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A CANONICAL RESOLUTION OF THE SINGULARITIES OF A TRIPLE COVERING OF ALGEBRAIC SURFACES

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We propose a canonical resolution of singularities for a triple covering $f : X \to Y$ of algebraic surfaces, where X is normal and Y is smooth.

1. Introduction.

A canonical resolution of singularities for double coverings of algebraic surfaces is described in [2].

If $f : X \to Y$ is such a cover, with X normal and Y smooth, then one can blow up Y at the singularities of the branching curve, form a fibre square and then normalize the new X we get in this manner. After finitely many such transformations we get a new double cover $f' : X' \to Y'$, with X' and Y' smooth, plus X' birational to X and Y' birational to Y.

Essentially, this procedure is made possible by the very explicit local equation of the double cover, $z^2 = F(x, y)$.

In the present paper we use a similar explicit local representation for triple coverings – worked out by R. Miranda in [5] – to show that also in this case the above procedure gives a canonical resolution.

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As pointed out by R. Miranda, this method fails when we are dealing with quadruple coverings; we included an example in this respect.

This work is divided in two parts; "the hypersurface case" and "the general case", corresponding to our triple covering being locally represented by an hypersurface or not (as discussed in [5]). The proof that the proposed algorithm works is by induction.

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2. Statement of the problem.

Let $f : X \to Y$ be a triple covering of algebraic surfaces. Assume Y to be smooth and X to be normal. This last condition implies that the singularities of X must be isolated. In what follows we are actually interested only in the behaviour of X near its singularities, so we can actually suppose that $Sing(X) = \{Q\}$.

Moreover, we will suppose that our singularity is in the triple ramification locus of the triple cover f. Indeed, over $f(Q)$ we have three possibilities for $f^{-1}(f(Q))$:

(1) three points, corresponding to étale covering;
(2) two points, corresponding to étale & double covering;
(3) a single point, corresponding to total ramification.

In the first case X must be nonsingular. In the second case, there exist a neighborhood U of Q in X such that $f_{|U} : U \to f(U)$ is a double covering; this case is treated in [2], and is known to have a canonical resolution of singularities. So we can assume that over $f(Q)$ we have total ramification.

To start with, let's observe that Q must lay over a singularity of the branching curve $\mathcal{C} \subset Y$ ([1], thm. III.5.2).

The canonical resolution of singularities that we propose is the following.

Let $f : X \to Y$ be as above. Blow up the smooth base Y at $f(Q)$, and continue blowing up until the reduced total transform of $\mathcal{C}$ will have only normal crossing divisors. This is possible by the Embedded Resolution of Singularities, see e.g. theorem [3].

Form then the fiber product $\tilde{X} = X \times_Y \tilde{Y}$ as in figure 1, in which we denote

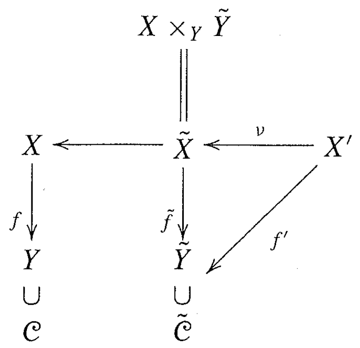

Fig. 1. blow-up, fiber product and normalization

by $\tilde{\mathcal{C}}$ the total transform of $\mathcal{C}$. Consider then the normalization $\nu : X' \to \tilde{X}$; the following proposition ensures that $f' := \tilde{f} \circ \nu$ is a triple covering (see also [1], thm. III.7.1).

Proposition 1. (refer to figure 1) *Let $f : X \to Y$ be a triple covering of surfaces, with Y smooth and X normal. Suppose that $\tilde{Y} \to Y$ is the blow-up of Y in one of its points; put $\tilde{X} = X \times_Y \tilde{Y}$ and let $\nu : X' \to \tilde{X}$ be the normalization. Then $\tilde{f} \circ \nu : X' \to \tilde{X}$ is a triple covering.*

Proof. We have to show that f' is a triple covering map, i.e. that it is flat and finite of degree 3. Since f is finite of degree 3, $\tilde{f}$ and then f' will be generically finite of degree 3. Then to complete the proof we have to show that f' is finite and flat.

Finiteness of f' follows from the following

Theorem 1. (Chevalley, [6] p. 124). *Let $f : X \to Y$ be a proper morphism with finite fibers. If Y is a Noetherian scheme, then f is finite.*

f is proper, so also $\tilde{f}$ is proper (properness is preserved by base change). Being normalization a finite map, also f' is proper. f has finite fibers, thus also $\tilde{f}$ and then f' have finite fibers. Y' is Noetherian because Y is such. Hypotheses of theorem 1 are thus fulfilled, and f' is proven to be finite.

