A SMALL RESOLUTION FOR TRIPLE COVERS IN ALGEBRAIC GEOMETRY

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Given a triple cover $\pi : X \longrightarrow Y$ of varieties, we produce a new variety $\mathfrak{G}_X$ and a birational morphism $\rho_X : \mathfrak{G}_X \longrightarrow X$ which is an isomorphism away from the fat-point ramification locus of π. The variety $\mathfrak{G}_X$ has a natural interpretation in terms of the data describing the triple cover, and the morphism ρ_X has an elegant geometric description.

1. Introduction.

The basic fact regarding a triple cover $\pi : X \longrightarrow Y$, proven in [2], is that any such cover is determined by a rank 2 locally free sheaf E on Y and a global section σ of $S^3(E)^* \otimes \Lambda^2(E)$. Furthermore, X can be realized as a subvariety of the geometric vector bundle $\mathbb{V}(E)$ equipped with the natural projection to Y.

In this article we give a necessary and sufficient criterion for X to be realized as a subvariety of a $\mathbb{P}^1$-bundle equipped with its natural projection to Y: we show that X can be so realized if and only if π has no fat triple ramification (a fat triple ramification point of π is a point $x \in X$ whose Zariski tangent space in the fibre of π has dimension 2).

Along the way, we show that to any triple cover $\pi : X \longrightarrow Y$, one can associate a subvariety $\mathfrak{G}_X$ of $\mathbb{P}(E^*)$ defined in terms of the global section σ. This variety $\mathfrak{G}_X$ is equipped with a birational morphism $\rho_X : \mathfrak{G}_X \longrightarrow X$, which has a nice geometric interpretation. In fact ρ_X is a sort of small resolution: it is the blow-up of a Weil divisor in X, and its fibre over any fat ramification point of π is a $\mathbb{P}^1$, but its exceptional set is in general of codimension larger than 1 in $\mathfrak{G}_X$. We construct this resolution first in a local case, and then we show that the construction globalizes. Throughout this article we make extensive use of Miranda's analyses in [2].

We note that our main result is not new: a more general statement (with a correspondingly more technical proof) can be found as Theorem 1.3 in the beautiful paper [1]. However, it is our hope that our simple geometric description in the case of triple covers can provide some insight into more general case.

The authors wish to thank Ciro Ciliberto, Alberto Calabri, Flaminio Flamini, Alfio Ragusa, and especially Rick Miranda for their extraordinary efforts during the PRAGMATIC 2001 summer school in Catania. The authors also wish to thank Mike Roth for teaching us about small resolutions, which turned out to be precisely the right objects for describing the results of our research.

2. Some examples of small resolutions.

Let $\mathbb{A}^5$ have coordinates x, y, z, w, t, and let $\mathbb{P}^4$ be its projectivization. Let $X \subset \mathbb{P}^4$ be the hypersurface defined by the equation $xw - yz = 0$. If we restrict our attention to the $\mathbb{P}^3 \subset \mathbb{P}^4$ where $t = 0$, this same equation defines a smooth quadric $Q \subset \mathbb{P}^3$; thus X is just the projective cone over Q with vertex $[0 : 0 : 0 : 0 : 1]$.

It is clear that X is smooth away from its vertex. If $\epsilon : \widetilde{X} \longrightarrow X$ is the blow-up of the vertex, then it is easy to see that $\widetilde{X}$ is a smooth variety, and that the exceptional divisor is isomorphic to $Q = \mathbb{P}^1 \times \mathbb{P}^1$. We are going to describe a method of resolving the singular point of X with a morphism $\rho : \Gamma \longrightarrow X$, where Γ is a smooth variety, but where the exceptional set is a $\mathbb{P}^1$. In particular, the exceptional set is "too small" to be a divisor; thus ρ will be an example of a *small resolution*.

We begin by choosing a line L belonging to one of the two rulings of the quadric Q; for ease in computation, we take L to be the line $x = y = 0$. Clearly L is a Weil divisor on Q. It is easy to check that L is also a Cartier divisor: since there is no point on Q at which x, y, z, w all vanish, the two open sets of Q where $z \neq 0$ and where $w \neq 0$ cover L. On the first set L is defined

by $x = 0$, and on the second set L is defined by $y = 0$.

