$, which is not the case.

The case $K \subset T_z^2X$. Since $T_x^2X \cap T_z^2X$ is a point and $\pi_{T_x^2X}(K)$ is a line,

$\dim\langle K\rangle \leq 2$. If $\dim\langle K\rangle = 2$, then $\langle K\rangle \supset T_x^2X \cap T_z^2X$. Take $K_1 \subset T_x^2X \cap X$, which is the preimage of the corresponding line under the projection $\pi_{T_z^2X}$. $K \neq K_1$ because $K \subset T_z^2X$, $K_1 \subset T_x^2X$ and $\dim T_x^2X \cap T_z^2X = 0$. We also have $\dim\langle K_1\rangle = 2$ and $\langle K_1\rangle \supset T_x^2X \cap T_z^2X$. So, $\langle K\rangle \cap \langle K_1\rangle \neq \emptyset$ and we have a family of planes, intersecting each other. Then (see e.g. [8]) these planes either lie in $\mathbb{P}^5$, either intersect one fixed plane L by lines or pass through one point p. By Lemma 3 $X \subset Cone_p(TC)$ or $X \subset Cone_L(C)$.

If $\dim\langle K\rangle = 1$, then X is swept out by one-dimensional family of lines $R \subset G(1,N)$.

Lemma 4. *If $X \subset \mathbb{P}^N$ is swept out by one-dimensional family of lines, $\dim T_x^2X \cap T_y^2X = 0$ for general points $x, y \in X$ and $N \geq 9$, then either $X \subset Cone_p(TC)$ or $X \subset Cone_L(C)$ where $p \in \mathbb{P}^N$ is a point, $L \subset \mathbb{P}^N$ is a plane, C is a curve.*

Proof. For general $\alpha \in R$ put $T_\alpha = \overline{\cup_{z\in l_\alpha \cap Smooth(X)} T_zX}$ is a linear space, $2 \leq \dim T_\alpha \leq 3$. If $\dim T_\alpha = 2$, then $\dim TX \leq 2+1 = 3$, which is not the case. So, $\dim T_\alpha = 3$. Since for a general point $\gamma \in R_x$ one has T_γ^1 is a plane and $T_\gamma^1 = \pi(T_\alpha)$ for some $\alpha \in R$, $T_x^2X \cap T_\alpha \neq \emptyset$. Now take a general point $\alpha \in R$ and consider the projection from T_α to $\mathbb{P}^{N-4}$. Then again by Corollary 3, $\pi_{T_\alpha}(X)$ is a surface in $\mathbb{P}^3$, $Cone_q(K)$, or TK for a certain point $q \in \mathbb{P}^{N-4}$ and a certain curve K. Since $N \geq 9$, then the first is not the case. In all other cases again for a general line $l \in \pi_{T_\alpha}(X)$ and for general points $s, t \in l$, $T_s\pi_{T_\alpha}(X) = T_t\pi_{T_\alpha}(X)$. So, for general $\beta \in R$, $\dim \pi_{T_\alpha}(T_\beta) = 2$. Hence, for general $\alpha, \beta \in R$, $T_\alpha \cap T_\beta \neq \emptyset$. Knowing that consider again the projection π_{T_α}.

If $\pi_{T_\alpha}(X) = TK$, then $\pi_{T_\alpha}(T_\beta)$ is a osculating plane to K of order 2 for general $\beta \in R$. For general $\gamma \in R$, $\dim \pi_{\pi_{T_\alpha}(T_\beta)}(\pi_{T_\alpha}(T_\gamma)) = \dim_2 \pi_{\pi_{T_\alpha}(T_\beta)}(K) = \min\{2, \dim \pi_{\pi_{T_\alpha}(T_\beta)}(\pi_{T_\alpha}(\mathbb{P}^N))\} = \min\{2, (N-3-1)-2-1\} = \min\{2, N-7\}$ by Proposition 4. Since $N \geq 9$, $\dim \pi_{\pi_{T_\alpha}(T_\beta)}(\pi_{T_\alpha}(T_\gamma)) = 2 = \dim \pi_{T_\alpha}(T_\gamma)$. Hence, $\pi_{T_\alpha}(T_\beta) \cap \pi_{T_\alpha}(T_\gamma) = \emptyset$. Since $T_\beta \cap T_\gamma \neq \emptyset$, $T_\beta \cap T_\gamma \subset T_\alpha$. So, there exists a point $p \in \mathbb{P}^N$ such that for general $\alpha \in R$ and general $z \in l_\alpha$ one has $T_z^2X \supset T_\alpha \ni p$. By Lemma 3, $X \subset Cone_p(TC)$ for a certain curve C.

