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ON A DEGENERATION OF THE SYMMETRIC PRODUCT OF A CURVE WITH GENERAL MODULI

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Let C be a smooth curve of genus $g \geq 1$ which degenerates to a rational g-cuspidal curve C_0, and let $\mathcal{L}_{n,\gamma}$ be a line bundle of type $(n+\gamma)x - \gamma(\delta/2)$ over $C^{(2)}$, where x is the tautological class and δ is the diagonal class. We study the degeneration of $\mathcal{L}_{n,\gamma}$ to $C_0^{(2)}$. The case with nodes instead of cusps has been studied by Franchetta and Ciliberto-Kouvidakis.

1. Introduction.

Let C be a smooth curve of genus $g \geq 1$, and let $C^{(2)}$ be its second symmetric product, which is defined as the quotient of the cartesian product by the natural involution. The map $C \times C \to C^{(2)}$ is ramified along the diagonal, hence if Δ is the diagonal divisor in $C^{(2)}$, the class of Δ in the Neron-Severi group of $C^{(2)}$ is divisible by two. If P, Q are points of C, we denote by $P + Q$ the corresponding point on $C^{(2)}$. It is well known that if C has general moduli the Neron-Severi group of $C^{(2)}$ is generated by the following classes: the class of the curve $X_P := \{P + Q \quad | Q \in C\}$, which is independent of the choice of P, and the class of $\Delta/2$ (cf. [5], Ch. 2 Sect. 5). Following [1] we denote x the class of X_P and by δ the class of Δ.

Let now $\mathcal{L}$ be a line bundle on $C^{(2)}$; then $\mathcal{L}$ is algebraically equivalent to a line bundle of class $(n + \gamma)x - \gamma(\delta/2)$; the number γ is called the *valence* of the line bundle, and we denote by $\mathcal{L}_{n,\gamma}$ a line bundle of this type. Since $x^2 = 1$, $(\delta/2)^2 = 1 - g$, $x.(\delta/2) = 1$, we have that $\mathcal{L}^2_{n,\gamma} = n^2 - \gamma^2 g$.

We are interested in the cone of effective divisors on $C^{(2)}$ in the x, $(\delta/2)$-plane. By standard Mori theory, a class of a curve is on the boundary of this cone if it has non positive square (see [2] Lemma 4.5). This is the case of the diagonal. To obtain the description of the effective cone it is then sufficient to see whether or not there is a curve $D \subset C^{(2)}$ with non-positive self intersection and whose numerical equivalence class is not proportional to δ (see [8] for the description of the cone of curves on higher symmetric products). To prove that such a curve cannot exist Ciliberto and Kouvidakis use in [1] the degeneration of a curve of genus g with general moduli to a rational curve with g nodes; this technique has been worked out by Franchetta in [4]. The idea of the degeneration goes as follows. When the curve C degenerates to a curve C_0 with g nodes, the symmetric product $C^{(2)}$ degenerates to $C_0^{(2)}$; let $\nu : \tilde{C}_0 \to C_0$ be the normalization map. For any $i = 1, \dots, g$, let P_i^1, P_i^2 be the two points which are the preimage of a node P_i under the normalization map ν. The curve $\tilde{C}_0$ is isomorphic to $\mathbb{P}^1$, so its symmetric product is isomorphic to $\mathbb{P}^2$. If we identify $\tilde{C}_0$ with a smooth conic $\Gamma \subset \mathbb{P}^2$, then the isomorphism $\tilde{C}_0^{(2)} \tilde{\to} \mathbb{P}^2$ sends $P + Q$, if $P \neq Q$, to the intersection of the two tangents $T_P\Gamma \cap T_Q\Gamma$, it sends the diagonal of $\tilde{C}_0^{(2)}$ to the conic Γ itself, and the curve X_P to the line tangent to Γ at the point $2P$; we will abuse notation and denote this line again by X_P. Clearly we have a birational map $\psi : C_0^{(2)} \dashrightarrow \mathbb{P}^2$. If $\sigma : W \to C_0^{(2)}$ is the desingularization of $C_0^{(2)}$, then the birational map ψ lifts to a birational morphism $\phi : W \to \mathbb{P}^2$, and it is possible to describe the limit on $C_0^{(2)}$ of a line bundle of type $\mathcal{L}_{n,\gamma}$ giving the linear system on the plane $\mathbb{P}^2$ induced by ϕ. For more details see [1]; here we explain only the main result. For any i define an isomorphism ω_i between the lines $X_{P_i^1}$, $X_{P_i^2}$ as follows: given a point $P \in \tilde{C}_0$ take the tangent line to conic Γ at the point $2P$ – this tangent intersects the above pair of lines in a pair of points which correspond to each other under ω_i. It is possible to prove the following statement (see [1], Prop. 2.1):