Now let $D \subset X$ be the cone over L. Then D is a Weil divisor in X, but it is not Cartier: D cannot be defined by only one equation in any open neighborhood of the origin. Our small resolution $\rho : \Gamma \longrightarrow X$ is the blow-up of X along D. (Note that once we show that Γ is not isomorphic to X, we will have proven indirectly that D is not Cartier: the blow-up of a Cartier divisor is always an isomorphism.)

Since D can be defined by the two equations $x = y = 0$ in X, the blow-up of X along D can be defined as the (closed) graph of the rational map $\phi : X - - \succ \mathbb{P}^1$, where $\phi([x : y : z : w : t]) = [x : y]$. This graph Γ is a closed subvariety of the product $X \times \mathbb{P}^1$, and the morphism $\rho : \Gamma \longrightarrow X$ is the restriction of the first projection map. Note that there is an open set of X on which the map ϕ agrees with the map ψ sending $[x : y : z : w : t]$ to $[z : w]$; that these two maps agree is a consequence of the defining equation for X. Also note that at any point of X other than its vertex $[0 : 0 : 0 : 0 : 1]$, at least one of ϕ and ψ is defined. This observation enables us to write down the defining equations for $\Gamma \subset X \times \mathbb{P}^1$: if u, v are coordinates on the $\mathbb{P}^1$ factor, then Γ is defined by $uy - vx = 0$ and $uw - vz = 0$. It follows that ρ is an isomorphism away from the vertex of X, and the exceptional set over the vertex is $\mathbb{P}^1$. It is also easy to check that Γ is smooth.

We now present an alternative way of describing the variety Γ. We may regard Γ as a subvariety of $\mathbb{P}^4 \times \mathbb{P}^1$, defined by the three equations $xw - yz = 0$, $uy - vx = 0$, and $uw - vz = 0$. These three equations may be expressed in a single matrix equation:

$$\Gamma = \left\{ [x : y : z : w : t] \times [u : v] \in \mathbb{P}^4 \times \mathbb{P}^1 \text{ such that } \begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} -v \\ u \end{bmatrix} = 0 \right\}.$$

The first of the three equations is now seen to express the vanishing of a determinant, which is necessary and sufficient for the second and third equations to have a nonzero solution. Note that in this description of Γ, the morphism ρ is the restriction of the natural projection from $\mathbb{P}^4 \times \mathbb{P}^1$ to $\mathbb{P}^4$.

As a second example, we take $X \subset \mathbb{P}^6$ to be the projective cone over $\mathbb{P}^2 \times \mathbb{P}^1$, with vertex $[\overset{\smile}{0} : 1]$. By analogy with the previous example, we consider a divisor $D \subset X$ which is the cone over one of the $\mathbb{P}^1 \times \mathbb{P}^1$ "rulings" of $\mathbb{P}^2 \times \mathbb{P}^1$; for example, we can take D to be defined by $x_0 = x_1 = 0$. As before, blowing up this divisor gives a small resolution $\rho : \Gamma \longrightarrow X$ which is an isomorphism away from the vertex of X, and whose fibre over the vertex is a $\mathbb{P}^1$. A computation similar to the previous one gives that Γ may be realized as

a subvariety of $\mathbb{P}^6 \times \mathbb{P}^1$, using a matrix condition:

$$\Gamma = \left\{ [\vec{x} : t] \times [u : v] \in \mathbb{P}^6 \times \mathbb{P}^1 \text{ such that } \begin{bmatrix} x_0 & x_1 \\ x_2 & x_3 \\ x_4 & x_5 \end{bmatrix} \begin{bmatrix} -v \\ u \end{bmatrix} = 0 \right\}.$$

This single matrix equation expresses six quadratic conditions: the vanishing of the three 2×2 minors, which are the defining equations for $X \subset \mathbb{P}^6$, and the three row equations, which come from the blow-up computation. As before, the morphism ρ is just the restriction of the natural projection from $\mathbb{P}^6 \times \mathbb{P}^1$ to $\mathbb{P}^6$.