If $\pi_{T_\alpha}(X) = Cone_q(K)$ then for general $\beta \in R$, $\pi_{T_\alpha}(T_\beta) \ni q$. Hence, after projection from q one has $\pi_q(\pi_{T_\alpha}(X)) = K$ and $\pi_q(\pi_{T_\alpha}(T_\beta))$ is a tangent line to K. Since $\dim\langle K\rangle = (N-3-1)-1 = N-5 \geq 4$, for general $\beta, \gamma \in R$, $\pi_q(\pi_{T_\alpha}(T_\beta)) \cap \pi_q(\pi_{T_\alpha}(T_\gamma)) = \emptyset$. But $T_\beta \cap T_\gamma \neq \emptyset$. So, $(T_\beta \cap \pi_{T_\alpha}^{-1}(q)) \cap (T_\gamma \cap \pi_{T_\alpha}^{-1}(q)) \neq \emptyset$, and we have an one-dimensional family of lines $(T_\beta \cap \pi_{T_\alpha}^{-1}(q))$, $\beta \in R$, intersecting each other. Hence, either there exists a point $p \in \mathbb{P}^N$ such that for general $\beta \in R$, $p \in T_\beta$ or there exists a plane $L \subset \mathbb{P}^N$ such that for general $\beta \in R$, $\dim L \cap T_\beta = 1$. By Lemma

3, in the first case $X \subset Cone_p(TC)$, in the second case $X \subset Cone_L(C)$. If $X \subset Cone_p(TC)$, then $\pi_{T_\alpha}(X) = TK$ for a certain curve K, which is not the case. So, $X \subset Cone_L(C)$. $\square$

10. Proof of Theorem 3, the case $\dim TX = 4$, $h > 1$.

Lemma 5. *Suppose that X is non-degenerate, TX is not-1-defective, $h \geq 2$ and for a general point $p \in TX$ one has $\pi_{T_pTX}(X) \subset Cone_L(TC) \subset \mathbb{P}^{N_1}$, where C is a curve, $\dim L \leq h-2$, $N_1 \geq \dim L + 4h + 1$, $\pi_L(\pi_{T_pTX}(X)) = TC$. Then $X \subset Cone_M(TK) \subset \mathbb{P}^N$, K is a curve, $\dim M \leq h-1$, $N \geq \dim M + 4h + 5$, $\pi_M(X) = TK$.*

