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GRASSMAN DEFECTIVITY À LA TERRACINI

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This work is a modern rivisitation of a classical paper by Alessandro Terracini, going back to 1915, wich suggests an elementary but powerful method for studying Grassmann defective varieties. In particular, the case of Veronese surfaces is completely understood, giving a positive answer to the so-called Waring problem for pairs of homogeneous polynomials in three variables.

1. Introduction.

Let $V \subset \mathbb{P}^r$ be an irreducible nondegenerate projective variety of dimension n defined over the complex field $\mathbb{C}$.

Definition 1.1. The *h-secant variety* $\mathrm{Sec}_h(V)$ of V is the irreducible variety given by the Zariski closure of the set:
$\{p \in \mathbb{P}^r$ s.t. p lies in the span of $h+1$ independent points of $V\}$.

Just counting parameters we get

$$\dim \mathrm{Sec}_h(V) \leq \min\{(n+1)(h+1)-1, r\} \tag{1}$$

where the right hand side is called the *expected dimension of* $\mathrm{Sec}_h(V)$. If strict inequality holds in (1), then V is said to be *h-defective* and the positive integer

$$\delta_h(V) := \min\{(n+1)(h+1)-1, r\} - \dim \mathrm{Sec}_h(V)$$

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is called the *h-defect* of V.

A basic tool for understanding defective varieties is the classical Terracini's lemma (see [15] for the original version and [8] for a modern proof), which says that, for $P_1, \dots, P_{h+1}$ general points and P general in their span one has

$$T_P(Sec_h(V)) = < T_{P_1}(V), \dots, T_{P_{h+1}}(V) > .$$

Therefore V is h-defective if and only if

$$\dim(< T_{P_1}(V), \dots, T_{P_{h+1}}(V) >) < \min\{(n+1)(h+1) - 1, r\}.$$

The systematic study of defective varieties goes back to the old Italian school: we wish to mention at least the contributions of Francesco Palatini ([10], [11]), Gaetano Scorza ([12], [13], [14]) and Alessandro Terracini ([15], [17]). This great amount of work was recently rediscovered by various authors, among whom Luca Chiantini and Ciro Ciliberto; we refer to their papers [2] and [3] for rigorous proofs and powerful generalizations of the classical results in the field.

In the present paper, instead, we focus on another kind of defectivity, the so-called Grassmann-defectivity.

Definition 1.2. The *(k, h)-Grassmann secant variety* $\mathrm{Sec}_{k,h}(V)$ of V is the Zariski closure of the set:
$\{l \in \mathbb{G}(k, r)$ s.t. l lies in the span of $h + 1$ independent points of $V\}$.

As above, we have an obvious inequality

$$\dim \mathrm{Sec}_{k,h}(V) \leq \min\{(h+1)n + (k+1)(h-k), (k+1)(r-k)\} \tag{2}$$

and we may introduce in a natural way the definitions of *expected dimension* of $\mathrm{Sec}_{k,h}(V)$, $(k, h) - \mathit{defectivity}$ and $(k, h) - \mathit{defect}\ \delta_{k,h}(V)$ of V.

Unfortunately there seems to be no easy form of a Terracini type lemma which may help in this situation. As a consequence, the problem of classifying Grassmann-defective varieties is rather hard (see [5] for a first step in this direction; moreover, [4] shows that there are no Grassmann defective curves). However, in the memoir [16], going back to 1915, Terracini suggests that the condition of (h, k)-defectivity for a variety of dimension n may be translated into the condition of h-defectivity for a variety of dimension $n + k$. More precisely, we have the following statement (to be proved in section 2):

Proposition 1.3. *Let $V \subset \mathbb{P}^r$ be an irreducible nondegenerate projective variety of dimension n. Let $\sigma : \mathbb{P}^k \times V \to \mathbb{P}^{r(k+1)+k}$ be the Segre embedding of $\mathbb{P}^k \times V$. Then V is (k,h)-defective with defect $\delta_{k,h}(V) = \delta$ if and only if $\sigma(\mathbb{P}^k \times V)$ is h-defective with defect $\delta_h(\sigma(\mathbb{P}^k \times V)) = \delta$.*