To prove that f' is flat we can use the following

Theorem 2. ([4]). *Let $f : X \to Y$ be a morphism with X Cohen-Macaulay and Y smooth. If all the fibers of f have the same dimension, then f is flat.*

Indeed, our surface is Cohen-Macaulay since it is normal, $\tilde{Y}$ is smooth and f' is finite; hence all of its fibers are 0-dimensional. □

As a consequence we can thus make the following

Assumption. *$f : X \to Y$ is a triple covering, X is normal, Y smooth, $\mathrm{Sing}(X) = \{Q\}$, f is totally ramified over $f(Q)$, the singularities of the branching curve $\mathcal{C}$ are only nodes with multiplicity (that is, the reduced branching curve has only double points).*

Our claim is that we can resolve these nodes by blowing them up and "partially normalizing" $\tilde{X}$ along some of the exceptional divisors that we get, or along suitable divisors corresponding to the tangents at the nodes. Since at the end we get a new branching curve which is smooth, the corresponding $\tilde{X}$ must be nonsingular and thus normal.

So far we have worked without any assumption on Y, except smoothness. To make the computations a little less cumbersome, we assume initially that $Y = \mathbb{A}^2$, since the result we want to prove is local in nature; later we will show that the general case can be worked out quite similarly as this one.

3. The hypersurface case.

Assumption. *Since now – and unless otherwise specified – we assume that $f : X \to Y$ is the triple covering such that $Y = \mathbb{A}^2_{x,y}$, $X = \{z^3 + 3zg(x, y) + 2h(x, y) = 0\} \subset \mathbb{A}^3_{x,y,z}$, f is given by the projection and the branching curve has an unique singularity at $(0, 0)$ which is a reducible node with multiplicity.*

By using Cardano equations, together with the fact that $(0, 0, 0) \in \mathrm{Sing}(X)$ is totally ramified, we get that

$$g(0,0) = h(0,0) = \frac{\partial h}{\partial x}(0,0) = \frac{\partial h}{y}(0,0) = 0.$$

In this case, the equation of the branching curve is given by $\Delta : g^3 + h^2 = x^m y^n$ near $(0, 0)$; by making a renormalization we can actually write

$$\Delta : g^3 + h^2 = x^m y^n.$$

Let p be a polynomial; with ord p we denote the multiplicity of p at the origin. If ord $p = d$, then by writing $p = p_d + \dots$ we mean that p_d is the lowest-degree nonzero homogeneous part, that is, the dots denote the sum of all monomials having degree greater than d. Using this notation, we define F_{n_1} and F_{n_2} by

$$g = F_{n_1} + \dots \qquad \text{and} \qquad h = F_{n_2} + \dots .$$

Definition 1. *The* node-order of $q \in X$ *is the order of* $f(q)$ *in the branching curve of the covering* f.

We will use induction on $n + m = N$, the node-order of $(0, 0, 0)$; that is, we will resolve the singularity $(0, 0, 0)$ supposing that any singularity $q_0 \in X_0$, for any covering $f_0 : X_0 \to Y_0$ whose node-order is smaller than N has a canonical resolution.

Proposition 2. *Let X, Y, f, N be as described above. Then there exists a sequence of blow-ups of points and partial normalizations along suitable divisors such that the corresponding node-orders are all strictly smaller than N.*

Proof. The proof is split in some cases; nevertheless, before examining them in detail, it is useful to state notations and to make some general computations.

Let $\tilde{Y} = \mathcal{B}\ell_{(0,0)}Y$; $\tilde{Y}$ is covered by two affine charts. In the chart C_x given by

$$\begin{array}{cccc} \sigma_x : & \mathbb{A}^2_{x,t} & \to & Y \\ & (x, t) & \mapsto & (tx, t) \end{array}$$

the equation defining $\tilde{X}$ in the pull-back is

$$\tilde{X}_x : z^3 + 3zg(tx, t) + 2h(tx, t) = 0 ;$$

by defining g_x, h_x with $g(tx, t) = t^{n_1} g_x(x, t)$ and $h(tx, t) = t^{n_2} h_x(x, t)$ we get the equations

$$\tilde{X}_x : z^3 + 3zt^{n_1} g_x + 2t^{n_2} h_x = 0 ,$$
$$\tilde{\mathcal{C}}_x : t^{3n_1} g_x^3 + t^{2n_2} h_x^2 = x^m t^N = 0 .$$