Proof. Since $\pi_{T_pTX}(X) \subset Cone_L(TC) \subset \mathbb{P}^{N_1}$, for a general point $x \in X$ one has $L \cap T^2_{\pi_{T_pTX}(x)}\pi_{T_pTX}(X) \neq \emptyset$. So, $\pi^{-1}_{T_pTX}(L) \cap T^2_xX \neq \emptyset$. Denote $\pi^{-1}_{T_pTX}(L)$ by L_p. For another general point $q \in TX$ take also $L_q = \pi^{-1}_{T_qTX}(L)$ for the corresponding L. Then also $L_q \cap T^2_xX \neq \emptyset$ for general $x \in X$, $\dim L_p = \dim L_q$. More, $\dim L_p = \dim L + 5$, $N = N_1 + 5 \geq \dim L + 4h + 6$. Since $\dim \pi_{L_p}(T_qTX) = \dim T_{\pi_{L_p}(q)}T\pi_{L_p}(X) = \dim T\pi_{L_p}(X) = \dim T(\pi_L(\pi_{T_pTX}(X))) = \dim T(TC) = \dim T^2C = 3$, $L_p \cap T_qTX \neq \emptyset$. Also $L_q \cap T_pTX \neq \emptyset$, and since $L_p \supset T_pTX$, $L_q \supset T_qTX$ and $T_pTX \cap T_qTX = \emptyset$ (TX is not-1-defective), one has $\dim L_q \cap L_p \geq 1$. As we saw above, $\pi_{L_p}(X) = TC$. If for general $x \in X$, $T^2_xX \cap L_q \not\subset L_p$, then $\pi_{L_p}(L_q) \cap T^2_{\pi_{L_p}(x)}\pi_{L_p}(X) \neq \emptyset$. But $T^2_{\pi_{L_p}(x)}\pi_{L_p}(X) = T^3_yC$, where $y \in C$ is a point for which $\pi_{L_p}(x) \in T_yC$. So, $\pi_{L_p}(L_q) \cap T^3_yC \neq \emptyset$ for general $y \in C$. Therefore $\dim_3 \pi_{\pi_{L_p}(L_q)}(C) \leq 2$. Hence, by Proposition 4, $\dim\langle \pi_{\pi_{L_p}(L_q)}(C)\rangle \leq 2$. So, $\dim \pi_{L_p}(L_q) \geq (N - \dim L_p - 1) - 3 \geq (\dim L + 4h + 6) - (\dim L + 5) - 1 - 3 = 4h - 3$. On the other hand, $\dim \pi_{L_p}(L_q) = \dim L_q - \dim L_p \cap L_q - 1 \leq (\dim L + 5) - 1 - 1 \leq (h - 2 + 5) - 2 = h + 1$. So, we have $4h - 3 \leq h + 1$, or $h \leq \frac{4}{3}$, which is not the case. Therefore, for general $x \in X$, $T^2_xX \cap L_q \subset L_p$. So, if we take $M = L_p \cap L_q$, then $\dim \pi_M(T^2_xX) = \dim \pi_{L_p}(T^2_xX) = \dim T^2_{\pi_{L_p}(x)}\pi_{L_p}(X) = \dim_2 TC = 3$, and $\dim T^2_{\pi_M(x)}\pi_M(X) = 3$. Hence, by Corollary 3, $\pi_M(X)$ is a surface in $\mathbb{P}^3$ or $Cone_r(K)$ or TK, where K is a curve. Since $\pi_{\pi_M(L_p)}(\pi_M(X)) = TC$, $\pi_M(X) = TK$ for a certain curve K. Let us calculate the dimension of M. Since $L_q \supset T_qTX$, $\pi_{L_p}(L_q) \supset T_{\pi_{L_p}(q)}T\pi_{L_p}(X) = T_{\pi_{L_p}(q)}T^2C$, and $\dim \pi_{L_p}(L_q) \geq \dim T^2C = 3$. So, $\dim M \leq \dim L_p - 3 - 1 = (\dim L + 5) - 3 - 1 = \dim L + 1 \leq h - 2 + 1 = h - 1$; $N \geq \dim L + 4h + 6 \geq (\dim M - 1) + 4h + 6 = \dim M + 4h + 5$. $\square$

Lemma 6. *Suppose that X is non-degenerate, TX is not-1-defective, $h \geq 2$*

and for a general point $p \in TX$ *one has* $\pi_{T_pTX}(X) \subset Cone_L(C) \subset \mathbb{P}^{N_1}$, *where* C *is a curve,* $\dim L \leq 2h-2$, $N_1 \geq \dim L + 3h + 1$. *Then* $X \subset Cone_M(K) \subset \mathbb{P}^N$, K *is a curve,* $\dim M \leq 2h$, $N \geq \dim M + 3h + 4$.

