LE MATEMATICHE
Vol. LXXVII (2022) – Issue II, pp. 229–263
doi: 10.4418/2022.77.2.1

HERZ-TYPE SOBOLEV SPACES ON DOMAINS

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We introduce Herz-type Sobolev spaces on domains, which unify and generalize the classical Sobolev spaces. We will give a proof of the Sobolev-type embedding for these function spaces. All these results generalize the classical results on Sobolev spaces. Moreover, some remarks on Caffarelli–Kohn–Nirenberg inequality are given.

1. Introduction

Function spaces have been widely used in various areas of analysis such as harmonic analysis and partial differential equations. Some example of these spaces can be mentioned such as Sobolev spaces. The interest in these spaces comes not only from theoretical reasons but also from their applications in mathematical analysis. We refer to the monographs [1], [2], [4] and [16] for further details, historical remarks and references on Sobolev spaces.

It is well known that Herz spaces play an important role in harmonic analysis. After they have been introduced in [17], the theory of these spaces had a remarkable development in part due to its usefulness in applications. For instance, they appear in the characterization of multipliers on Hardy spaces [3],

Received on January 1, 2022

AMS 2010 Subject Classification: 46B70, 46E35
Keywords: Herz spaces, Sobolev spaces, Caffarelli–Kohn–Nirenberg inequality, Embeddings
This work was supported by the General Direction of Higher Education and Training under Grant No. C00L03UN280120220004 and by The General Directorate of Scientific Research and Technological Development, Algeria.

in the summability of Fourier transforms [8], in regularity theory for elliptic equations in divergence form [20]. Also [21], studied the Cauchy problem for Navier-Stokes equations on Herz spaces and weak Herz spaces. Recently, Herz spaces appear in the study of semilinear parabolic equations [7] and summability of Fourier transforms on mixed-norm Lebesgue spaces [9]. For important and latest results on Herz spaces, we refer the reader to the papers [19], [23] and to the monograph [11].

Based on Sobolev and Herz spaces we present a class of function spaces, called Herz-type Sobolev spaces, which generalize the classical Sobolev spaces. These type of function spaces, but over $\mathbb{R}^n$, are introduced by Lu and Yang [14] were gave some applications to partial differential equations.

In this paper our spaces defined over a domain. More precisely the domain is often assumed to satisfy a cone condition.

The paper is organized as follows. First we give some preliminaries where we fix some notation and recall some basics facts on Herz spaces, where the approximation by smooth functions is given. In particular, we will prove the Herz type version of Caffarelli–Kohn–Nirenberg-type inequalities.

In Section 3, we present basics facts on Herz-type Sobolev spaces in analogy to the classical Sobolev spaces and we prove a Sobolev embedding theorem for these spaces. In particular we prove that

$$\dot{K}^{\alpha_2,r}_{p,m}(\Omega) \hookrightarrow \dot{K}^{\alpha_1,r}_{q}(\Omega) \tag{1}$$

with some appropriate assumptions on the parameters. The surprise here is that the embedding (1) is true if $1 < q < p < \infty, \alpha_2 + \frac{n}{p} \geq \alpha_1 + \frac{n}{q} > 0$ and

$$\max\left(\frac{n}{p}, \frac{n}{p} + \alpha_2, \frac{n}{p} - \frac{n}{q} + \alpha_2 - \alpha_1\right) < m < n.$$

The proof based on a local estimate and on the boundedness of maximal function and Riesz potential operator on Herz spaces. Other properties of these function spaces such interpolation inequalities, extension and compact embeddings are postponed to the future work.

2. Herz spaces

As usual, $\mathbb{R}^n$ denotes the n-dimensional real Euclidean space, $\mathbb{N}$ the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The letter $\mathbb{Z}$ stands for the set of all integer numbers. For any $u > 0$, $k \in \mathbb{Z}$ we set $R(u) = \{x \in \mathbb{R}^n : \frac{u}{2} \leq |x| < u\}$ and $R_k = R(2^k)$. For $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B(x,r)$ the open ball in $\mathbb{R}^n$ with center x and radius r. Let χ_k, for $k \in \mathbb{Z}$, denote the characteristic function of the

set R_k. If $1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, then p' is called the conjugate exponent of p.
We denote by $|\Omega|$ the n-dimensional Lebesgue measure of $\Omega \subseteq \mathbb{R}^n$. For any measurable subset $\Omega \subseteq \mathbb{R}^n$ the Lebesgue space $L^p(\Omega)$, $0 < p \le \infty$ consists of all measurable functions for which

$$\|f\|_{L^p(\Omega)} = \Big(\int_\Omega |f(x)|^p \, dx \Big)^{1/p} < \infty, \quad 0 < p < \infty$$

and

$$\|f\|_{L^\infty(\Omega)} = \operatorname*{ess\text{-}sup}_{x \in \Omega} |f(x)| < \infty.$$

If $\Omega = \mathbb{R}^n$ then we put $\|f\|_{L^p(\mathbb{R}^n)} = \|f\|_p$.