Note that by simply eliminating the variable t in the above matrix description, we can define Γ as a subvariety of $\mathbb{A}^6 \times \mathbb{P}^1$, where $\mathbb{A}^6$ is the finite ($t \neq 0$) part of $\mathbb{P}^6$. This construction is the one that will provide us with a sort of universal local picture of our resolution for triple covers.

3. The local picture of the resolution.

Consider the affine space $\mathbb{A}^4$ with coordinates A, B, C, D, and let F be the free sheaf of rank 2 on this affine space. Then $\mathbb{V}(F)$ is nothing more than the affine space $\mathbb{A}^6$ with coordinates A, B, C, D, z, w; here z, w are global sections that generate F. Let $\mathfrak{X}$ be the subvariety of $\mathbb{V}(F)$ defined by the three quadrics

$$\begin{aligned}
z^2 &= Az + Bw + 2(A^2 - BD) \\
zw &= -Dz - Aw + (BC - AD) \\
w^2 &= Cz + Dw + 2(D^2 - AC).
\end{aligned}$$

By the results in Miranda's paper [2], we know that the projection of $\mathbb{V}(F)$ to $\mathbb{A}^4$ sending (A, B, C, D, z, w) to (A, B, C, D) restricts to a triple cover $\Pi : \mathfrak{X} \longrightarrow \mathbb{A}^4$.

Now, as pointed out in [2], the variety $\mathfrak{X}$ is determinantal: it is the locus in $\mathbb{V}(F)$ where the matrix

$$\begin{bmatrix} z + A & B \\ C & w + D \\ w - 2D & z - 2A \end{bmatrix}$$

has rank at most one. By a result in [2], the rank of this matrix is zero if and only if the map Π has fat triple ramification over the point (A, B, C, D); it is clear from the matrix description that this happens only over the point $(0, 0, 0, 0)$.

This determinantal representation is familiar: up to a change of coordinates on $\mathbb{A}^6$, we see that $\mathfrak{X}$ is just the affine cone over $\mathbb{P}^2 \times \mathbb{P}^1$. Furthermore, we see that the vertex of this cone – its only singular point – is exactly the fat triple

point where $A = B = C = D = z = w = 0$. The temptation to compute its small resolution is overwhelming, and so we define $\Gamma \subset \mathfrak{X} \times \mathbb{P}^1$ to be the subvariety of $\mathbb{V}(F) \times \mathbb{P}^1$ defined by the matrix condition

$$\begin{bmatrix} z+A & B \\ C & w+D \\ w-2D & z-2A \end{bmatrix} \begin{bmatrix} -v \\ u \end{bmatrix} = 0.$$

We know from our previous computation that the natural projection $\rho : \Gamma \longrightarrow \mathfrak{X}$ is an isomorphism away from the fat point, and that the fibre over this point is all of $\mathbb{P}^1$. We will refer to the morphism $\rho : \Gamma \longrightarrow \mathfrak{X}$ as the *resolution of the triple cover* Π.

We note that Γ comes equipped with a morphism ϕ to $\mathbb{A}^4 \times \mathbb{P}^1$: ϕ is just the product of $\Pi \circ \rho$ with the second projection of Γ to $\mathbb{P}^1$. We are going to compute the image of ϕ. To do this, we·first note that if $(A, B, C, D, z, w) \times [u : v]$ is a point of Γ, then we can solve for z and w in terms of the other coordinates, using the first two rows of the matrix:

$$(1) \qquad \begin{aligned} z &= B\left(\tfrac{u}{v}\right) - A \\ w &= C\left(\tfrac{v}{u}\right) - D. \end{aligned}$$

Here we assume that both u and v are nonzero; in the case that either vanishes, the third row of the matrix can be used instead of one of the first two. Continuing under the assumption that both u and v are nonzero, we use the third row of the matrix to compute that

$$-v\left(C\left(\frac{v}{u}\right) - D - 2D\right) + u\left(B\left(\frac{u}{v}\right) - A - 2A\right) = 0,$$

and since $uv \neq 0$, we conclude that:

$$(2) \qquad Bu^3 - 3Au^2v + 3Duv^2 - Cv^3 = 0.$$

We note that the same equation results from the computations in the cases where $u = 0$ or $v = 0$.