Proof. Since $\pi_L(\pi_{T_pTX}(X)) = C$, for a general point $x \in X$ one has $L \cap T^2_{\pi_{T_pTX}(x)}\pi_{T_pTX}(X) \neq \emptyset$. So, $\pi^{-1}_{T_pTX}(L) \cap T^2_xX \neq \emptyset$. Denote $\pi^{-1}_{T_pTX}(L)$ by L_p. For another general point $q \in TX$ take also $L_q = \pi^{-1}_{T_qTX}(L)$ for the corresponding L. Then also $L_q \cap T^2_xX \neq \emptyset$ for general $x \in X$, $\dim L_p = \dim L_q$. More, $\dim L_p = \dim L + 5$, $N = N_1 + 5 \geq \dim L + 3h + 1 + 5 = \dim L + 3h + 6$. Since $\dim \pi_{L_p}(T_qTX) = \dim T_{\pi_{L_p}(q)}T\pi_{L_p}(X) = \dim T\pi_{L_p}(X) = \dim T(\pi_L(\pi_{T_pTX}(X))) = \dim TC = 2$, $\dim L_p \cap T_qTX = 1$. Also $\dim L_q \cap T_pTX = 1$, and since $L_p \supset T_pTX$, $L_q \supset T_qTX$ and $T_pTX \cap T_qTX = \emptyset$ (TX is not-1-defective), one has $\dim L_q \cap L_p \geq 3$. As we saw above, $\pi_{L_p}(X) = C$. If for general $x \in X$, $T^2_xX \cap L_q \not\subset L_p$, then $\pi_{L_p}(L_q) \cap T^2_{\pi_{L_p}(x)}\pi_{L_p}(X) \neq \emptyset$. But $T^2_{\pi_{L_p}(x)}\pi_{L_p}(X) = T^2_yC$, where $y \in C$, $y = \pi_{L_p}(x)$. So, $\pi_{L_p}(L_q) \cap T^2_yC \neq \emptyset$ for general $y \in C$. Therefore $\dim_2 \pi_{\pi_{L_p}(L_q)}(C) \leq 1$. Hence, by Proposition 4, $\dim\langle\pi_{\pi_{L_p}(L_q)}(C)\rangle \leq 1$. So, $\dim \pi_{L_p}(L_q) \geq (N - \dim L_p - 1) - 2 \geq (\dim L + 3h + 6) - (\dim L + 5) - 1 - 2 = 3h - 2$. On the other hand, $\dim \pi_{L_p}(L_q) = \dim L_q - \dim L_p \cap L_q - 1 \leq \dim L + 5 - 3 - 1 \leq 2h - 2 + 5 - 3 - 1 = 2h - 1$. So, we have $3h - 2 \leq 2h - 1$, or $h \leq 1$, which is not the case. Therefore, for general $x \in X$, $T^2_xX \cap L_q \subset L_p$. So, if we take $M = L_p \cap L_q$, then $\dim \pi_M(T^2_xX) = \dim \pi_{L_p}(T^2_xX) = \dim T^2_{\pi_{L_p}(x)}\pi_{L_p}(X) = \dim_2 C = 2$, and $\dim T^2_{\pi_M(x)}\pi_M(X) = 2$. Hence, $\pi_M(X) = K$ is a curve. Let us calculate the dimension of M. Since $L_q \supset T_qTX$, $\pi_{L_p}(L_q) \supset T_{\pi_{L_p}(q)}T\pi_{L_p}(X) = T_{\pi_{L_p}(q)}TC$, and $\dim \pi_{L_p}(L_q) \geq \dim TC = 2$. So, $\dim M \leq \dim L_p - 2 - 1 = \dim L + 5 - 2 - 1 \leq 2h - 2 + 5 - 2 - 1 = 2h$. At the end, $N \geq \dim L + 3h + 6 \geq (\dim M - 5 + 2 + 1) + 3h + 6 = \dim M + 3h + 4$. □

Proposition 11. *Suppose that for a non-degenerate surface* $X \subset \mathbb{P}^N$ *the* h*-defect* $\delta_h(TX) > 0$, $h \geq 2$. *Then one of the following conditions holds:*

(1) $X \subset Cone_L(TC)$, *where* $L \subset \mathbb{P}^N$ *is a linear subspace,* $\dim L \leq h - 1$, C *is a curve,* $N \geq \dim L + 4h + 5$, $\pi_L(X) = TC$;

(2) $X \subset Cone_L(C)$, *where* $L \subset \mathbb{P}^N$ *is a linear subspace,* $\dim L \leq 2h$, C *is a curve,* $N \geq \dim L + 3h + 4$.

Proof. Take general points $x_0, \dots, x_h \in X$, $p_0, \dots, p_h \in TX$ such that $\forall i$, $0 \leq i \leq h$, $p_i \subset T_{x_i}X$, $q_1 \in \langle p_0, \dots, p_{h-1}\rangle$, $q \in \langle q_1, p_h\rangle$. By the Terracini lemma $T_qS^h(TX) = \langle T_{p_0}TX, \dots, T_{p_h}TX\rangle$, $T_{q_1}S^{h-1}(TX) = \langle T_{p_0}TX, \dots, T_{p_{h-1}}TX\rangle$.