Let $\Omega \subseteq \mathbb{R}^n$ be open. For any nonnegative integer m let $C^m(\Omega)$ be the vector space consisting of all functions f, which, together with all their partial derivatives $D^\beta f$ of orders $|\beta| \le m$, are continuous on Ω. We put $C^0(\Omega) = C(\Omega)$ and $C^\infty(\Omega) = \cap_{m \ge 0} C^m(\Omega)$. We denote by $C_c(\Omega)$ the set of all functions in $C(\Omega)$ which have compact support in Ω.

In this section we present some fundamental properties of Herz spaces. We start by recalling the definition and some of the properties of the homogenous Herz spaces.

Definition 2.1. Let $\alpha \in \mathbb{R}$ and $1 \le p, q \le \infty$. The homogeneous Herz space $\dot{K}_p^{\alpha,q}(\mathbb{R}^n)$ is defined as the set of all $f \in L^p_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ such that

$$\|f\|_{\dot{K}_p^{\alpha,q}(\mathbb{R}^n)} = \Big(\sum_{k=-\infty}^{\infty} 2^{k\alpha q} \|f\chi_k\|_p^q \Big)^{1/q} < \infty$$

with the usual modifications when $p = \infty$ and/or $q = \infty$.

The spaces $\dot{K}_p^{\alpha,q}(\mathbb{R}^n)$ are Banach spaces. If $\alpha = 0$ and $1 \le p = q \le \infty$ then $\dot{K}_p^{0,p}(\mathbb{R}^n)$ coincides with the Lebesgue spaces $L^p(\mathbb{R}^n)$. If $1 \le q_1 \le q_2 \le \infty$, then we may derive the embedding $\dot{K}_p^{\alpha,q_1}(\mathbb{R}^n) \hookrightarrow \dot{K}_p^{\alpha,q_2}(\mathbb{R}^n)$. In addition

$$\dot{K}_p^{\alpha,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n, |\cdot|^{\alpha p}), \quad \text{(Lebesgue space equipped with power weight)},$$

where

$$\|f\|_{L^p(\mathbb{R}^n, |\cdot|^{\alpha p})} = \Big(\int_{\mathbb{R}^n} |f(x)|^p \, |x|^{\alpha p} dx \Big)^{1/p}.$$

If $\Omega \subset \mathbb{R}^n$ is open and $f : \Omega \to \mathbb{R}$ a measurable function, then we write $f \in \dot{K}_p^{\alpha,q}(\Omega)$ if $f\chi_\Omega \in \dot{K}_p^{\alpha,q}(\mathbb{R}^n)$ and we put $\|f\|_{\dot{K}_p^{\alpha,q}(\Omega)} = \|f\chi_\Omega\|_{\dot{K}_p^{\alpha,q}(\mathbb{R}^n)}$.

Various important results have been proved in the space $\dot{K}_p^{\alpha,q}(\mathbb{R}^n)$ under some assumptions on α, p and q. The conditions $-\frac{n}{p} < \alpha < n(1 - \frac{1}{p})$, $1 < p < \infty$

and $1 \leq q \leq \infty$ is crucial in the study of the boundedness of classical operators in $\dot{K}_p^{\alpha,q}(\mathbb{R}^n)$ spaces. This fact was first realized by Li and Yang [10] with the proof of the boundedness of the maximal function. As usual, we put

$$\mathcal{M}(f)(x) = \sup_Q \frac{1}{|Q|} \int_Q |f(y)|\, dy, \quad f \in L^1_{\mathrm{loc}}(\mathbb{R}^n), x \in \mathbb{R}^n,$$

where the supremum is taken over all cubes with sides parallel to the axis and $x \in Q$. Also we set

$$\mathcal{M}_t(f) = \big(\mathcal{M}(|f|^t)\big)^{\frac{1}{t}}, \quad 0 < t < \infty.$$

Lemma 2.2. *Let $1 < p < \infty$ and $1 \leq q \leq \infty$. If f is a locally integrable functions on $\mathbb{R}^n$ and $-\frac{n}{p} < \alpha < n(1-\frac{1}{p})$, then*

$$\big\|\mathcal{M}(f)\big\|_{\dot{K}_p^{\alpha,q}(\mathbb{R}^n)} \leq c\big\|f\big\|_{\dot{K}_p^{\alpha,q}(\mathbb{R}^n)}.$$

A detailed discussion of the properties of these spaces my be found in the recent monograph [22], the papers [13], [15], [18], and references therein.

The next lemma is a Hardy-type inequality which is basically a consequence of Young's inequality in the sequence Lebesgue space ℓ^q.