Let $\mathfrak{G} \subset \mathbb{A}^4 \times \mathbb{P}^1$ be the subvariety defined by equation (2). Let $\Pi' : \mathfrak{G} \longrightarrow \mathbb{A}^4$ be the obvious projection; this projection is compatible via ϕ with the composite map $\Pi \circ \rho : \Gamma \longrightarrow \mathbb{A}^4$. In fact, we have the following result:

Proposition 1. *The morphism* $\phi : \Gamma \longrightarrow \mathfrak{G}$ *is an isomorphism of varieties over* $\mathbb{A}^4$.

Proof. The fact that $\Pi \circ \rho = \Pi' \circ \phi$ is clear from the definition of ϕ, so we only need to show that ϕ is an isomorphism. To do this, note that the equations (1) for z and w in terms of A, B, C, D define regular functions on all of $\mathfrak{G}$; this is easily checked using equation (2). From the definition

$$\phi((A, B, C, D, z, w) \times [u : v]) = (A, B, C, D) \times [u : v],$$

we see that ϕ is surjective, and also that the regular functions for z and w are sufficient to define the inverse morphism. Thus ϕ is an isomorphism, as needed. $\square$

Corollary 1.1. *Away from the point* $(0, 0, 0, 0) \in \mathbb{A}^4$, *the three morphisms* $\Pi : \mathfrak{X} \longrightarrow \mathbb{A}^4$, $\Pi \circ \rho : \Gamma \longrightarrow \mathbb{A}^4$, *and* $\Pi' : \mathfrak{G} \longrightarrow \mathbb{A}^4$ *are isomorphic triple cover maps.*

Now we are in a position to construct a triple cover resolution for any sufficiently local triple cover $\pi : X \longrightarrow Y$. By "sufficiently local" we mean that Y is affine, E is a free sheaf of rank 2 on Y, and X is the subvariety of $\mathbb{V}(E)$ defined by the three quadrics

$$(3) \qquad \begin{aligned} z^2 &= az + bw + 2(a^2 - bd) \\ zw &= -dz - aw + (bc - ad) \\ w^2 &= cz + dw + 2(d^2 - ac); \end{aligned}$$

here the coefficients a, b, c, d are regular functions on Y, and z, w are global sections that generate E. It follows from Miranda's analysis in [2] that this is in fact the local situation for any triple cover.

Given such a sufficiently local triple cover, we define a morphism $f : Y \longrightarrow \mathbb{A}^4$ by the formula $f(y) = (a(y), b(y), c(y), d(y))$. This is equivalent to requiring that $f^*(A) = a$, and so on; thus we have the following commutative diagram:

$$\begin{array}{ccc} f^*\Gamma & \longrightarrow & \Gamma \\ \downarrow{\scriptstyle f^*\rho} & & \downarrow{\scriptstyle \rho} \\ X = f^*\mathfrak{X} & \longrightarrow & \mathfrak{X} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \Pi} \\ Y & \xrightarrow{\ f\ } & \mathbb{A}^4 \end{array}$$

It is proven in [2] that the fat points of $X \subset \mathbb{V}(E)$ are precisely the points where $a = b = c = d = 0$; it follows that the morphism $f^*\rho$ is an isomorphism away

from the fat-point ramification locus of π, and has a $\mathbb{P}^1$ fibre over any fat point in X. We will refer to $f^*\rho$ as the *resolution of the triple cover π*.

In this way we may view the right-hand side of the above diagram as a sort of universal local picture of our triple cover resolution. The reader with some skill in visualizing three-dimensional commutative diagrams[1] will see that the isomorphism $\phi : \Gamma \longrightarrow \mathfrak{G}$ pulls back via f to an isomorphism $f^*\phi : f^*\Gamma \longrightarrow f^*\mathfrak{G}$; here $f^*\mathfrak{G}$ is the subvariety of $Y \times \mathbb{P}^1$ defined by the equation

$$bu^3 - 3au^2v + 3duv^2 - cv^3 = 0.$$

This is in fact a variety over Y, whose structure morphism $\pi' : f^*\mathfrak{G} \longrightarrow Y$ is none other than $f^*\Pi'$. We can similarly "pull back" our other result:

Corollary 1.2. *Let $B \subset Y$ be the (set-theoretic) image under π of the fat-point ramification locus in X. Away from B, the three morphisms $\pi : X \longrightarrow Y$, $\pi \circ f^*\rho : f^*\Gamma \longrightarrow Y$, and $\pi' : f^*\mathfrak{G} \longrightarrow Y$ are isomorphic triple cover maps.*

Thus we reach the following interesting conclusion:

Proposition 2. *Let $\pi : X \longrightarrow Y$ be a sufficiently local triple cover; as before, this means that X is defined as a subvariety of a free rank 2 vector bundle on Y. If π has no fat-point ramification, then in fact X is isomorphic as a triple cover to a subvariety of a (trivial) $\mathbb{P}^1$-bundle over Y equipped with the natural projection.*

Proof. This is just a restatement of the isomorphism between X and $f^*\mathfrak{G}$ from the previous corollary. $\square$

4. Geometric description of the resolution.

In this section we are going to describe geometrically the isomorphism appearing in the preceding proposition. Along the way, we will also describe the geometric meaning of the $\mathbb{P}^1$-bundle appearing there.

To begin, let $\pi : X \longrightarrow Y$ be any triple cover. For convenience of notation, we set $F = \pi_*(\mathcal{O}_X)$. Following [2], we have that $F = \mathcal{O}_Y \oplus E$, where E is a rank 2 locally free sheaf on Y. If $U \subset Y$ is any open set over which E is generated freely by two sections z, w, then over U we have that X is defined as a subvariety of $\mathbb{V}(E)$ by the quadrics (3); this makes sense, because local sections of E correspond to local coordinates on $\mathbb{V}(E)$. Over a fixed $y \in Y$, the fibre of $\mathbb{V}(E)$ is an affine plane, and the quadrics (3) cut out one, two, or three points in this plane.

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[1] At least, with more skill than the authors have in drawing them.

Our idea is to consider the function on X that is defined by sending a point in a fibre of π to the line through the other two points in the fibre; clearly we need to work a bit to understand this idea. For one thing, this definition only makes sense for fibres containing three distinct points of X. Still, we may hope to define a rational map on X whose locus of indeterminacy is contained in the ramification locus of π. The greater difficulty is understanding what the range of this function should be: we need to map to a bundle whose fibre over a fixed point in Y is the $\mathbb{P}^1$ of lines in the fibre of $\mathbb{V}(E)$.

It turns out that we can make this idea work by considering the inclusion $E \hookrightarrow \mathcal{O}_Y \oplus E = F$. This inclusion allows us to identify the fibre of $\mathbb{V}(E)$ with the "finite part" of the fibre of $\mathbb{P}(F)$, which is a projective plane. The "line at infinity" in the fibre of $\mathbb{P}(F)$ is identified with the fibre of $\mathbb{P}(E)$. (These identifications follow from applying the functors **Spec** and **Proj** to the stated inclusion.) As sets, we have that $\mathbb{P}(F) = \mathbb{V}(E) \cup \mathbb{P}(E)$, and so we may view $X \subset \mathbb{V}(E)$ as a subvariety of $\mathbb{P}(F)$ that does not intersect $\mathbb{P}(E)$.

Now we are in a position to describe our putative rational map on X. Over a point y not in the branch locus of π, the fibre of π consists of three distinct points x_1, x_2, x_3. The line through any two of these points, say x_2 and x_3, is a line in the fibre of $\mathbb{P}(F)$ over y. Such a line corresponds to a point $p_{2,3}$ in the fibre of $\mathbb{P}(F^*)$ over y. There is a natural projection $\mathbb{P}(F^*) \longrightarrow \mathbb{P}(E^*)$ which dualizes the inclusion $\mathbb{P}(E) \hookrightarrow \mathbb{P}(F)$; over the point y, this map is the projection whose center is the point corresponding to the line at infinity in the fibre of $\mathbb{P}(F)$ over y. We have that $p_{2,3}$ is never equal to the center of this projection, because X does not meet $\mathbb{P}(E)$; thus we can project $p_{2,3}$ to a point $q_{2,3}$ in the fibre of $\mathbb{P}(E^*)$ over y. We define a rational map $\psi : X -- \rightarrow \mathbb{P}(E^*)$ by setting $\psi(x_1) = q_{2,3}$, and similarly for x_2, x_3.