This fact was pointed out by Terracini (see p. 97 of [16]) only for Veronese surfaces, but it turns out to hold in complete generality with almost the same proof. The interest of Terracini in the case of Veronese varieties is explained by the simple observation that the Veronese variety $V_{n,d}$ of dimension n and degree d is *not* (k, h)-defective if and only if the following Waring type problem admits an affirmative answer:

Given positive integers d, n, k, h, may we write any $(k+1)$ homogeneous polynomials $f_j(x_0, \dots, x_n)$, $j = 0, \dots k$, of degree d as linear combinations of the same $(h+1)$ d-th powers of linear forms $l_i(x_0, \dots, x_n)$, $i = 0, \dots, h$?

Along these lines, Terracini's approach leads to the following conclusions:

Theorem 1.4. *If $(d, h) \neq (3, 4)$ then $V_{2,d}$ is not $(1, h)$-defective.*

Indeed the paper [16] contains a proof of this result under the additional numerical hypothesis:

$$h + 1 \geq \frac{(d+1)(d+2)}{4}$$

which can be removed using a subtler argument.

We stress moreover that Theorem 1.4 is sharp, since the Veronese surface $V_{2,3}$ is (1,4)-defective, as already noticed by London in [9] (see Remark 2.2).

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2. The Proofs.

Proof of Proposition 1.3. consider the natural map

$$\phi : \underbrace{\mathbb{P}^h \times \ldots \times \mathbb{P}^h}_{k+1} \times V^{h+1} \to (\mathbb{P}^r)^{k+1}$$

$$((\lambda_{ij})_{\substack{i=0,\dots,k\\ j=0,\dots,h}}, (p^{(j)})_{j=0,\dots,h}) \mapsto (\sum_{j=0}^{h} \lambda_{ij} p^{(j)})_{i=0,\dots,k}$$

which to a set of coefficients and a $(h+1)$-ple of points lying on V associates a $(k+1)$-ple of points in $\mathbb{P}^r$ each one contained in the linear span of the $h+1$ points we started with.
By definition, V is *not* (k,h)-defective if and only if dim Im (ϕ) has maximal dimension. If $p : U \subseteq \mathbb{C}^n \to V$ is a local parametrization of V, we may introduce $\mathrm{Jac}(\phi \circ p)$, the Jacobian matrix of $\phi \circ p$, given by

$$\Big(\frac{\partial(\phi \circ p)}{\partial \lambda_{00}} \cdots \frac{\partial(\phi \circ p)}{\partial \lambda_{k0}} \cdots\cdots \frac{\partial(\phi \circ p)}{\partial \lambda_{0h}} \cdots \frac{\partial(\phi \circ p)}{\partial \lambda_{kh}}$$

$$\frac{\partial(\phi \circ p)}{\partial u_1^{(0)}} \cdots \frac{\partial(\phi \circ p)}{\partial u_n^{(0)}} \cdots \frac{\partial(\phi \circ p)}{\partial u_1^{(h)}} \cdots \frac{\partial(\phi \circ p)}{\partial u_n^{(h)}}\Big) =$$

$$\begin{pmatrix} p^{(0)}\ 0 \dots 0 \dots p^{(h)}\ 0 \dots 0 & \lambda_{00} p^{(0)}_{u_1} \dots \lambda_{00} p^{(0)}_{u_n} \dots \lambda_{0h} p^{(h)}_{u_1} \dots \lambda_{0h} p^{(h)}_{u_n} \\ 0\ \ p^{(0)} \dots 0 \dots 0\ \ p^{(h)} \dots 0 & \lambda_{10} p^{(0)}_{u_1} \dots \lambda_{10} p^{(0)}_{u_n} \dots \lambda_{1h} p^{(h)}_{u_1} \dots \lambda_{1h} p^{(h)}_{u_n} \\ \vdots \quad \vdots \qquad \vdots & \vdots \qquad \vdots \qquad \vdots \\ 0\ \ 0 \dots p^{(0)} \dots 0\ \ 0 \dots p^{(h)} & \lambda_{k0} p^{(0)}_{u_1} \dots \lambda_{k0} p^{(0)}_{u_n} \dots \lambda_{kh} p^{(h)}_{u_1} \dots \lambda_{kh} p^{(h)}_{u_n} \end{pmatrix}$$

and apply the inverse function theorem to conclude that V is *not* (k,h)-defective (resp., (h,k)-defective with defect δ) if and only if $\mathrm{Jac}(\phi \circ p)$ has maximal rank (resp., rank equal to the maximal one minus δ) at a general point.