Lemma 2.3. *Let $0 < a < 1$ and $0 < q \leq \infty$. Let $\{\varepsilon_k\}_{k\in\mathbb{Z}}$ be a sequences of positive real numbers and denote $\delta_k = \sum_{j=k}^{\infty} a^{j-k}\varepsilon_j$, $k \in \mathbb{Z}$. Then there exists a constant $c > 0$ depending only on a and q such that*

$$\big\|\{\delta_k\}_{k\in\mathbb{Z}}\big\|_{\ell^q} \leq c\big\|\{\varepsilon_k\}_{k\in\mathbb{Z}}\big\|_{\ell^q}.$$

Let $V_{\alpha,p,q}$ be the set of $(\alpha,p,q) \in \mathbb{R}\times[1,\infty]^2$ such that:
• $\alpha < n - \frac{n}{p}$, $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$,
• $\alpha = n - \frac{n}{p}$, $1 \leq p \leq \infty$ and $q = 1$,

The next lemma gives a necessary and sufficient condition on the parameters α, p and q, in order to make sure that

$$\langle T_f, \varphi\rangle = \int_\Omega f(x)\varphi(x)dx, \quad \varphi \in \mathcal{D}(\Omega), f \in \dot{K}_p^{\alpha,q}(\Omega)$$

generates a regular distribution $T_f \in \mathcal{D}'(\Omega)$.

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^n$ be open, $0 \in \Omega$ and $1 \leq p,q \leq \infty$. Then*

$$\dot{K}_p^{\alpha,q}(\Omega) \hookrightarrow L^1_{\mathrm{loc}}(\Omega),$$

if and only if $(\alpha,p,q) \in V_{\alpha,p,q}$.

Proof. We divide the proof into two steps.

Step 1. Assume that $(\alpha,p,q)\in V_{\alpha,p,q}$, $f\in \dot{K}_p^{\alpha,q}(\Omega)$ and $B(0,2^N)\subset\Omega, N\in\mathbb{Z}$. By similarity we only consider the first case. Hölder's inequality gives

$$\begin{aligned}\|f\|_{L^1(B(0,2^N))} &= \sum_{i=-\infty}^{N}\|f\chi_{R_i\cap\Omega}\|_1\\ &\lesssim \sum_{i=-\infty}^{N}2^{i(n-\frac{n}{p})}\|f\chi_{R_i\cap\Omega}\|_p\\ &= c2^{N(n-\frac{n}{p}-\alpha)}\sum_{i=-\infty}^{N}2^{(i-N)(n-\frac{n}{p}-\alpha)}2^{i\alpha}\|f\chi_{R_i\cap\Omega}\|_p\\ &\lesssim \|f\|_{\dot{K}_p^{\alpha,q}(\Omega)}.\end{aligned}$$

Step 2. Assume that $(\alpha,p,q)\notin V_{\alpha,p,q}$. We distinguish two cases.

Case 1. $\alpha>n-\frac{n}{p}$. Let $r>0$ be such that $B(0,r)\subset\Omega$ and set $f(x)=|x|^{-n}\chi_{0<|\cdot|<r}(x)$. We obtain $f\in\dot{K}_p^{\alpha,q}(\Omega)$ for any $1\le p,q\le\infty$ whereas $f\notin L^1_{\mathrm{loc}}(\Omega)$. Indeed, we find

$$\begin{aligned}\|f\|^q_{\dot{K}_p^{\alpha,q}(\Omega)} &= \sum_{k\in\mathbb{Z}:2^k<2r}2^{k\alpha q}\|f\chi_{R_k\cap\Omega}\|_p^q\\ &\lesssim \sum_{k\in\mathbb{Z}:2^k<2r}2^{k(\alpha-n)q}\|\chi_{R_k}\chi_{0<|\cdot|<r}\|_p^q\\ &\lesssim \sum_{k\in\mathbb{Z}:2^k<2r}2^{k(\alpha-n+\frac{n}{p})q}\\ &<\infty,\end{aligned}$$

with the usual modification if $p=\infty$ and/or $q=\infty$. Obviously, $f\notin L^1_{\mathrm{loc}}(\Omega)$.

Case 2. $\alpha=n-\frac{n}{p}$, $1\le p\le\infty$ and $1<q\le\infty$. By similarity we can assume that $B(0,\frac12)\subset\Omega$. We consider the function f defined by

$$f(x)=|x|^{-n}(|\log|x||)^{-1}\chi_{0<|\cdot|<\frac12}(x).$$

An easy computation yields that

$$\|f\|^q_{\dot{K}_p^{\alpha,q}(\Omega)}\lesssim\sum_{k=1}^{\infty}k^{-q}<\infty,$$

which gives that $f\in\dot{K}_p^{n-\frac{n}{p},q}(\Omega)$, with the usual modifications when $q=\infty$. It is easily seen that f does not belong to $L^1_{\mathrm{loc}}(\Omega)$. □

Remark 2.5. We easily see that in general if $0\notin\Omega$ then the set $V_{\alpha,p,q}$ is not optimal. From this lemma it thus makes sense to talk about weak derivatives of functions in $\dot{K}_p^{\alpha,q}(\Omega)$, in addition the assumption $(\alpha,p,q)\in V_{\alpha,p,q}$ is optimal.