We have the following result:

Proposition 3. *If $\pi : X \longrightarrow Y$ is sufficiently local, then the map ψ is in fact rational, and its image is contained in $f^*\mathfrak{G} \subset \mathbb{P}(E^*) = Y \times \mathbb{P}^1$. The restricted map*

$$\psi : X\, f^* -- \rightarrow \mathfrak{G}$$

is birational, and the isomorphism $f^*\phi : f^*\Gamma \longrightarrow f^*\mathfrak{G}$ is the resolution of indeterminacy of this birational map.

All of these claims can be checked easily (by the reader!) once we establish the expression for ψ in terms of local coordinates on $X \subset \mathbb{V}(E)$:

Lemma 1. *The local expression for ψ over a point $y \in Y$ is*

$$
\begin{aligned}
\psi(y \times (z, w)) &= y \times [z + a(y) : b(y)] \\
&= y \times [c(y) : w + d(y)] \\
&= y \times [w - 2d(y) : z - 2a(y)],
\end{aligned}
$$

where z, w are coordinates on the fibre of π over y, and a, b, c, d are the sections of $\mathcal{O}_Y$ appearing as coefficients in the equations (3).

Note that the equivalence of the three expressions for ψ is a consequence of the equations (3) which define X as a subvariety of $\mathbb{V}(E)$.

Proof. In order to proceed, we need to recall a fact from [2] regarding the sheaf E: the $\mathcal{O}_Y$-algebra $\pi_*(\mathcal{O}_X)$ is in fact a rank 3 $\mathcal{O}_Y$-module, and E is the rank 2 submodule consisting of sections that have zero trace over $\mathcal{O}_Y$. This means that if we take local generators z, w of E as local coordinates on $\mathbb{V}(E)$, then the vector sum of the points (z_i, w_i) in any fibre of π must be zero.

Now we can prove the lemma. Let $y \in Y$ be any point not in the branch locus of π. Then the fibre of π over y consists of three distinct points, which we denote $(z_1, w_1), (z_2, w_2), (z_3, w_3)$. The fibre of $\mathbb{V}(E)$ over y is an affine plane with coordinates z, w; inside this plane, the line containing (z_2, w_2) and (z_3, w_3) is given by

$$
-(w_3 - w_2)z + (z_3 - z_2)w + (z_2 w_3 - z_3 w_2) = 0.
$$

In local coordinates, then, we have

$$
\psi(z_1, w_1) = [-(w_3 - w_2) : (z_3 - z_2)];
$$

this is understood to be a point in the fibre of $\mathbb{P}(E^*)$ over y.

We claim that this expression agrees with the first one given in the statement of the lemma. To see this, we use the equations (3) and the zero trace observation above to compute that

$$
\begin{aligned}
(z_1 + a(y))(z_3 - z_2) &= z_1 z_3 - z_1 z_2 + a(y)(z_3 - z_2) \\
&= z_1 z_3 - z_1 z_2 + (z_3^2 - z_2^2 - b(y)(w_3 - w_2)) \\
&= (z_1 + z_2 + z_3)(z_3 - z_2) - b(y)(w_3 - w_2) \\
&= 0 - b(y)(w_3 - w_2).
\end{aligned}
$$

This proves that $\psi(z_1, z_2) = [z_1 + a(y) : b(y)]$ on the open set where π is unramified and where this expression is defined. Since we only require ψ to be a rational map, the lemma is proved. $\square$

This result shows that the locus of indeterminacy of ψ is precisely the fat-point ramification locus of π, which in general is a proper subset of the ramification locus of π. This is consistent with the fact that reasonable definitions of the rational map ψ can be made for double ramification points and for curvilinear triple ramification points; in these cases the Zariski tangent spaces to the ramification points determine lines in the fibres of π. At a fat point, the dimension of the Zariski tangent space in the fibre is equal to 2, so there is no reasonable way to define ψ at such a point.

5. Globalization.

Now we are going to define our resolution for an arbitrary triple cover $\pi : X \longrightarrow Y$. The idea is straightforward: we know from [2] that Y is covered by open affine sets Y_i for which the restricted triple covers $\pi : X_i \longrightarrow Y_i$ are sufficiently local, and we have already defined the resolution for sufficiently local triple covers. It remains to check that these local definitions patch together compatibly to define a global resolution.