Theorem. (Ciliberto - Kouvidakis). *Let $\mathcal{L}_{n,\gamma}$ be a line bundle on $C^{(2)}$ with $\gamma > 0$; then it induces under the Franchetta degeneration a linear system $\mathfrak{C}_{\mathbb{P}^2}$ on $\mathbb{P}^2$ given by curves of degree n which pass through the g points $P_i^1 + P_i^2$, $i = 1, \dots, g$ with multiplicity γ. Moreover :*

(*) a limit curve in $\mathfrak{C}_{\mathbb{P}^2}$ intersects the lines $X_{P_i^1}$, $X_{P_i^2}$ in points which correspond to each other under ω_i (see fig. 1).

Now, if there were a curve $D_{n,\gamma} \subset C^{(2)}$ of type (n, γ), with $D^2 \leq 0$ (that is $n^2 \leq \gamma^2 g$), it would give rise, by the result above, to a curve D_0 in a planar linear system which, according to the Nagata conjecture (see [7]), should be

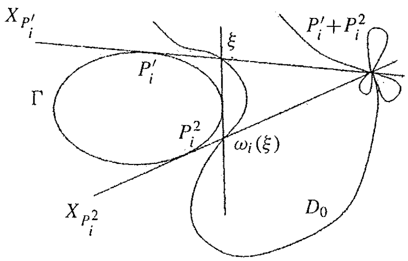

Fig. 1. Picture of condition $(*)$

empty. Since the conjecture is known to be true for a quadratic number of points $g = m^2 \geq 10$, Ciliberto and Kouvidakis obtain in this way the description of the cone of curves of $C^{(2)}$, for quadratic genera ≥ 16 (for the case $g = 9$ they provide a supplementary argument). For another proof of this result see also [6].

The curves in the limit linear system satisfy moreover the property $(*)$. This condition *a priori* could be used when $g \neq m^2$, but, in practice, it is difficult to handle. Here we let the genus g curve C degenerate to a rational curve C_0 with g cusps. The main motivation for this work is to extend Ciliberto and Kouvidakis' result to the case of the cuspidal degeneration in order to obtain another extra condition, hopefully more useful; the idea is that the cusp is the "limit" of a node when the two tangents to the branches of the node approach each other. In the above notations our main result is the following:

Theorem 1. *Let $\mathcal{L}_{n,\gamma}$ be a line bundle on $C^{(2)}$ with $\gamma > 0$; then it induces, under the Franchetta degeneration to a rational g-cuspidal curve C_0, a linear system $\mathcal{C}'_{\mathbb{P}^2}$ on $\mathbb{P}^2$ given by curves of degree n which pass through the g points $2P_i$ with multiplicity γ. Moreover :*

$(**)$ *if a limit curve $D_0 \in \mathcal{C}'_{\mathbb{P}^2}$ intersects the line X_{P_i} at a point $P_i + Q$, then it is tangent to the line X_Q at $P_i + Q$ (see fig. 2).*