The case $\delta_1(TX) > 0$. By the proved part of Theorem 3 (the case $h = 1$) either $X = v_3(\mathbb{P}^2)$, either $X \subset Cone_p(TC)$ $(\pi_p(X) = TC,\ X \neq TK)$ or $X \subset Cone_L(C)$, $\dim L \leq 2$.

If $X = v_3(\mathbb{P}^2) \subset \mathbb{P}^9$ then $\forall h > 1$, $S^h(TX) = \mathbb{P}^9$, $\delta_h(TX) = 0$.

If $X \subset Cone_p(TC)$, $\pi_p(X) = TC$, $X \neq TK$, then $TX = Cone_p(T^2C)$. Hence, $S^h(TX) = Cone_p(S^h(T^2C))$. Since $C \subset \mathbb{P}^{N-1}$, by Theorem 1, $\dim S^h(T^2C) = \min\{N-1, \dim T^2C \cdot (h+1) + h\} = \min\{N-1, 4h+3\}$ and $\dim S^h(TX) = \min\{N-1, 4h+3\} + 1 = \min\{N, 4h+4\}$. Therefore $\delta_h(TX) = \min\{N, 5h+4\} - \min\{N, 4h+4\}$ and $\delta_h(TX) > 0$ iff $N > 4h+4$, i. e. $N \geq 4h+5 = \dim\{p\} + 4h + 5$.

If $X \subset Cone_L(C)$, $\dim L \leq 2$, then for a general point $p \in TX$, $\dim T_pTX \cap L = 1$ $(\dim \pi_L(T_pTX) = \dim T_{\pi_L(p)}TC = 2)$. If $\dim L = 0$ then $\dim TX = 3$, which is not possible. If $\dim L = 1$ then $T_pTX \supset L$, we will consider this case next. If $\dim L = 2$ and for general points $p_0, p_1 \in TX$, $L \neq \langle L \cap T_{p_0}TX, L \cap T_{p_1}TX\rangle$, then $M = L \cap T_pTX$ does not depend on $p \in TX$ and $X \subset Cone_M(K)$ for a certain curve K, put $L = M$. So, we can assume that $L = \langle L \cap T_{p_0}TX, L \cap T_{p_1}TX\rangle$. Therefore in any case for $h > 1$, $T_qS^h(TX) = \langle T_{p_0}TX, \ldots, T_{p_h}TX\rangle \supset L$. Hence, $T_qS^h(TX) = \pi_L^{-1}(\pi_L(\langle T_{p_0}TX, \ldots, T_{p_h}TX\rangle)) = \pi_L^{-1}(\langle T_{\pi_L(p_0)}TC, \ldots, T_{\pi_L(p_h)}TC\rangle) = \pi_L^{-1}(T_{\pi_L(q)}S^h(TC))$. Since $\dim\langle C\rangle = N - \dim L - 1$ by Theorem 1, $\dim T_{\pi_L(q)}S^h(TC) = \dim S^h(TC) = \min\{N - \dim L - 1, 2(h+1) + h\} = \min\{N - \dim L - 1, 3h+2\}$ and $\dim S^h(TX) = \dim \pi_L^{-1}(T_{\pi_L(q)}S^h(TC)) = \dim L + 1 + \min\{N - \dim L - 1, 3h+2\} = \min\{N, \dim L + 3h + 3\}$. So, $\delta_h(TX) = \min\{N, 5h+4\} - \dim T_qS^h(TX) = \min\{N, 5h+4\} - \min\{N, \dim L + 3h + 3\}$. Since $\dim L + 3h + 3 \leq (h+1) + 3h + 3 = 4h + 4 < 5h + 4$, $\delta_h(TX) > 0$ iff $N > \dim L + 3h + 3$, i. e. $N \geq \dim L + 3h + 4$.

The case $\delta_1(TX) = 0$. If $d_1(TX) > 0$ then $N = \dim S^1(TX)$. Hence $\mathbb{P}^N = S^1(TX) = S^h(TX)$, which is not the case because $\delta_h(TX) > 0$. So, $d_1(TX) = 0$. Then by Proposition 1 for $Y = TX$, $m = 1$ and a point $p \in TX$, $\delta_{h-1}(T\pi_{T_pTX}(X)) = \delta_h(TX) > 0$.