Let E be the exceptional divisor and $k \in \mathbb{N}$, $k > 0$. The partial normalization ν along $-kE$ is dual to the substitution $\nu_k : \frac{z}{t^k} \mapsto z$, after which we obtain the equations

$$(1) \qquad \begin{array}{l} X'_x : z^3 + 3zt^{n_1-2k} g_x + 2t^{n_2-3k} h_x = 0 , \\ \mathcal{C}'_x : t^{3n_1-6k} g_x^3 + t^{2n_2-6k} h_x^2 = x^m t^{N-6k} = 0 ; \end{array}$$

doing the same computations in the other affine chart

$$\begin{array}{cccc} \sigma_y : & \mathbb{A}^2_{t,y} & \to & Y \\ & (t, y) & \mapsto & (t, ty) \end{array}$$

we get the equations

$$X'_y : z^3 + 3zt^{n_1-2k}g_y + 2x^{n_2-3k}h_y = 0\,,$$
$$\mathcal{C}'_y : t^{3n_1-6k}g_y^3 + t^{2n_2-6k}h_y^2 = t^{N-6k}y^n = 0\,.$$

We will call *fundamental transformation* each step of the above type, that is: the composition of the blow up in $(0,0)$, the greatest partial normalization which can be done and the restriction to both the affine charts.

Each fundamental transformation gives us two affine triple coverings, respectively given by the pair $(g'_x, h'_x) = (t^{n_1-2k}g_x, t^{n_2-3k}h_x)$ over $\mathbb{A}^2_{x,t}$ and by $(g'_y, h'_y) = (t^{n_1-2k}g_y, t^{n_2-3k}h_y)$ over $\mathbb{A}^2_{t,y}$. With the pair (new g, new h) we will denote one of those two pairs, and (new N) will be the corresponding node-order of $(0,0)$.

Case 1: $3n_1 = 2n_2$. We have $N \geq 3n_1 = 2n_2 = 6k$ for some positive integer k. After a fundamental transformation we look in a chart, say in C_x, and we get

$$g_x^3 + h_x^2 = x^m t^{N-6k}$$

as the equation of the branching curve.

If that transformation doesn't break the equality, that is, if 3 ord(new g)=2 ord(new h), then we repeat the fundamental transformation; we remark that in this case (new g) = 0 and (new h) = 0 are exactly the equations of the proper strict transforms of the curves $g = 0, h = 0$.

This shows that we have two possibilities:

- after finitely many steps both (new g) = 0 and (new h) = 0 are nonsingular. This means that (e.g. in C_x) $\frac{\partial h}{\partial x}(0,0)$ and $\frac{\partial h}{\partial y}(0,0)$ cannot both be zero, so the ramification above $(0,0)$ is not total, and we are done (see the remark at the beginning of the hypersurface case).
- after a finite number of steps the equality doesn't hold anymore: for example, 3 ord(new g) <2 ord(new h). We first notice that, since we were getting proper transforms (until last step), we get ord(new g) $\leq$ ord g and ord(new h) $\leq$ ord h; since the first inequality is forced to be strict, the degree of the branching curve at this step will be 3 ord(new g) $< N$, and we are done by induction. The other inequality is treated in an analogous manner.

Case 2: $3n_1 < 2n_2$ $N = 3n_1$ and $\mathcal{C} : x^m y^n = (F_{n_1})^3 = (x^\alpha y^\beta)^3$; remembering (1) we blow up and normalize, with $k = \lfloor \frac{N}{6} \rfloor = \lfloor \frac{n_1}{2} \rfloor$.

In the chart C_x we get

$$\mathcal{C}'_x : (x^\alpha t^{n_1-2k})^3 = 0.$$

If $n_1 = 2k$, this curve is smooth; so we can assume that $n_1 = 2k+1$ and $\mathcal{C}'_x : (x^\alpha t)^3 = 0$. If $\alpha + 1 < n_1$ we are done by induction; otherwise, since $\alpha + \beta = n_1$ and $\alpha, \beta > 0$, we necessarily have $\alpha = 2k$ and $\beta = 1$, that is

$$\mathcal{C} : x^{6k} y^3 = 0 , \qquad \mathcal{C}'_x : x^{6k} t^3 = 0 ,$$

We have $N =$ new N; two possible cases arise:

- x^2 divides g. Since x^6 divides $x^{6k}y^3$, x^6 divides both g^3 and $g^3 + h^2 = x^{6k}y^3$, thus $x^6 | h^2$ i.e. $x^3 | h$. A routine computation allows to verify that, instead of the substitution $\frac{z}{t^k} \mapsto z$, we can do the (greater) partial normalization which corresponds to the substitution $\frac{z}{xt^k} \mapsto z$. So we perform this alternate partial normalization and we get a branch curve $\mathcal{C}''_x : x^{6k-6}t^3 = 0$ in which the node-order of $(0, 0)$ is strictly less than N.
- x^2 does not divide g. This means that g contains a nonzero term of type cy^a or cxy^a; it is trivial to note that such a term has been transformed into a term of (new g) which has strictly smaller degree. Now, since $\operatorname{ord}(g'^3_x + h'^2_x) = \operatorname{ord}(x^{6k}t^3)$ is odd, it can't be that $3 \operatorname{ord} g'_x > 2 \operatorname{ord} h'_x$; so either we fall in case 1 (=) with (new N) $< N$ or we fall again in this case: $3 \operatorname{ord} g_x < 2 \operatorname{ord} h_x$ with $N =$ new $N = 6k+3$. The previous argument ensures that after a finite number of steps each one falling in this case we have ord(new g) $< 2k+1$; thus new $N = 3$ ord(new g) $< 6k+3$ and we are done by induction.