Theorem 2.6. *Let $\Omega \subset \mathbb{R}^n$ be open and $0 \in \Omega$, $1 < p < \infty, 1 \leq q < \infty$ and $\alpha > -\frac{n}{p}$. Then $C_c(\Omega)$ is dense in $\dot{K}_p^{\alpha,q}(\Omega)$.*

Proof. First observe that $C_c(\Omega) \subset \dot{K}_p^{\alpha,q}(\Omega)$ if and only if $\alpha > -\frac{n}{p}$. Indeed, let $\varphi \in C_c(\Omega)$ be such that $\varphi(x) = 1, x \in B(0, 2^N) \subset \Omega, N \in \mathbb{Z}$. We have

$$\begin{aligned} \|\varphi\|^q_{\dot{K}_p^{\alpha,q}(\Omega)} &= \sum_{k=-\infty}^{\infty} 2^{k\alpha q} \|\varphi \chi_{\Omega \cap R_k}\|_p^q \\ &\geq \sum_{k=-\infty}^{N} 2^{k\alpha q} \|\chi_{B(0,2^N)\cap R_k}\|_p^q \\ &= \sum_{k=-\infty}^{N} 2^{k\alpha q} \|\chi_{R_k}\|_p^q \\ &= c \sum_{k=-\infty}^{N} 2^{k(\alpha + \frac{n}{p})q} \end{aligned}$$

and this series is divergent if $\alpha \leq -\frac{n}{p}$. It is clear that $C_c(\Omega) \subset \dot{K}_p^{\alpha,q}(\Omega)$ whenever $\alpha > -\frac{n}{p}$. Let $\dot{K}_{p,c}^{\alpha;q}(\Omega)$ be the set of all $g \in \dot{K}_p^{\alpha,q}(\Omega)$ such that $g = 0$ outside a compact. As in [24, Proposition 3.1] we obtain that $\dot{K}_{p,c}^{\alpha;q}(\Omega)$ is dense in $\dot{K}_p^{\alpha,q}(\Omega)$. Therefore we prove the density of $C_c(\Omega)$ in $\dot{K}_{p,c}^{\alpha,q}(\Omega)$. Let $f \in \dot{K}_{p,c}^{\alpha,q}(\Omega)$ with $f(x) = 0$ if $x \notin A \subset \Omega$ compact. As in [2, Theorem 2.19], the proof can be restricted to the case f is real-valued and nonnegative. Since f is measurable, there exists a monotonically increasing sequence $\{u_i\}_{i\in\mathbb{N}_0}$ of nonnegative simple functions converging pointwise to f on Ω and

$$0 \leq u_i \leq f, \quad i \in \mathbb{N}_0.$$

Since

$$0 \leq f - u_i \leq f, \quad i \in \mathbb{N}_0,$$

by dominated convergence theorem $\{u_i\}_{i\in\mathbb{N}_0}$ converge to f in $\dot{K}_p^{\alpha,q}(f)$. Therefore we find an $u \in \{u_i\}_{i\in\mathbb{N}_0}$ such that

$$\|f - u\|_{\dot{K}_p^{\alpha,q}(\Omega)} < \frac{\varepsilon}{2}.$$

Since $0 \leq u \leq f$, $\mathrm{supp} u \subset A$. Let $\theta > 0$ be such that $\max(0, \frac{-\alpha p}{n}) < \theta < 1$. Assume that $A \subset V \subset \bar{V} \subset \Omega$ with $\bar{V}$ compact. We set

$$E = \sum_{k\in\mathbb{Z}: R_k \cap \bar{V} \neq \emptyset} 2^{k\alpha q} |R_k|^{\frac{\theta q}{p}}.$$

By Lusin's theorem we can find that $\varphi \in C_c(\Omega)$ such that

$$|\varphi(x)| \leq \|u\|_\infty$$

for any $x \in \Omega$, supp$\varphi \subset \bar{V}$ and

$$|H| \leq \Big(\frac{\varepsilon}{4\|u\|_\infty E^{\frac{1}{q}}}\Big)^{\frac{p}{1-\theta}},$$

where $H = \{x \in \Omega : \varphi(x) \neq u(x)\}$. We set $B = \{x \in \bar{V} : \varphi(x) \neq u(x)\}$. Observe that $H = B$. We have

$$\begin{aligned}
\|u-\varphi\|^q_{\dot{K}^{\alpha,q}_p(\Omega)} &= \sum_{k=-\infty}^{\infty} 2^{k\alpha q}\|(u-\varphi)\chi_{R_k\cap\Omega}\|^q_p \\
&= \sum_{k=-\infty}^{\infty} 2^{k\alpha q}\|(u-\varphi)\chi_{R_k\cap H}\|^q_p \\
&= \sum_{k=-\infty}^{\infty} 2^{k\alpha q}\|(u-\varphi)\chi_{R_k\cap B}\|^q_p.
\end{aligned}$$

Therefore

$$\|u-\varphi\|^q_{\dot{K}^{\alpha,q}_p(\Omega)} = \sum_{k\in\mathbb{Z}:R_k\cap B\neq\emptyset} 2^{k\alpha q}\|(u-\varphi)\chi_{R_k\cap B}\|^q_p.$$