Let us use an induction on h. Suppose that $h = 2$. Then $\delta_1(T\pi_{T_pTX}(X)) > 0$. By the proved part of Theorem 3 (the case $h = 1$) either $T\pi_{T_pTX}(X) = v_3(\mathbb{P}^2)$, either $T\pi_{T_pTX}(X) \subset Cone_p(TC)$ $(\pi_p(T\pi_{T_pTX}(X)) = TC,\ T\pi_{T_pTX}(X) \neq TK)$ other $T\pi_{T_pTX}(X) \subset Cone_L(C)$, $\dim L \leq 2$.

If $\pi_{T_pTX}(X) = v_3(\mathbb{P}^2)$ then $\dim_2 \pi_{T_pTX}(X) = 5$. Hence, $\dim_2(X) = 5$. By Theorem 2, $T_qS^2(TX)$ contains $T^2_{x_i}X$ for $0 \leq i \leq 2$. If $T^2_{x_2}X \cap T^2_{x_1}X = \emptyset$, then after the projection π from $\langle T^2_{x_2}X, T^2_{x_1}X\rangle$ we have $\dim \pi(T^2_{x_0}(X)) = \dim T_qS^2(TX) - \dim\langle T^2_{x_2}X, T^2_{x_1}X\rangle - 1 = \min\{N, 5 \cdot 2 + 4\} - \delta_2(TX) - 11 - 1 = \min\{N - 12, 2\} - \delta_2(TX) \leq 1$. Therefore $\dim T^2_{\pi(x_0)}(\pi(X)) \leq$

1, and $\pi(X) = \langle\pi(X)\rangle = T^2_{\pi(x_0)}(\pi(X))$. So, $\mathbb{P}^N = \pi^{-1}(\langle\pi(X)\rangle) = \pi^{-1}(T^2_{\pi(x_0)}(\pi(X))) = \langle T^2_{x_0}X, T^2_{x_1}X, T^2_{x_2}X\rangle = T_qS^2(TX)$ and $S^2(TX) = \mathbb{P}^N$. But this is not the case because $\delta_2(TX) > 0$. Hence, $T^2_{x_2}X \cap T^2_{x_1}X \neq \emptyset$ and $T^2_xX \cap T^2_yX \neq \emptyset$ for general points $x, y \in X$. Take $p \in T_xX$. Since $\dim T^2_{\pi_{T_pTX}(y)}(\pi_{T_pTX}(X)) = 5$, $T^2_yX \cap T_pTX = \emptyset$. Therefore, $\pi_{T_pTX}(T^2_xX)$, which is a point, belongs to $\pi_{T_pTX}(T^2_yX) = T^2_{\pi_{T_pTX}(y)}\pi_{T_pTX}(X)$. So, for a general point $y \in X$, $\pi_{T_pTX}(T^2_xX) \in T^2_{\pi_{T_pTX}(y)}\pi_{T_pTX}(X)$. But if $\pi_{T_pTX}(X)$ is $v_3(^2)$, this is not so.

If $\pi_{T_pTX}(X) \subset Cone_p(TC)$, $\dim\langle\pi_{T_pTX}(X)\rangle \geq \dim\{p\} + 4\cdot 1 + 5 = 9$, $\pi_p(X) = TC$, then by Lemma 5 applied to $h = 2$, $X \subset Cone_M(TK)$, $N \geq \dim M + 4\cdot 2 + 5$, $\dim M \leq 2 - 1$, $\pi_M(X) = TK$, K is a curve.

If $\pi_{T_pTX}(X) \subset Cone_L(C)$, $\dim L \leq 2 = 2\cdot 1$, $\dim\langle\pi_{T_pTX}(X)\rangle \geq \dim L + 7$, then by Lemma 6 applied to $h = 2$, $X \subset Cone_M(K)$, $\dim M \leq 2\cdot 2$, $N \geq \dim M + 3\cdot 2 + 4$, K is a curve.

So, the statement is proved for $h = 2$.

In the case $h > 2$ Lemmas 5 and 6 give the proof of the step from $h-1$ to h.

$\square$

11. Proof of Corollary 5.

To prove Corollary 5 one should notice that by definition TX is h-defective iff $\delta_h(TX) > 0$ and not $\delta_{h-1}(TX) > 0$, apply Theorem 3 and Proposition 8.

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