In the other chart C_y the situation is completely symmetrical, so the proof is identical provided that we rename some symbols accordingly.

Case 3: $3n_1 > 2n_2N = 2n_2$ and $\mathcal{C} : x^m y^n = (F_{n_2})^2 = (x^\alpha y^\beta)^2$; we blow up and normalize with $k = \lfloor \frac{N}{6} \rfloor = \lfloor \frac{n_2}{3} \rfloor$.

We look only into C_x; for the other chart a symmetrical proof holds. The branching curve is given by

$$\mathcal{C}'_x : (x^\alpha t^{n_2 - 3k})^2 = 0.$$

Since if $n_2 = 3k$ this curve is smooth, we have two possible subcases.

Subcase 3.1: $n_2 = 3k+1$ The hypotheses imply that $\mathcal{C}'_x : (x^\alpha t)^2 = 0$, $N = 6k+2$; thus if $\alpha + 1 < n_2$ we are done by induction. Otherwise, since $\alpha + \beta = n_2$ and $\alpha, \beta > 0$ it must be $\alpha = n_2 - 1 = 3k$ and $\beta = 1$, that is

$$\mathcal{C} : x^{6k} y^2 = 0 , \qquad \mathcal{C}'_x : x^{6k} t^2 = 0 ,$$

and in particular $N =$ new N. As in case 2, we have two possibilities:

- x^3 divides h. Since x^6 divides $x^{6k}y^2$, x^6 divides both h^2 and $g^3 + h^2 = x^{6k}y^2$, thus $x^6|g^3$ i.e. $x^2|g$. So we could have done the greater partial normalization $\frac{z}{xt^k} \mapsto z$ getting as the branch locus the curve $\mathcal{C}''_x : x^{6k-6}t^2 = 0$ which gives (new N) $= N - 6 < N$.
- x^3 does not divide h. h contains a nonzero term of type cy^a, cxy^a or cx^2y^a; each one of them is brought by a fundamental transformation into a monomial having strictly smaller degree. Indeed, by definition

$$h_x(x,t) = \frac{h(tx,t)}{t^{n_2}} = \frac{h(tx,t)}{t^{3k+1}},$$

so they are transformed respectively into ct^{a-3k-1}, cxt^{a-3k} and cx^2t^{a-3k+2}. Since $\operatorname{ord}(g'^3_x + h'^2_x) = \operatorname{ord}(x^{6k}t^2)$ is not multiple of 3, it can't be that 3 ord $g'_x < 2$ ord h'_x; so either we fall in case 1 (=) with (new N) $< N$ or we fall again in subcase 3.1: 3 ord $g'_x > 2$ ord h'_x with $N =$ new $N = 6k+2$, which is resolved in the same way as case 2.

Subcase 3.2: $n_2 = 3k+2$ As in subcase 3.1 $\mathcal{C}'_x : (x^\alpha t^2)^2 = 0$, $N = 6k+4 = n_2$, new $N = 2\alpha + 4$, and if $\alpha + 2 < n_2$ we are done by induction; being $\alpha, \beta > 0$ and $\alpha + \beta = n_2$ it must be either $\alpha = 3k$ (the proof in this case is omitted since it is very similar to the one for subcase 3.1) or $\alpha = 3k+1$. Let us be in this last case, say

$$\mathcal{C} : (x^{3k+1}y)^2 = 0\,, \quad \mathcal{C}'_x : (x^{3k+1}t^2)^2 = 0\,, \qquad \text{new } N = N + 2\,.$$

We have that 3 ord $g \geq 2$ ord h (if not, the equation of $\mathcal{C}'_x : x^{6k+2}t^4 = 0$ must be a cube, which is impossible) so 2 ord $h'_x \leq$ new $N = 6k+6$. If ord $h'_x = 3k+3$, then (new N) is multiple of 3 and we are done by one more fundamental transformation. If ord $h'_x < 3k+3$, then 2 ord $h'_x < 6k+6 =$ new N, so in the equation of $\mathcal{C}'_x$ the lowest-degree monomial of h'_x must be canceled by some other monomial, which implies 3 ord $g'_x = 2$ ord h'_x and in particular ord $g'_x < 2k+2$, that is ord $g'_x \leq 2k+1$.