Let $k \in \mathbb{Z}$ be such that $R_k \cap B \neq \emptyset$. Then

$$\begin{aligned}
\|(u-\varphi)\chi_{R_k\cap B}\|_p &\leq 2\|u\|_\infty\|\chi_{R_k\cap B}\|_p \\
&= 2\|u\|_\infty\|\chi_{R_k\cap B}\|^{1-\theta}_p\|\chi_{R_k\cap B}\|^\theta_p \\
&\leq 2\|u\|_\infty\|\chi_{R_k}\|^\theta_p\|\chi_B\|^{1-\theta}_p \\
&\leq 2|R_k|^{\frac{\theta}{p}}\|u\|_\infty|B|^{\frac{1-\theta}{p}}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\|u-\varphi\|^q_{\dot{K}^{\alpha,q}_p(\Omega)} &\leq 2^q\, E\|u\|^q_\infty|B|^{\frac{1-\theta}{p}q} \\
&< (\frac{\varepsilon}{2})^q, \quad \varepsilon > 0
\end{aligned}$$

and that ends the proof. □

Theorem 2.7. *Let Ω be open, $1 \leq p < \infty$, $1 \leq q < \infty$ and $\alpha > -\frac{n}{p}$. Then $\dot{K}^{\alpha,q}_p(\Omega)$ is separable.*

Proof. As in [12, Lemma 2.17] it suffices to prove the theorem for $\Omega = \mathbb{R}^n$. For $j \in \mathbb{N}$ and $m = (m_1, ..., m_n) \in \mathbb{Z}^n$ let

$$Q_{j,m} = \left\{ x \in \mathbb{R}^n : 2^{-j} m_i \leq x < 2^{-j}(m_i + 1), i = 1, ..., n \right\}$$

be the dyadic cube. Put

$$F_j = \left\{ f : f = \sum_{m \in \mathbb{Z}^n} a_{j,m} \chi_{Q_{j,m}}, a_{j,m} \in \mathbb{Q} \right\}, \quad j \in \mathbb{N},$$

where $a_{j,m} = 0$ if $|m| \geq N, N \in \mathbb{N}$. We have $F = \cup_{j \in \mathbb{N}} F_j$, is a countable set. Let $f \in \dot{K}_p^{\alpha,q}(\mathbb{R}^n)$ and $\varepsilon > 0$. From Theorem 2.6 there exists $\varphi \in C_c(\mathbb{R}^n)$ such that

$$\left\| f - \varphi \right\|_{\dot{K}_p^{\alpha,q}(\mathbb{R}^n)} \leq \frac{\varepsilon}{2}.$$

Assume that supp$\varphi \subset Q_{-J,z}$, $J \in \mathbb{N}, z \in \mathbb{Z}^n$ with J large enough. Let $j \in \mathbb{N}, m \in \mathbb{Z}^n$ and

$$\varphi_{j,m}(x) = \begin{cases} 2^{-jn} \int_{Q_{j,m}} \varphi(y) dy, & \text{if} \quad x \in Q_{j,m} \subseteq Q_{-J,z}, \\ 0, & \text{if} \quad x \in Q_{j,m} \nsubseteq Q_{-J,z} \text{ or } x \notin Q_{j,m}. \end{cases}$$

Observe that

$$\left\| \varphi - \varphi_{j,m} \right\|_{\dot{K}_p^{\alpha,q}(\mathbb{R}^n)}^q = \sum_{k=-\infty}^{\infty} 2^{k\alpha q} \left\| (\varphi - \varphi_{j,m}) \chi_k \right\|_p^q.$$

But

$$\left\| (\varphi - \varphi_{j,m}) \chi_k \right\|_p^p = \int_{Q_{-J,z}} |\varphi(x) - \varphi_{j,m}(x)|^p \chi_k(x) dx$$

for any $j \in \mathbb{N}$ and any $m \in \mathbb{Z}^n$. Since φ is uniformly continuous on $Q_{-J,z}$, for each $\varepsilon' > 0$ there is a $\delta > 0$ such that

$$|\varphi(x) - \varphi(y)| < (\varepsilon')^p$$

whenever $|x - y| < \delta$. Let $x \in Q_{-J,z}$. We can find a dyadic cube Q_{j,m_1} such that $x \in Q_{j,m_1} \subseteq Q_{-J,z}$ for any $j \in \mathbb{N}$. We have

$$|\varphi(x) - \varphi_{j,m_1}(x)| \leq 2^{-jn} \int_{Q_{j,m_1}} |\varphi(x) - \varphi(y)| dy, \quad x \in Q_{j,m_1} \subseteq Q_{-J,z}$$

for any $j \in \mathbb{N}$. Taking j large enough be such that $|x - y| \leq \sqrt{n} 2^{-j} \leq \delta$, $x, y \in Q_{j,m_1}$. Let j_1 one of them. Therefore

$$|\varphi(x) - \varphi_{j_1,m_1}(x)| < (\varepsilon')^p, \quad x \in Q_{j_1,m_1} \subseteq Q_{-J,z}.$$