On the other hand, consider the Segre embedding

$$\sigma : \mathbb{P}^k \times V \to \mathbb{P}^{r(k+1)+k}$$

$$(\lambda_0, \dots, \lambda_k, p) \mapsto (\lambda_0 p, \dots, \lambda_k p).$$

If $t_i = \frac{\lambda_i}{\lambda_0}$, locally we have

$$(\sigma \circ p)(t_1, \dots, t_k, u_1, \dots, u_n) =$$

$$= (p(u_1, \dots, u_n), t_1 p(u_1, \dots, u_n), \dots, t_k p(u_1, \dots, u_n))$$

and we may compute:

$$\frac{\partial(\sigma \circ p)}{\partial t_i} = (0, \dots, 0, p, 0 \dots, 0) \qquad \text{for } i = 1, \dots k$$

$$\frac{\partial(\sigma \circ p)}{\partial u_j} = (p_{u_j}, t_1 p_{u_j}, \dots, t_k p_{u_j}) = (\lambda_0 p_{u_j}, \dots, \lambda_k p_{u_j}) \qquad \text{for } j = 1, \dots n.$$

Hence $T(\sigma(\mathbb{P}^k \times V))$ is spanned by the columns of the matrix

$$\begin{pmatrix} \lambda_0 p & 0 & \dots & 0 & \lambda_0 p_{u_1} & \dots & \lambda_0 p_{u_n} \\ \lambda_1 p & p & \dots & 0 & \lambda_1 p_{u_1} & \dots & \lambda_1 p_{u_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_k p & 0 & \dots & p & \lambda_k p_{u_1} & \dots & \lambda_k p_{u_n} \end{pmatrix}$$

$$\sim \begin{pmatrix} p & 0 & \dots & 0 & \lambda_0 p_{u_1} & \dots & \lambda_0 p_{u_n} \\ 0 & p & \dots & 0 & \lambda_1 p_{u_1} & \dots & \lambda_1 p_{u_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & p & \lambda_k p_{u_1} & \dots & \lambda_k p_{u_n} \end{pmatrix}$$

where the relations $\sim$ between these two matrices means that their columns span the same vector space $A(\lambda_0, \dots, \lambda_k, p)$.

by Terracini's lemma,

$$T(\mathrm{Sec}_h(\sigma(\mathbb{P}^k \times V)) =< A(\lambda_{00}, \dots, \lambda_{k0}, p^{(0)})| \dots |A(\lambda_{0h}, \dots, \lambda_{kh}, p^{(h)}) >$$

and $\sigma(\mathbb{P}^k \times V)$ is *not* h-defective (resp., h-defective with defect δ) if and only if this matrix has maximal rank (resp., rank equal to the maximal one minus δ).

Since $\mathrm{Jac}(\phi \circ p) \sim< A(\lambda_{00}, \dots, \lambda_{k0}, p^{(0)})| \dots |A(\lambda_{0h}, \dots, \lambda_{kh}, p^{(h)}) >$ the thesis follows. □

Lemma 2.1. *Let $V \subset \mathbb{P}^r$ be an irreducible nondegenerate projective variety of dimension n. Let $\sigma : \mathbb{P}^k \times V \to \mathbb{P}^{r(k+1)+k}$ be the Segre embedding of $\mathbb{P}^k \times V$. Fix $p^{(0)}, \dots, p^{(k)}$ general points on V and $\lambda^{(0)}, \dots, \lambda^{(h)}$ general points in $\mathbb{P}^k$, so that $P^{(j)} := (\lambda_0^{(j)} p^{(j)}, \dots, \lambda_k^{(j)} p^{(j)})$ is a general point on $\sigma(\mathbb{P}^k \times V) \subset \mathbb{P}^{r(k+1)+k}$ for $j = 0 \dots h$; finally, take a general point $P \in \, < P^{(0)}, \dots, P^{(h)} >$. Then there is a natural identification between:*