Then, proceeding as in case 1, after enough many fundamental transformations either we get one of (new g) or (new h) nonsingular (which excludes total ramification), or one of ord (new g) and ord(new h) must decrease.

If it is ord(new h), then (new N) $\leq 2((3k+2)-1) = 2(3k+1) < 2(3k+2)$; if it is ord(new g) then $N \leq 3((2k+1)-1) = 6k$. In both cases we are done by induction.

□

To ensure that the present proof is working, we have to show that blowing-up and normalizing at each step of our procedure produces the same result as blowing up several times and normalizing only at the end; this is accomplished by the following lemma (which holds also in a broader context).

Lemma 1. *Let $f : X \to Y$ be a surjective and flat map of surfaces which is finite of degree 3. Consider the diagram in figure 2, where σ denotes the blow-up of Y in one of its points and ν, ν' and ν_1 are normalization maps; then $\tilde{X}'_1$ is isomorphic to $\tilde{X}'$.*

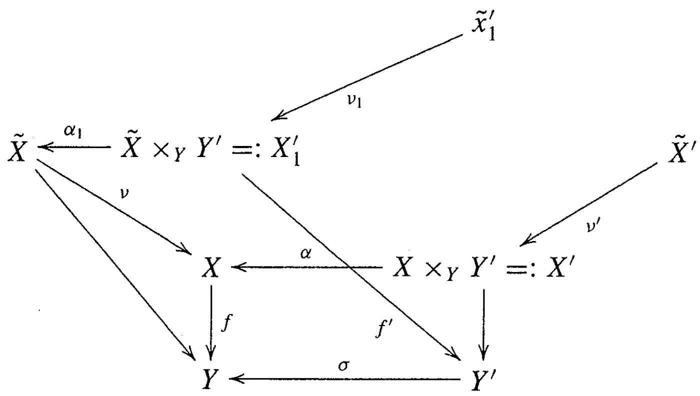

Fig. 2 the diagram referred by lemma 1

Proof. The proof is carried out by adjoining maps to the diagram without violating commutativity, by five applications of universal properties.

(1) From $f' : X'_1 \to Y'$ and $\nu \circ \alpha_1 : X'_1 \to X$ we obtain $\varphi : X'_1 \to X' = X \times_Y Y'$.

(2) Lift $\varphi \circ \nu_1 : \tilde{X}'_1 \to X'$ to $\tilde{\varphi} : \tilde{X}'_1 \to \tilde{X}'$.

(3) Lift $\alpha \circ \nu' : \tilde{X}' \to X$ to $\eta : \tilde{X}' \to \tilde{X}$.

(4) η, together with $\tilde{X}' \to Y'$, gives $\psi : \tilde{X}' \to X'_1 = \tilde{X} \times_Y Y'$.

(5) Lift ψ to $\tilde{\psi} : \tilde{X}' \to \tilde{X}'_1$.

The diagram obtained by adjoining all of those maps is commutative; thus $\tilde{\varphi}$ and $\tilde{\psi}$ are each one the inverse of the other, giving the isomorphism between $\tilde{X}'_1$ and $\tilde{X}'$. □

So far, we have proven the following

Proposition 3. *Let $f : X \to Y$ be a triple covering of algebraic surfaces, such that each $y \in Y$ has a neighborhood U such that*

- *U is isomorphic to $\mathbb{A}^2$;*
- *$V := f^{-1}(U)$ is isomorphic to an hypersurface in $\mathbb{A}^3$;*
- *$f_{|V}$ is given by the projection.*

Then there is a canonical resolution of singularities as follows:

(1) *Blow-up Y in all the singularities of the branching curve $\mathcal{C}$.*
(2) *Form a fiber product and normalize, getting a new covering as described in proposition 1; repeat from step 1 until all singularities of $\mathcal{C}$ are at worst nodes with multiplicities.*
(3) *Blow up the nodal singularities, form a fiber product and normalize, as above, to get a nonsingular covering (proven by induction using proposition 2).*

4. The general case.

R. Miranda proved in [5] that any triple cover can be given locally by a projection

$$\frac{Y \times \mathbb{A}^2_{z,w}}{(F, G, H)} \to Y \tag{2}$$

where

$$\begin{aligned} F &= z^2 - az - bw - 2A, & A &= a^2 - bd, \\ G &= zw + dz + aw + B, & B &= ad - bc, \\ H &= w^2 - cz - dw - 2C, & C &= d^2 - ac, \end{aligned} \tag{3}$$

and $a, b, c, d \in \mathcal{O}_Y$. For now, we still suppose that every point in Y has a neighborhood isomorphic to $\mathbb{A}^2$, that is: we assume that $Y = \mathbb{A}^2$ and $p = (0, 0)$ is the unique singularity of the branch curve; we denote by $\mathfrak{m}_p$ the maximal ideal in $\mathcal{O}_{Y,p}$.