Hence

$$\begin{aligned}\|\varphi-\varphi_{j_1,m_1}\|^q_{\dot K_p^{\alpha,q}(\mathbb{R}^n)} &= \sum_{2^k\lesssim(1+|z|)2^J} 2^{k\alpha q}\|(\varphi-\varphi_{j_1,m_1})\chi_{R_k\cap Q_{-J,z}}\|_p^q \\ &\le \sum_{2^k\lesssim(1+|z|)2^J} 2^{k(\alpha+\frac{n}{p})q}\sup_{x\in Q_{j_1,m_1}}|\varphi(x)-\varphi_{j_1,m_1}(x)|^{\frac{q}{p}} \\ &\le c(\varepsilon')^q((1+|z|)2^J)^{(\alpha+\frac{n}{p})q},\end{aligned}$$

with the help of the fact that $\alpha>-\frac{n}{p}$. Since $\varphi_{j_1,m_1}(x)\in\mathbb{R}$ we can find that $\tilde\varphi_{j_1,m_1}(x)\in\mathbb{Q}$ be such that

$$|\varphi_{j_1,m_1}(x)-\tilde\varphi_{j_1,m_1}(x)|<\varepsilon',\quad x\in Q_{j_1,m_1}\subseteq Q_{-J,z}.$$

Now

$$\begin{aligned}\|\varphi-\tilde\varphi_{j_1,m_1}\|_{\dot K_p^{\alpha,q}(\mathbb{R}^n)} &\le \|\varphi-\varphi_{j_1,m_1}\|_{\dot K_p^{\alpha,q}(\mathbb{R}^n)}+\|\tilde\varphi_{j_1,m_1}-\varphi_{j_1,m_1}\|_{\dot K_p^{\alpha,q}(\mathbb{R}^n)} \\ &\le C\varepsilon'((1+|z|)2^J)^{(\alpha+\frac{n}{p})q}.\end{aligned}$$

We choose ε' be such that $C\varepsilon'((1+|z|)2^J)^{\alpha+\frac{n}{p}}<\frac{\varepsilon}{2}$, which yields that

$$\|f-\tilde\varphi_{j_1,m_1}\|_{\dot K_p^{\alpha,q}(\mathbb{R}^n)}\le\varepsilon.$$

This completes the proof. □

Let $J\in\mathcal{D}(\mathbb{R}^n)$ be a real-valued function such that

$$J(x)\ge 0,\quad \text{if}\quad x\in\mathbb{R}^n,\quad J(x)=0\quad\text{if}\quad |x|\ge 1\quad\text{and}\quad \int_{\mathbb{R}^n}J(x)dx=1.$$

We put $J_\varepsilon(x)=\varepsilon^{-n}J(\frac{x}{\varepsilon})$, $x\in\mathbb{R}^n$.

Theorem 2.8. *Let $\Omega\subset\mathbb{R}^n$ be open, $0\in\Omega, 1\le p<\infty$, $1\le q<\infty$ and $-\frac{n}{p}<\alpha<n-\frac{n}{p}$. Let $f\in\dot K_p^{\alpha,q}(\Omega)$ be a function defined on $\mathbb{R}^n$ and vanishes identically outside Ω. Then*

$$\lim_{\varepsilon\to0_+}\|J_\varepsilon*f-f\|_{\dot K_p^{\alpha,q}(\Omega)}=0. \tag{2}$$

Proof. We will do the proof into two steps.

Step 1. We will prove that

$$\lim_{\varepsilon\to0_+}\|J_\varepsilon*\varphi-\varphi\|_{\dot K_p^{\alpha,q}(\Omega)}=0 \tag{3}$$

for any $\varphi\in C_c(\Omega)$ and $\alpha\in\mathbb{R}$.

Substep 1.1. $\alpha > -\frac{n}{p}$. Assume that $\mathrm{supp}\varphi \subset B(0,2^N) \subset \Omega$, $N \in \mathbb{N}$. Using the fact that $|x-y| > 2^N \geq \varepsilon$ for any $x \in \mathbb{R}^n \backslash B(0,2^{N+1})$ and any $y \in B(0,2^N)$ we obtain

$$J_\varepsilon * \varphi(x) = 0, \quad x \in \mathbb{R}^n \backslash B(0,2^{N+1}), \quad \varepsilon \leq 2^N,$$

which yields

$$\left\|(J_\varepsilon * \varphi - \varphi)\chi_k\right\|_{L^p(\Omega)}^p = \int_{\Omega \cap R_k \cap B(0,2^{N+1})} \left|J_\varepsilon * \varphi(x) - \varphi(x)\right|^p dx, \quad k \in \mathbb{Z}.$$

Observe that

$$J_\varepsilon * \varphi(x) - \varphi(x) = \int_{B(0,1)} J(z)(\varphi(x-\varepsilon z) - \varphi(x))dz.$$