2. The cuspidal Franchetta degeneration.

Let $\pi : X \to U$ be a flat family of curves over the complex disk $U = \{|t| \leq 1\}$; let us suppose that the fiber X_t is a smooth genus g curve for $t \neq 0$, whereas the central fiber X_0 is a rational curve with g cusps. We can consider the second symmetric product of this family, and we obtain the family

$$p : Y \to U$$

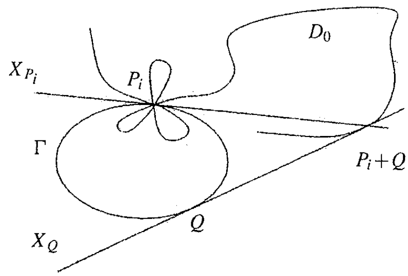

Figure 2. Picture of condition $(**)$

such that $Y_t = X_t^{(2)}$; obviously Y_t is a smooth surface for $t \neq 0$, whereas Y_0 is a singular surface isomorphic to $X_0^{(2)}$. Since the problem is local around the cusps, we can restrict to the case $g = 1$; then X_0 is a rational curve with a single cusp P, and the singular locus of $Y_0 = X_0^{(2)}$ is a cuspidal curve given by $\{P + Q \quad | Q \in X_0\}$. Let us denote by S the central fiber of Y; exactly as in [1], S is birational to $\mathbb{P}^2$ and we will denote again by ψ the natural birational map; in this case ψ is bijective and bicontinuous, but it is not an isomorphism, because it is not regular on the cuspidal curve of S. On the other hand, the inverse map ψ^{-1} is regular everywhere: actually, it is a desingularization of S. We want describe the 3-fold Y and its central fiber Y_0 in a neighborhood of the point $2P$.

First of all, we write equations for the variety $Sym^2\mathbb{C}^2$. This is done in [1], pg. 328, in the following way. We start with the ordinary product $\mathbb{C}^2 \times \mathbb{C}^2$ with coordinates $(x_1, x_2; y_1, y_2)$; the symmetric product is the quotient of this variety by the equivalence relation $(x_1, x_2; y_1, y_2) \sim (y_1, y_2; x_1, x_2)$. Under the change of coordinates:

$$s_i = \frac{(x_i + y_i)}{2}, \quad r_i = \frac{(x_i - y_i)}{2}$$

the equivalence relation becomes $(s_1, s_2; r_1, r_2) \sim (s_1, s_2; -r_1, -r_2)$. In other words, we have the isomorphism

$$Sym^2\mathbb{C}^2 \cong \mathbb{C}^2 \times \frac{\mathbb{C}^2}{\sim}.$$

Since $\frac{\mathbb{C}^2}{\sim}$ is clearly isomorphic to a quadric cone in $\mathbb{C}^3$, we have an embedding:

$$\iota : Sym^2\mathbb{C}^2 \hookrightarrow \mathbb{C}^2 \times \mathbb{C}^3 \cong \mathbb{C}^5$$

induced by the map: $\mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^5$ given by:

$$(s_1, s_2; r_1, r_2) \to (s_1, s_2, r_1^2, r_2^2, r_1 r_2)$$

In a neighborhood of the cusp of the zero fiber the family $\pi : X \to U$ is given in $\mathbb{C}^2 \times U$ with coordinates (x_1, x_2, t) by the equation $x_2^2 - x_1^3 = t$ and π is given by the projection to the t coordinate; the equations of the symmetric product of this family in $\mathbb{C}^2 \times \mathbb{C}^2 \times U$ are $x_2^2 - x_1^3 = t$, $y_2^2 - y_1^3 = t$. Then the image under ι of this family is a local description of Y in a neighborhood of the point $2P$. Recalling the expressions for x_i, y_i in terms of s_i, r_i we obtain the following equations:

$$\text{(1)} \qquad \begin{cases} (s_2 + r_2)^2 - (s_1 + r_1)^3 = t \\ (s_2 - r_2)^2 - (s_1 - r_1)^3 = t \end{cases}$$

The coordinates in $\mathbb{C}^5$ are:

$$z_1 = s_1, \quad z_2 = s_2, \quad z_3 = r_1^2, \quad z_4 = r_2^2, \quad z_5 = r_1 r_2.$$