Now we need two results from [5].

Proposition 4. ([5], Corollary 4.6). *The locus in Y over which there is total ramification is defined by the ideal (A, B, C) in $\mathcal{O}_Y$.*

Proposition 5. ([5], Proposition 5.2). *X is singular over p if and only if one of the following conditions holds:*

i) $a, b, c, d \in m_p$;
ii) $a, c \in m_p$, $b \in m_p^2$;
iii) $b, d \in m_p$, $c \in m_p^2$;
iv) $b \notin m_p$, $A \in m_p$, $bB - 2aA \in m_p^2$;
v) $c \notin m_p$, $C \in m_p$, $cB - 2dC \in m_p^2$;
vi) $b, A \notin m_p$, $D \in m_p^2$;
vii) $c, C \notin m_p$, $D \in m_p^2$,

where $D = B^2 - 4AC$.

If $b \notin m_p$ or $c \notin m_p$, then we can solve for w in terms of z (or vice-versa) by tensoring with $K(Y)$; thus we get a local equation for X of the form

$$z^3 + 3g(x, y)z + 2h(x, y) = 0 \quad \text{in } Y \times \mathbb{A}^1_z, \text{ with } g, h \in \mathcal{O}_Y$$

where – once again – the covering is just the projection (to be more precise, the "new X" that we get will be birational with the "old X", but the branching curve in Y will remain locally unchanged). So we can suppose $b, c \in m_p$.

Also, if $a \notin m_p$ or $d \notin m_p$ then one of A, B or C is not in m_p so by proposition 4 we don't have total ramification. The next lemma shows that these cases are enough to complete the argument.

Lemma 2. *There exists a chain of blow-ups and partial normalizations such that none of the singularities of the new covering is of type i).*

Let

$$f = \sum_{(\alpha,\beta)\in I_f} c_{\alpha,\beta} x^\alpha y^\beta \in \mathbb{K}[x, y]$$

be a polynomial, where $I_f = \{(\alpha, \beta) | c_{\alpha,\beta} \neq 0\} \subset \mathbb{N}^2$; we put

$$\mu(f) := \bigcup_{(\alpha,\beta)\in I_f} x^\alpha y^\beta \subset \mathbb{K}[x, y].$$

If $f_1, \ldots, f_k \in \mathbb{K}[x, y]$, we denote by $M(f_1, \ldots, f_k) = \mu(f_1) \cup \ldots \cup \mu(f_k)$, and let $m_1, \ldots, m_h$ be the minimal set of generators for the (monomial) ideal generated by the elements of $M(f_1, \ldots, f_k)$.

The minimality of the m_i implies that we can eventually reorder them in such a way that

$$\begin{array}{lllllll} \log_x(m_1) & < & \log_x(m_2) & < & \ldots & < & \log_x(m_h) \quad \text{and} \\ \log_y(m_1) & > & \log_y(m_2) & > & \ldots & > & \log_y(m_h) \end{array}$$

We can represent graphically a set of monomials $S \subset \mathbb{K}[x, y]$ in the plane with coordinates (α, β) by putting a point at (α, β) if and only if S contains a nonzero monomial of the form $c_{\alpha,\beta}x^{\alpha}y^{\beta}$.

This is the setting in which we express the following

Definition 2. *A* box *is a rectangle which edges are either horizontal or vertical. The* box associated to $f_1, \dots, f_k$ *is the smallest box which contains all the points which represent the minimal generators of the ideal* $\big(M(m_1, \dots, m_h)\big)$. *The* box-size of $f_1, \dots, f_k$ *is the area of the box associated to* $f_1, \dots, f_k$, *i.e. the non-negative integer*

$$\mathbf{BS}(f_1, \dots, f_k) = \big(\log_x(m_h) - \log_x(m_1)\big) \times \big(\log_y(m_1) - \log_y(m_h)\big) .$$

See figure 4 for an example relevant to this definition.

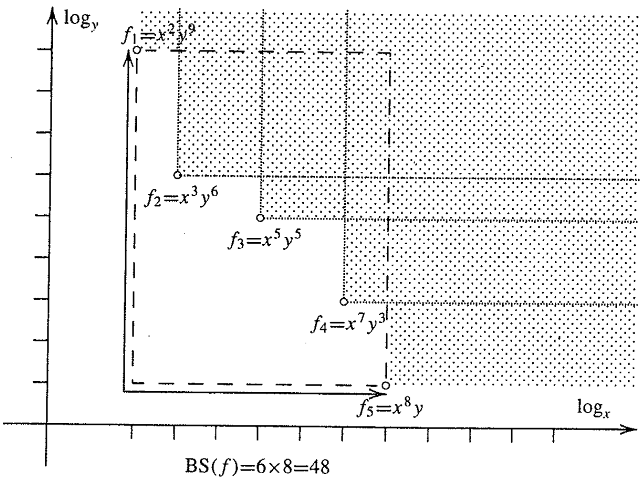

Fig. 3. The box-size of $x^2y^9 + x^3y^6 + x^5y^5 + x^7y^3 + x^8y$ is 48

Proof of lemma 2. A simple computation shows that the partial normalization corresponding to the substitutions