- *hyperplanes $H \subset \mathbb{P}^{r(k+1)+k}$ such that $T_p(\mathrm{Sec}_h(\sigma(\mathbb{P}^k \times V))) \subset H$;*
- *k-dimensional linear systems $\mathcal{H}$ of hyperplane sections of $V \subset \mathbb{P}^r$ with a projectivity $\omega : \mathcal{H} \to \mathbb{P}^k$ such that all the elements of the linear system pass through the points $p^{(j)} \in V$ and for every j the hyperplane section of the linear system corresponding to $\lambda^{(j)}$ is tangent to V at $p^{(j)}$.*

Proof. If $X_{\alpha\beta} (\alpha = 0 \dots k, \beta = 0 \dots r)$ are coordinates on $\mathbb{P}^{r(k+1)+k}$, the points of $\sigma(\mathbb{P}^k \times V)$ are exactly those of the form

$$X_{\alpha\beta} = \lambda_\alpha p(u_1, \dots, u_n)_\beta$$

in the notation of the proof of Proposition 1.3. A hyperplane H in $\mathbb{P}^{r(k+1)+k}$ is given by a linear equation

$$\sum_{\alpha,\beta} a_{\alpha\beta} X_{\alpha\beta} = 0$$

and if H is general we may assume that $a_{\alpha\beta} \neq 0$ for every α, β. So a hyperplane section of the variety $\sigma(\mathbb{P}^k \times V)$ is of the form

$$\sum_{\alpha,\beta} a_{\alpha\beta} \lambda_\alpha p(u_1, \dots, u_n)_\beta = 0$$

i.e.

$$\sum_{\alpha} \lambda_\alpha \sum_{\beta} a_{\alpha\beta} p(u_1, \dots, u_n)_\beta = 0.$$

Hence every general hyperplane section of the variety $\sigma(\mathbb{P}^k \times V)$ corresponds to a k-dimensional linear system $\mathcal{H}$ of hyperplane sections of the variety V with a fixed projectivity $\omega : \mathcal{H} \to \mathbb{P}^k$; conversely, the data of such a linear system $\mathcal{H}$ and a projectivity $\omega : \mathcal{H} \to \mathbb{P}^k$ uniquely determine a hyperplane section of $\sigma(\mathbb{P}^k \times V)$. Moreover, if

$$T_P(\mathrm{Sec}_h(\sigma(\mathbb{P}^k \times V))) =< T_{P^{(0)}}(\sigma(\mathbb{P}^k \times V)), \dots, T_{P^h}(\sigma(\mathbb{P}^k \times V)) >\subset H$$

then

$$\sum_{\beta} a_{\alpha\beta} p^{(j)}(u_1, \dots, u_n)_\beta = 0 \quad \forall \alpha = 0, \dots, k$$

$$\sum_{\alpha} \lambda_\alpha^{(j)} \sum_{\beta} a_{\alpha\beta} p^{(j)}_{u_\gamma \beta} = 0 \qquad \forall \gamma = 1, \dots, n.$$

In other words, all the elements of the linear system pass through the points $p^{(j)} \in V$ and for every j the hyperplane section of the linear system corresponding to the values $\{\lambda_\alpha^{(j)}\}_{\alpha=0\dots k}$ of parameters is tangent to V at $p^{(j)}$. □

Proof of theorem 1.4. Assume by contradiction that $V_{2,d}$ is $(1, h)$-defective. By proposition 1.3, $\sigma(\mathbb{P}^1 \times V_{2,d}) \subset \mathbb{P}^{d(d+3)+1}$ has to be h-defective, i. e.

(3) $\qquad \dim \mathrm{Sec}_h(\sigma(\mathbb{P}^1 \times V_{2,d})) = d(d+3) + 1 - \delta < 4(h+1) - 1$

with $\delta \geq 1$.