Summing the two equations in (1) we have $2s_2^2 + 2r_2^2 - 2s_1^3 - 6s_1 r_1^2 = 2t$; subtracting them after multiplication by r_1 we have $2s_2(r_1 r_2) - 3s_1^2 r_1^2 - r_1^4 = 0$; subtracting them after multiplication by r_2 we have $2s_2 r_2^2 - 3s_1^2 r_1 r_2 - r_1^2(r_1 r_2) = 0$. Hence the equations defining Y in $\mathbb{C}^5 \times U$ are the following:

$$\text{(2)} \qquad \begin{cases} z_5^2 - z_3 z_4 = 0 \\ z_2^2 + z_4 - z_1^3 - 3z_1 z_3 = t \\ 2z_2 z_4 - 3z_1^2 z_5 - z_3 z_5 = 0 \\ 2z_2 z_5 - 3z_1^2 z_3 - z_3^2 = 0 \end{cases}$$

It is easy to verify that these equations define a fibration of surfaces over the disk that we denote by $p : Y \to U$, and that the total space Y is singular only at the origin. For $t \neq 0$ the fiber Y_t is a smooth surface, whereas for $t = 0$ the fiber Y_0 is a singular surface isomorphic to the second symmetric product $X_0^{(2)}$. In the paper of Ciliberto and Kouvidakis the central fiber was reducible with three irreducible components, since they studied the degeneration to a nodal curve and the local equation of the node is $xy = t$; in our case the central fiber is an irreducible surface singular along the cuspidal curve of the symmetric product. It is easy to find equations in $\mathbb{C}^5$ for the singular curve of Y_0; in fact it is the image in $Sym^2\mathbb{C}^2$ of the curve in $\mathbb{C}^2 \times \mathbb{C}^2$ defined parametrically by:

$$\{(0, 0; u^2, u^3) | u \in \mathbb{C}\} \cup \{(u^2, u^3; 0, 0) | u \in \mathbb{C}\}.$$

In coordinates s_1, s_2, r_1, r_2 this becomes:

$$s_1 = r_1 = \frac{u^2}{2} \quad s_2 = r_2 = \frac{u^3}{2}$$

so that the parametric equations for the singular curve of Y_0 in $\mathbb{C}^5$ are:

$$\{(\frac{u^2}{2}, \frac{u^3}{2}, \frac{u^4}{4}, \frac{u^6}{4}, \frac{u^5}{4}) \quad | \, u \in \mathbb{C}\}$$

This gives the following equations:

$$(3) \qquad \begin{cases} z_1^2 - z_3 = 0 \\ z_2^2 - z_4 = 0 \\ z_1 z_2 - z_5 = 0 \\ 2z_1 z_3 - z_4 = 0 \end{cases}$$

Now we consider the diagonal $\mathcal{D}$ of the fibered symmetric product, which passes through the point $2P$ of the surface S; it is quite easy to write equations for $\mathcal{D}$; in fact, if $(x_1, x_2) + (y_1, y_2)$ is a point of $Sym^2\mathbb{C}^2$, it belongs to $\mathcal{D}$ if and only if $x_1 = y_1$, $x_2 = y_2$, so that $s_1 + r_1 = s_1 - r_1$, $s_2 + r_2 = s_2 - r_2$ and this implies $r_1 = r_2 = 0$. Then we have $z_3 = z_4 = z_5 = 0$, and substituting in the equations (2) it follows that the diagonal $\mathcal{D}$ of Y is expressed by the equations:

$$(4) \qquad \begin{cases} z_3 = z_4 = z_5 = 0 \\ z_2^2 - z_1^3 = t \end{cases}$$

Hence the intersection Δ_0 of $\mathcal{D}$ with the central fiber is given by:

$$(5) \qquad \begin{cases} z_3 = z_4 = z_5 = 0 \\ z_2^2 - z_1^3 = 0 \end{cases}$$

which is a copy of the curve X_0 in the (z_1, z_2) -plane. Now we blow up the origin in $\mathbb{C}^5 \times U$; the exceptional divisor E is obviously a $\mathbb{P}^5$. Let $\lambda_1, \dots, \lambda_5, \lambda$ be the homogeneous coordinates in the exceptional divisor; then by (2) it follows that the intersection of E with $\tilde{Y}$ is given by the system:

$$(6) \qquad \begin{cases} \lambda_5^2 - \lambda_3 \lambda_4 = 0 \\ 2\lambda_2 \lambda_5 - \lambda_3^2 = 0 \\ 2\lambda_2 \lambda_4 - \lambda_3 \lambda_5 = 0 \\ \lambda_4 - \lambda = 0 \end{cases}$$