Recall that in defining the universal local resolution $\rho : \Gamma \longrightarrow \mathfrak{X}$, we constructed Γ as a subvariety of $\mathbb{V}(F) \times \mathbb{P}^1$. The unidentified factor of $\mathbb{P}^1$ is an obstruction to globalization: if each sufficiently local resolution variety is defined as a subvariety of $\mathbb{V}(E)_{|Y_i} \times \mathbb{P}^1$, then it is not clear how to interpret the second factor as the restriction of a globally defined object. Fortunately, we have seen how to remedy this: using the isomorphism $\phi : \Gamma \longrightarrow \mathfrak{G}$, we will take $\mathfrak{G}$ to be our resolution variety instead of Γ. Then we take the resolution variety for a sufficiently local triple cover to be $\mathfrak{G}_i = f_i^* \mathfrak{G}$ instead of $f_i^* \Gamma$. In the previous section we showed that each variety $\mathfrak{G}$ is naturally a subvariety of $\mathbb{P}(E^*)_{|Y_i}$. Thus we may hope to patch together the varieties $\mathfrak{G}_i$ to construct a subvariety $\mathfrak{G}_X$ of $\mathbb{P}(E^*)$.

Now we will invoke a beautiful result of Miranda from [2] to finish our construction. Miranda shows that any triple cover $\pi : X \longrightarrow Y$ is determined by a rank 2 locally free sheaf E on Y and a global section σ of $S^3(E)^* \otimes \Lambda^2(E)$. In fact, Miranda shows that if a, b, c, d are the coefficients appearing in the quadrics (3) that define X_i as a subvariety of $\mathbb{V}(E)_{|Y_i}$, then the local expression for σ over Y_i is

$$-b(z^3)^* + a(z^2)^* w^* - dz^*(w^2)^* + c(w^3)^*.$$

Using the natural isomorphism $S^3(E)^* \cong S^3(E^*)$, we get the following local expression for σ:

$$-\frac{1}{6}b(z^*)^3 + \frac{1}{2}a(z^*)^2 w^* - \frac{1}{2}dz^*(w^*)^2 + \frac{1}{6}c(w^*)^3.$$

Up to a constant factor, this is just the cubic defining $\mathfrak{G}_i$ as a subvariety of $\mathbb{P}(E^*)_{|Y_i}$. Since σ is a global section, we conclude that the varieties $\mathfrak{G}_i$ must patch together to form a variety $\mathfrak{G}_X \subset \mathbb{P}(E^*)$. It is clear that the structure morphisms patch together compatibly, and so we have the following result:

Proposition 4. *Let* $\pi : X \longrightarrow Y$ *be any triple cover, and let* E *be a rank 2 locally free sheaf on* Y *such that* $X \subset \mathbb{V}(E)$. *Then there is a variety* $\mathfrak{G}_X \subset \mathbb{P}(E^*)$ *and a birational morphism* $\rho_X : \mathfrak{G}_X \longrightarrow X$ *which is an isomorphism away from the fat-point ramification locus of* π, *and whose fibre over every fat point is a* $\mathbb{P}^1$.

We refer to the morphism $\rho_X : \mathfrak{G}_X \longrightarrow X$ as the *resolution of the triple cover* π. Now we get the following global result:

Proposition 5. *Let* $\pi : X \longrightarrow Y$ *be a triple cover.* X *is isomorphic as a triple cover to a subvariety of a* $\mathbb{P}^1$-*bundle on* Y *equipped with the natural projection if and only if* π *has no fat-point ramification.*

Proof. One implication is the global version of Proposition 2; the other implication follows from the fact that the fibre of a subvariety of a $\mathbb{P}^1$-bundle cannot have a two-dimensional Zariski tangent space. $\square$

REFERENCES

[1]   G. Casnati - T. Ekedahl, *Covers of Algebraic Varieties I,* J. Alg. Geom., 5 (1966), pp. 439–460.

[2]   R. Miranda, *Triple covers in algebraic geometry,* Amer. J. Math., 107 - 5 (1985), pp. 1123–1158.

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