Hence if we take a general point P as in the statement of Lemma 2.1, then $T_P(\mathrm{Sec}_h(\sigma(\mathbb{P}^1 \times V_{2,d})))$ is contained in δ independent hyperplanes $H_t (1 \leq t \leq \delta)$, which we may assume to be general since P is general. By Lemma 2,1, each

H_t gives rise to a pencil $\mathcal{F}_{Ht}$ of plane curves of degree d all passing through $h+1$ general points $p^{(0)}, \ldots, p^{(h)}$ in such a way that for every $p^{(j)}$ at least one curve of the pencil passes doubly through $p^{(j)}$.

By an infinitesimal Lemma already known to Terracini and reproved in modern times by Ciliberto and Hirschowitz in [7], we have that every H_t is tangent to $\sigma(\mathbb{P}^1 \times V_{2,d})$ along a positive dimensional variety C_t passing through $P^{(0)}, \ldots, P^{(h)}$. The points of C_t are indeed Segre images of pairs (λ, p) such that the plane curve of $\mathcal{F}_{Ht}$ corresponding to λ has a singular point in p. If all the C_t are one-dimensional for each H_t we have *a priori* two cases:

(i) $\sigma^{-1}(C_t) = \bigcup_{j=0}^{h} \mathbb{P}^1 \times \{p^{(j)}\}$, so that all the curves of $\mathcal{F}_{Ht}$ pass doubly through $p^{(0)}, \ldots, p^{(h)}$;

(ii) $\sigma^{-1}(C_t)$ surjects on $\mathbb{P}^1$ and injectively projects to $\mathbb{P}^2$ over a plane curve passing through $p^{(0)}, \ldots, p^{(h)}$, so that all the curves of $\mathcal{F}_{Ht}$ have a double point.

Since the hyperplanes H_t are general, by symmetry the same case occurs for all of them. Moreover, if the C_t's are higher dimensional we fall *a fortiori* in case (ii).

In case (i) we have

$$\mathcal{F}_{Ht} \subseteq \mathcal{L}_{2,d}(2^{h+1})$$

for every t, where $\mathcal{L}_{2,d}(2^{h+1})$ as usual denotes the linear system of plane curves of degree d with $h+1$ assigned general double points. We claim that

$$\dim < \mathcal{F}_{H1} \ldots \mathcal{F}_{H\delta} > \geq \frac{\delta}{2}.$$

To check the claim, let $< \mathcal{F}_{H1} \ldots \mathcal{F}_{H\delta} >$ be spanned by the columns of a matrix

$$\begin{pmatrix} f_{01} \cdots f_{0\delta} \\ f_{11} \cdots f_{1\delta} \end{pmatrix}$$

Just making elementary operations on columns, we obtain (up to reindexing)

$$\begin{pmatrix} f_{01} \cdots f_{0\delta} \\ f_{11} \cdots f_{1\delta} \end{pmatrix} \sim \begin{pmatrix} f_{01} \cdots f_{0x} & 0 \ \ldots \ 0 \\ f_{11} \cdots f_{1x} & g_{1x+1} \cdots g_{1\delta} \end{pmatrix}$$

where

$$x = \dim < f_{01} \ldots f_{0\delta} > .$$

Since both $f_{01} \dots f_{0x}$ and $g_{1x+1} \dots g_{1\delta}$ are linearly independent, we deduce

$$\dim < \mathcal{F}_{H1} \dots \mathcal{F}_{H\delta} > \geq \max\{x, \delta - x\} \geq \frac{\delta}{2}$$

and the claim is checked.