These are clearly the equations of the cone Θ with vertex $v \cong [1:0:0:0:0:0]$ over the rational normal cubic curve defined by the equations (6) in $\mathbb{P}^4$. Let $\tilde{S}$, $\tilde{\mathcal{D}}$, $\tilde{\Delta}_0$, $\tilde{X}_P$ be the strict transforms of S, $\mathcal{D}$, Δ_0, X_P respectively. It is a straightforward computation to show that $\tilde{S} \cap \Theta = \tilde{\mathcal{D}} \cap \Theta = l$ where l is the line $\lambda = \lambda_3 = \lambda_4 = \lambda_5 = 0$; moreover $\tilde{\Delta}_0$ and $\tilde{X}_P$ intersect the cone Θ at the vertex $v \cong [1:0:0:0:0:0]$ and they are tangent at v to the line l. Unfortunately the 3-fold $\tilde{Y}$ is still singular, and we must blow up at the point v once more; in this way we obtain a fibration $\pi : \tilde{\tilde{Y}} \to U$ with smooth total space. Let $\tilde{\tilde{S}}$ be the strict transform of $\tilde{S}$, and let $\tilde{\psi} : \tilde{S} \longrightarrow \mathbb{P}^2$, $\tilde{\tilde{\psi}} : \tilde{\tilde{S}} \longrightarrow \mathbb{P}^2$ be the birational maps induced by ψ; moreover, let $\rho : W \to \tilde{\tilde{S}}$ be the normalization of $\tilde{\tilde{S}}$, obtained unfolding along the cuspidal curve $\tilde{\tilde{X}}_P$ of $\tilde{\tilde{S}}$. We will denote by X_W the preimage of this curve in W. It is clear that the birational map ψ lifts to a birational morphism $\phi : W \to \mathbb{P}^2$.

Lemma 1. *The only indeterminacy of the map ϕ^{-1} is the point $2P$. In other words the map ϕ is the blow-up of $\mathbb{P}^2$ at the points $2P, \xi_1, \ldots, \xi_k$ where the ξ_i's are points infinitely near to $2P$.*

Proof. The map $\psi : S \longrightarrow \mathbb{P}^2$ is bijective and bicontinuous, but it is not regular along the cuspidal curve of S. The surface $\tilde{S}$ is the blow-up of S at the point $2P \in S$, and the induced map $\tilde{\psi} : \tilde{S} \longrightarrow \mathbb{P}^2$ contracts only the curve l (at the point $2P$). Since $\tilde{\tilde{S}}$ is the blow-up of $\tilde{S}$ at the point $v \in l$ and W is the surface obtained unfolding along the double curve of $\tilde{\tilde{S}}$, it is clear that all the exceptional curves of ϕ are contracted to the point $2P$, and this completes the proof. $\square$

Corollary 1. *Let G_1, G_2 be two curves on W which are the preimages of two curves G'_1, G'_2 under the morphism ρ and which pass simply through the point $p \in X_W$. If G'_1, G'_2 both have a cusp at the point $q = \rho(p) \in \tilde{\tilde{X}}_P$, then G_1, G_2 have the same tangent at p.*

Proof. It is sufficient to show that if $p \in X_W$, there is only one direction $\mathbf{v} \in T_q W$ such that if G is a curve in W which passes simply through p with tangent direction $\mathbf{v}$, the image curve $G' = \rho(G)$ has a cusp at the point $q = \rho(p)$.