Therefore

$$|J_\varepsilon * \varphi(x) - \varphi(x)| \leq \sup_{z \in B(0,1)} |\varphi(x-\varepsilon z) - \varphi(x)|$$

which tend to zero as $\varepsilon \to 0$. Hence

$$\begin{aligned}
\left\|J_\varepsilon * \varphi - \varphi\right\|_{\dot{K}_p^{\alpha,q}(\Omega)}^q &\leq \sum_{k=-\infty}^{N+2} 2^{k\alpha q}\left\|(J_\varepsilon * \varphi - \varphi)\chi_k\right\|_p^q \\
&\leq \sup_{|x| \leq 2^{N+2}} \sup_{z \in B(0,1)} |\varphi(x-\varepsilon z) - \varphi(x)|^q \sum_{k=-\infty}^{N+2} 2^{k(\alpha+\frac{n}{p})q} \\
&\lesssim \sup_{|x| \leq 2^{N+2}} \sup_{z \in B(0,1)} |\varphi(x-\varepsilon z) - \varphi(x)|^q.
\end{aligned}$$

Letting ε tend to zero, we obtain (3) for any $\varphi \in C_c(\Omega)$ and $\alpha > -\frac{n}{p}$.

Substep 1.2. $\alpha < -\frac{n}{p}$. By duality

$$\left\|J_\varepsilon * \varphi - \varphi\right\|_{\dot{K}_p^{\alpha,q}(\Omega)} = \sup\left|\int_\Omega (J_\varepsilon * \varphi(x) - \varphi(x))g(x)dx\right|,$$

where the supremum is taken over all continuous functions of compact support g such that $\left\|g\right\|_{\dot{K}_{p'}^{-\alpha,q'}(\Omega)} = 1$. It is easily seen that

$$\int_\Omega (J_\varepsilon * \varphi(x) - \varphi(x))g(x)dx = \int_\Omega (\tilde{J}_\varepsilon * g(x) - g(x))\varphi(x)dx,$$

where $\tilde{J}_\varepsilon(x) = J_\varepsilon(-x), x \in \mathbb{R}^n$. We have

$$\left|\int_\Omega (\tilde{J}_\varepsilon * g(x) - g(x))\varphi(x)dx\right| \leq \left\|\tilde{J}_\varepsilon * g - g\right\|_{\dot{K}_{p'}^{-\alpha,q'}(\Omega)}\left\|\varphi\right\|_{\dot{K}_p^{\alpha,q}(\Omega)}.$$

Observe that $-\alpha > -\frac{n}{p'}$. Using Substep 1.1, we see that

$$\|\tilde{J}_\varepsilon * g - g\|_{\dot{K}_{p'}^{-\alpha,q'}(\Omega)} \leq \frac{\eta}{\|\varphi\|_{\dot{K}_p^{\alpha,q}(\Omega)}}$$

for any $\eta > 0$ and any ε small enough. Hence

$$\|J_\varepsilon * \varphi - \varphi\|_{\dot{K}_p^{\alpha,q}(\Omega)} \leq \eta$$

for any $\eta > 0$ and any ε small enough.

Substep 1.3. $\alpha = -\frac{n}{p}$. Let $\alpha_0 > -\frac{n}{p}$ and $\alpha_1 < -\frac{n}{p}$ be such that $\alpha = \theta\alpha_0 + (1-\theta)\alpha_1, 0 < \theta < 1$. Hölder's inequality yields

$$\begin{aligned}\|J_\varepsilon * \varphi - \varphi\|_{\dot{K}_p^{\alpha,q}(\Omega)} &\leq \|J_\varepsilon * \varphi - \varphi\|^{\theta}_{\dot{K}_p^{\alpha_0,q}(\Omega)} \|J_\varepsilon * \varphi - \varphi\|^{1-\theta}_{\dot{K}_p^{\alpha_1,q}(\Omega)} \\ &\leq \eta\end{aligned}$$

for any $\eta > 0$ and any ε small enough.

Step 2. We prove (2). By Theorem 2.6 we can find $\varphi \in C_c(\Omega)$ such that

$$\|f - \varphi\|_{\dot{K}_p^{\alpha,q}(\Omega)} \leq \frac{\eta}{3}$$

for any $\eta > 0$ small enough. So, for any $\eta_1 > 0$ small enough

$$\|J_\varepsilon * f - J_\varepsilon * \varphi\|_{\dot{K}_p^{\alpha,q}(\Omega)} \leq c\|\mathcal{M}(f-\varphi)\|_{\dot{K}_p^{\alpha,q}(\mathbb{R}^n)} \leq c\eta_1$$

by Lemma 2.2, because of $-\frac{n}{p} < \alpha < n - \frac{n}{p}$. We choose η_1 be such that $c\eta_1 < \frac{\eta}{3}$. From Step 1,

$$\|J_\varepsilon * \varphi - \varphi\|_{\dot{K}_p^{\alpha,q}(\Omega)} \leq \frac{\eta}{3}$$

by choosing ε sufficiently small, which prove (2) but with $p > 1$. Let $s > 1$. Hölder's inequality and the fact that $-n < \alpha < 0$ yield

$$\|J_\varepsilon * f - f\|_{\dot{K}_1^{\alpha,q}(\Omega)} \leq \|J_\varepsilon * f - f\|_{\dot{K}_s^{\alpha+n-\frac{n}{s},q}(\Omega)},$$

which tends to zero as $\varepsilon \to 0_+$.