By the claim, to get a contradiction it will be sufficient to show that $\dim \mathcal{L}_{2,d}(2^{h+1}) < \frac{\delta}{2}$. Using (3) we compute:

$$\operatorname{expdim}\mathcal{L}_{2,d}(2^{h+1}) = \frac{d(d+3)}{2} - 3(h+1) \leq \frac{d(d+3)}{2} - \frac{3}{4}(d(d+3)+3-\delta) =$$

$$= \frac{-d(d+3) - 9 + 3\delta}{4} < \frac{\delta}{2}.$$

In order to conclude, just notice that if $d \geq 5$ then $\mathcal{L}_{2,d}(2^{h+1})$ is nonspecial by the Alexander-Hirschowitz theorem (see [1]), so

$$\dim \mathcal{L}_{2,d}(2^{h+1}) = \operatorname{expdim}\mathcal{L}_{2,d}(2^{h+1}) < \frac{\delta}{2};$$

if instead $d \leq 4$, the only special system arises when $d = 2$ and $h = 1$ or $d = 3$ and $h = 4$, but also in these cases we have (see [6], Example 4.3)

$$\dim \mathcal{L}_{2,d}(2^{h+1}) = 0 < \frac{\delta}{2}.$$

So case (i) is over.

In case (ii), it follows from Bertini's theorem that all the pencils have a base curve of the same degree, say m. Since this curve has to pass through $p^{(0)}, \dots, p^{(h)}$, we have

$$\frac{m(m+3)}{2} \geq h + 1 \geq \frac{d(d+3)+3-\delta}{4} \tag{4}$$

If the base curve pass doubly through $p^{(0)}, \dots, p^{(h)}$, we may argue exactly as in case (i); otherwise, the moving parts of the δ independent pencils have to pass through $p^{(0)}, \dots, p^{(h)}$ in correspondence with prescribed general coefficients; hence the generic fiber of the natural map

$$\mathbb{G}(\mathbb{P}^1, \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-m))) \longrightarrow (\mathbb{P}^1)^{h+1}/\operatorname{Aut}(\mathbb{P}^1),$$

which associates to a one-dimensional linear system the $(h+1)$-ple of coefficients corresponding to its curves through $p^{(0)}, \dots, p^{(h)}$, must have dimension $\geq \delta - 1$. It follows that

$$\dim \mathbb{G}(\mathbb{P}^1, \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-m))) - (h+1) + \# \operatorname{Aut}(\mathbb{P}^1) \geq \delta - 1$$

i.e.

$$(d-m)(d-m+3)+1 \geq \frac{d(d+3)+3-\delta}{4} + \delta - 1. \tag{5}$$

Summing equations (4) and (5) we get

$$3\frac{m(m+3)}{2} + (d-m)(d-m+3)+1 \geq d(d+3)+2$$

and we may deduce that

$$d - m < \frac{d+3}{5}.$$

Substituting in (5) we obtain

$$\frac{(d+3)(d+18)}{25} + 1 > \frac{d(d+3)+2}{4}$$

hence $d \leq 3$.

If $d = 2$ then $m = 1$, so (4) implies $\delta \geq 5$, while (5) forces $\delta \leq \frac{11}{3}$, contradiction.

If $d = 3$ then $m = 1$, then (4) implies $\delta \geq 13$, while (5) forces $\delta \leq 9$, contradiction.

If $d = 3$ and $m = 2$, then (4) implies $h \leq 4$. Hence for $(d, h) \neq (3, 4)$(4) implies $\delta \geq 5$ while (5) forces $\delta \leq 1$, contradiction.

So the proof is over. □

Remark 2.2. To check that $V_{2,3}$ is indeed (1,4)-defective we may argue as in the proof of Theorem 1.4. In fact, since for $m = 2, h = 4, d = 3$ and $\delta = 1$ conditions (4) and (5) are verified, there exists a pencil of plane cubics with a fixed component of degree 2 passing through 5 general points and a moving part passing through the same points in correspondence with prescribed general coefficients. Hence by Lemma 2.1 we deduce that $T_P(\operatorname{Sec}_4(\sigma(\mathbb{P}^1 \times V_{2,3})))$ is contained in at least one hyperplane of $\mathbb{P}^{19}$, so $\dim \operatorname{Sec}_4(\sigma(\mathbb{P}^1 \times V_{2,3})) \leq 18$. Since $\operatorname{expdim} \operatorname{Sec}_4(\sigma(\mathbb{P}^1 \times V_{2,3})) = 19$, it turns out that $\sigma(\mathbb{P}^1 \times S)$ is 4-defective. Now the thesis directly follows from Proposition 1.3.

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