The problem being local, we can suppose that $\tilde{\tilde{S}}$ is the surface in $\mathbb{C}^3$ with coordinates (x, y, z) defined by the equation $y^2 = x^3$, and W is the blow-up of this surface along the cuspidal double curve $x = y = 0$. If $[a : b]$ are the homogeneous coordinates in the exceptional divisor, in the affine open set $b \neq 0$, which is a $\mathbb{C}^4$ with coordinates (x, y, z, a), the equations of the surface

W are the following:

$$a^2 - x = 0, \quad y = ax \tag{7}$$

The morphism ρ is given by the projection $(x, y, z, a) \to (x, y, z)$, and the curve X_W is given by the equation $a = 0$. If we choose (z, a) as local parameters for the surface W, we have $x = a^2, y = a^3$ and the differential of the map ρ at the point (x, y, z, a) is given by the matrix:

$$d\rho_{(x,y,z,a)} = \begin{pmatrix} 0 & 2a \\ 0 & 3a^2 \\ 1 & 0 \end{pmatrix} \tag{8}$$

This shows that $d\rho$ has rank 1 at the points of the curve X_W, and we are done. □

3. Limits of line bundles.

Now we want describe the limits of line bundles and sections under the cuspidal Franchetta degeneration, in the case of positive valence. The problem is that the 3-fold $\tilde{Y}$ is not smooth, so a priori it is not possible to extend a line bundle on $\tilde{Y} - \tilde{Y}_0$ to a line bundle on $\tilde{Y}$. A possible approach is blowing up the singular point v: in this way (we omit the straightforward computation since it is not necessary for understand what follows) we obtain a smooth 3-fold $\tilde{\tilde{Y}}$ and then we can work with the techniques of [3]; the problem is that after the second blow-up the central fiber of $\tilde{\tilde{Y}}$ contains three irreducible components which do not intersect transversally, and the computation that we need in order to find the limit becomes rather complicated. Instead, we will use the following approach. Let C be a rational nodal curve with g nodes; then it is possible to find a degeneration of this curve to a curve with g cusps. If for simplicity we suppose $g = 1$, then such a degeneration locally will be of type:

$$y^2 = x^3 + \alpha x^2 \tag{9}$$

This is a family over U_α, where U_α is the disk with coordinate α, such that the general fiber C_α, $\alpha \neq 0$, is a rational curve with a node, and the central fiber C_0 is a rational curve with a cusp. Then we would use a degeneration of type (9) to describe the limit of a bundle of type $\mathcal{L}_{n,\gamma}$ in the cuspidal case as a degeneration of limits in the nodal case. For this, we need a family of surfaces over the disk U such that the general fiber is a family of smooth curves which degenerate to a nodal curve, whereas the central fiber is a family of smooth curves which degenerate to a cuspidal curve. This is very simple to obtain: for α fixed, we consider the symmetric product of the family $y^2 = x^3 + \alpha x^2 + t$; exactly as in section 2, it is given by the following equations:

$$\left\{\begin{array}{l} z_5^2 - z_3 z_4 = 0 \\ 2z_2 z_5 - 3z_1^2 z_3 - z_3^2 - \alpha^2 z_1 z_3 = 0 \\ 2z_2 z_4 - 3z_1^2 z_5 - z_3 z_5 - \alpha^2 z_1 z_5 = 0 \\ z_2^2 + z_4 - z_1^3 - 3z_1 z_3 - \alpha^2 (z_1 + z_3) = t \end{array}\right. \tag{10}$$

Obviously, these are equations for a family $\rho : \mathcal{Y} \to U_\alpha$ which has the desired properties. We will denote by $Y_t(\alpha)$ the fiber over α. The total space $\mathcal{Y}$ is singular along the line m in $\mathbb{C}^5 \times U_t \times U_\alpha$ given by the equations $z_1 = z_2 = z_3 = z_4 = z_5 = t = 0$. We remark that each fiber $Y_t(\alpha)$ is a 3-fold with only one singularity at the point $(0, 0, 0, 0, 0, 0, \alpha)$, so that m is exactly the locus of the singular points of the fibers of ρ. Now we blow up the line m; the resulting strict transform of the total space $\tilde{\mathcal{Y}}$ is singular only at a point of the central fiber, that we blow up to obtain a smooth 4-fold. We will denote it again by $\tilde{\mathcal{Y}}$, and use the following notations:

- $\mathcal{L} = \mathcal{L}_{n,\gamma}$ is a line bundle on $C^{(2)}$;
- $\overline{\mathcal{L}}(\alpha)$ is the extension to $\tilde{Y}_t(\alpha)$, $\alpha \neq 0$, of a line bundle which restricts to $(n+\gamma)x - \gamma(\delta/2)$ on every smooth fiber of $\tilde{Y}_t(\alpha)$ (this exists because $\tilde{Y}_t(\alpha)$ is a smooth surface);
- $\mathcal{C}_{\mathbb{P}^2}(\alpha)$ is the linear system on $\mathbb{P}^2$ induced by $\overline{\mathcal{L}}(\alpha)$ (see fig. 1).

Clearly each $\tilde{Y}_t(\alpha)$, $\alpha \neq 0$ is just the blow-up of $Y_t(\alpha)$ at its singular point, and the exceptional divisor of this blow-up is exactly the restriction to $\tilde{Y}_t(\alpha)$ of the exceptional divisor of $\tilde{\mathcal{Y}}$. Then we can "glue together" the line bundles $\overline{\mathcal{L}}(\alpha)$, $\alpha \neq 0$, finding a linear bundle $\mathcal{L}_{\tilde{\mathcal{Y}}}$ on $\tilde{\mathcal{Y}} - \tilde{\mathcal{Y}}_0$ which restricts to $\overline{\mathcal{L}}(\alpha)$ over each $\tilde{Y}_t(\alpha)$, $\alpha \neq 0$. Since $\tilde{\mathcal{Y}}$ is a smooth 4-fold, the line bundle $\mathcal{L}_{\tilde{\mathcal{Y}}}$ extends to a line bundle $\overline{\mathcal{L}}_{\tilde{\mathcal{Y}}}$ on all of $\tilde{\mathcal{Y}}$; then we are interested to describe the restriction of $\overline{\mathcal{L}}_{\tilde{\mathcal{Y}}}$ to the central fiber of $Y_0(t)$; we will denote this restriction by $\mathcal{L}^0_{cusp}$. If we consider $\tilde{\mathcal{Y}}$ as a family over $U_\alpha \times U_t$, restricting $\overline{\mathcal{L}}_{\tilde{\mathcal{Y}}}$ to $\{t = 0\}$ it is clear that $\mathcal{L}^0_{cusp}$ induces a linear system $\mathcal{C}_{\mathbb{P}^2}(0)$ on $\mathbb{P}^2$. This linear system $\mathcal{C}_{\mathbb{P}^2}(0)$ is the limit of the linear systems $\mathcal{C}_{\mathbb{P}^2}(\alpha)$ when the points $P^1(\alpha)$, $P^2(\alpha)$ corresponding to the node of $C_\alpha(\alpha \neq 0)$ approach the point P along Γ (recall that the point P is such that $2P$ corresponds to cusp of C_0). Then, by [1], it follows that the curves of $\mathcal{C}_{\mathbb{P}^2}(0)$ must have multiplicity γ at P.

In the nodal case the limit curves of the linear system induced on $\mathbb{P}^2$ must satisfy the additional property $(*)$. We want to understand the analogue of this property in the cuspidal case. Let us suppose that a curve A of $\mathcal{C}_{\mathbb{P}^2}(0)$ intersects the line X_P at the point $P+Q$. It is not difficult to see that the preimages on $\tilde{\tilde{S}}$ of A and X_Q have both a cusp at the point $\tilde{\tilde{\psi}}^{-1}(P+Q)$. Then Corollary 1 implies that the preimages of these curves in W have the same tangent line at the point $\rho^{-1}(P+Q)$. By Lemma 1 this means that X_Q is the tangent to the curve A at the point $P+Q$.

The generalization to the case with g cusps is obvious and we have therefore proved our theorem.

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