$$(z, w) \mapsto \left(\frac{z}{x^{\alpha}y^{\beta}}, \frac{w}{x^{\alpha}y^{\beta}}\right) \tag{4}$$

transforms the covering given by (a, b, c, d) in the one given by

$$\left(\frac{a}{x^{\alpha}y^{\beta}}, \frac{b}{x^{\alpha}y^{\beta}}, \frac{c}{x^{\alpha}y^{\beta}}, \frac{d}{x^{\alpha}y^{\beta}}\right).$$

If we represent the monomials of a, b, c, d graphically, the substitution corresponds to the translation $(h, k) \mapsto (h - \alpha, k - \beta)$, proving that **BS** remains unchanged under the action of such substitutions.

On the other hand, we will show that **BS** strictly decreases under blow-ups (see e.g. figure 4).

Let's first examine how the blow-up substitutions $x = \bar{x}$, $y = \bar{x}\,\bar{y}$ behave under this graphical paradigm (the case of the other substitutions $x = \bar{x}\,\bar{y}$, $y = \bar{y}$ is similar). A monomial $x^\alpha y^\beta$ becomes $\bar{x}^{\alpha+\beta}\bar{y}^\beta$, that is, the (α, β)-plane undergoes the linear transformation $(\alpha, \beta) \mapsto (\alpha + \beta, \beta)$.

Let D and D_0 be the boxes corresponding respectively to the minimal generators of the ideals $J = (M(a, b, c, d))$ and $J' = (M(a', b', c', d'))$, where $a' = a(x, xy), \dots, d' = d(x, xy)$; by definition **BS** (a, b, c, d) = **area** (D) and **BS** (a', b', c', d') = **area** (D_0).

Claim. area (D_0) < area (D).

Proof of the claim. Suppose that the minimal generators of J are $x^{\alpha_1}y^{\beta_1}, \dots, x^{\alpha_h}y^{\beta_h}$; since they are minimal, we can eventually reorder them and assume that $\alpha_1 < \dots < \alpha_h$ and $\beta_1 > \dots > \beta_h$.

The generators of J' will correspond to pairs $(\alpha_1 + \beta_1, \beta_1), \dots, (\alpha_h + \beta_h, \beta_h)$. A *minimal* set of generators for J' will then consist of a subset of these pairs, indexed by $1 \le i_1 < \dots < i_k \le h$, such that $\alpha_{i_1} + \beta_{i_1} < \dots < \alpha_{i_k} + \beta_{i_k}$ and $\beta_{i_1} > \dots > \beta_{i_k}$.

Then $\beta_{i_1} - \beta_{i_k} \le \beta_1 - \beta_h$, so height$(D_0) \le$ height(D).

Also width$(D_0) = (\alpha_{i_k} + \beta_{i_k}) - (\alpha_{i_1} + \beta_{i_1}) = (\alpha_{i_k} - \alpha_{i_1}) + (\beta_{i_k} - \beta_{i_1}) \le \alpha_h - \alpha_1$, with equality if and only if $i_1 = 1$, $i_k = h$, $\beta_{i_k} = \beta_{i_1}$, that is $\beta_h = \beta_1$. But **area** $(D) > 0 \Rightarrow \beta_h < \beta_1$, so width $(D_0) < \alpha_h - \alpha_1$.

Then **area** $(D_0) < (\alpha_h - \alpha_1)(\beta_1 - \beta_h) =$ **area** (D). □

Now we apply these facts to **BS** (a, b, c, d): by repeatedly blowing-up singularities and eventually doing partial normalizations of type (4), we necessarily obtain an X' which is covered by affine charts all of whose singularities are such that **BS** $(a, b, c, d) = 0$.

This last condition means that all monomials of $M(a, b, c, d)$ are multiples of the same monomial $x^\alpha y^\beta$; thus, after the partial normalization

$$(z, w) \mapsto \left(\frac{z}{x^\alpha y^\beta}, \frac{w}{x^\alpha y^\beta}\right)$$

M contains a constant monomial, that is, one of a, b, c, d is not in m_p. □

So far, the assumption that Y is locally isomorphic to $\mathbb{A}^2$ was used only to ensure the existence of local coordinates on Y; but this can actually be done for any smooth Y. Indeed, we can always choose local parameters u_1 and u_2 about a point $y \in Y$; moreover, we can suppose that they are given by some regular functions on a suitable neighborhood U of y in Y which are nonsingular at y and such that y is their only common zero in U. Blow-up and normalization can be done using u_1, u_2 as local coordinates instead of x, y.

We also notice that, since we have required the total transform of the branching curve of f to be a normal crossing divisor, we see that the singularities of this total transform are algebraically reducible nodes, and that they can be expressed locally as $u_1^n u_2^m = 0$, where u_i are local parameters around the singularity. So every computation done before is valid using local parameters, and we have the following

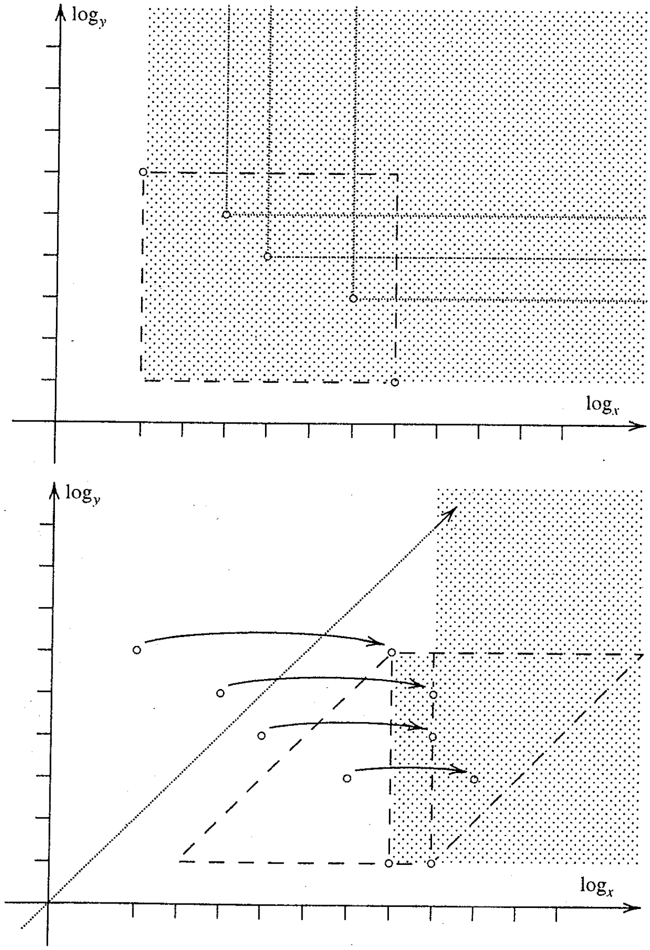

Fig. 4. Box-size decreases by blowing-up

Theorem 3. *Let $f : X \to Y$ be a triple covering of algebraic surfaces, with X normal and Y smooth. Then there exists a canonical resolution of singularities of the covering $f : X \to Y$ as described in proposition 3.*

Example. R. Miranda communicated to us an example which proves that the above approach doesn't work for 4-coverings. Indeed, let X be given by $z^4 = xy$ in $\mathbb{A}^3_{x,y,z}$, with (as usual) the covering map given by the projection.

We note that X is normal: indeed, it is complete intersection in the smooth $\mathbb{A}^3$; thus normality is equivalent to regularity in codimension 1, which holds for X, since Sing $X = (0, 0, 0)$.

To show that the algorithm doesn't resolve the singularity, it is enough to show a suitable sequence of substitutions, chosen among

$$\varphi_x : \begin{cases} \bar{x} = xy \\ \bar{y} = y \end{cases} \quad \text{and} \quad \varphi_y : \begin{cases} \bar{x} = x \\ \bar{y} = xy \end{cases}$$

under which action (after partial normalization) some singularity remains unchanged. But:

$$z^4 = xy \overset{\varphi_x}{\longmapsto} z^4 = x^2 y$$

and

$$z^4 = x^2 y \overset{\varphi_x}{\longmapsto} z^4 = x^2 y^3 \overset{\varphi_x}{\longmapsto} z^4 = x^2 y^5 \overset{\bar{z}=z/y}{\longmapsto} z^4 = x^2 y \,.$$

It remains to show that the normalization of $z^4 = x^2 y$ is singular. Define

$$A := \frac{\mathbb{K}[x, y, z]}{(z^4 - x^2 y)} \to \frac{\mathbb{K}[x, w, z]}{(z^2 - xw)} =: B$$

by

$$x \longmapsto x, \quad y \longmapsto w^2, \quad z \longmapsto z\,;$$

being B integral over A, it must be included in the normalization of A. But B is easily seen to be normal: the above inclusion is thus an equality, and B is the normalization of A. Finally, B is singular.

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