This completes the proof. □

Let $1 \leq q < \infty$. The Caffarelli–Kohn–Nirenberg inequality says that

$$\Big(\int_{\mathbb{R}^n} |x|^{\gamma p}|f(x)|^p dx\Big)^{\frac{1}{p}} \leq c\Big(\int_{\mathbb{R}^n} |x|^{\alpha q}|\nabla f(x)|^q dx\Big)^{\frac{1}{q}}$$

for any $f \in \mathcal{D}(\mathbb{R}^n)$, where

$$\alpha > -\frac{n}{q}, \quad \gamma > -\frac{n}{p}, \quad \alpha - 1 \leq \gamma \leq \alpha, \quad \frac{n}{p} - \frac{n}{q} = \alpha - \gamma - 1 \leq 0, \tag{4}$$

see [5]. This inequality plays an important role in theory of function spaces and PDE's. Our aim is to extend this result to Herz spaces.

Theorem 2.9. *Let* $1 \leq q \leq \frac{n}{n-1}, 0 < r \leq \infty$ *and*

$$\alpha_2 + n - 1 = \alpha_1 + \frac{n}{q} > 0.$$

Then

$$\big\|f\big\|_{\dot{K}_q^{\alpha_1,r}(\mathbb{R}^n)} \lesssim \big\|f\big\|_{\dot{W}_{1,1}^{\alpha_2,r}(\mathbb{R}^n)}, \quad f \in \mathcal{D}(\mathbb{R}^n), \tag{5}$$

holds, where

$$\big\|f\big\|_{\dot{W}_{1,1}^{\alpha_2,r}(\mathbb{R}^n)} = \Big(\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 r}\big\|(\nabla f)\chi_k\big\|_1^r\Big)^{\frac{1}{r}}. \tag{6}$$

Proof. Since $\alpha_2 + n > 0$, (6) is well defined and finite for any $f \in \mathcal{D}(\mathbb{R}^n)$. Let

$$I_{1,k} = \big[-2^k, 2^k\big], \quad I_{2,k} = \big[\frac{1}{\sqrt{n}}2^{k-1}, 2^k\big], \quad k \in \mathbb{Z},$$

and

$$I_{3,k} = \big[-2^k, -\frac{1}{\sqrt{n}}2^{k-1}\big], \quad I_{4,k} = \Big(-\frac{1}{\sqrt{n}}2^{k-1}, \frac{1}{\sqrt{n}}2^{k-1}\Big), \quad k \in \mathbb{Z}.$$

We set

$$J_k = \cup_{i=1}^{n-1} V_{i,k} \cup V_k, \quad k \in \mathbb{Z},$$

where

$$V_k = \big(I_{2,k} \times (I_{4,k})^{n-1}\big) \cup \big(I_{3,k} \times (I_{4,k})^{n-1}\big), \quad V_{i,k} = V_{i,k}^1 \cup V_{i,k}^2,$$

with $k \in \mathbb{Z}$, $i \in \{1, 2, ..., n-1\}$,

$$V_{i,k}^1 = (I_{1,k})^{n-i} \times I_{2,k} \times (I_{4,k})^{i-1} \quad \text{and} \quad V_{i,k}^2 = (I_{1,k})^{n-i} \times I_{3,k} \times (I_{4,k})^{i-1}.$$

If $i = 1$, then we put $V_{1,k}^1 = (I_{1,k})^{n-1} \times I_{2,k}$ and $V_{1,k}^2 = (I_{1,k})^{n-1} \times I_{3,k}$.

Let $x \in R_k, k \in \mathbb{Z}$. Assume that x does not belongs to the set J_k. Then $x \notin V_{i,k}$ and $x \notin V_k$ for any $i \in \{1, 2, ..., n-1\}$. Since x is not an element of $V_{1,k}^1 \cup V_{1,k}^2$, we have necessary that $(x_1, ..., x_{n-1})$ belongs in $(I_{1,k})^{n-1}$ and $x_n \in I_{4,k}$, otherwise x is not an element of R_k, which is a contradiction. Assume that there exists $x_{i_0} \notin I_{4,k}$ with $i_0 \in \{2, ..., n-1\}$. Observe that $x \notin V_{n-i_0+1,k}^1 \cup V_{n-i_0+1,k}^2$, which yields that

$$(x_1, ..., x_{i_0-1}) \in (I_{1,k})^{i_0-1}, \quad x_{i_0} \in I_{2,k} \cup I_{3,k}, \quad (x_{i_0+1}, ..., x_n) \notin (I_{4,k})^{n-i_0}.$$

Let

$$v = \max\big\{j : i_0 \leq j < n, x_j \notin I_{4,k}\big\}.$$

Hence

$$x_m \in I_{4,k}, \quad v+1 \le m < n. \tag{7}$$

Also $x \notin V^1_{n-v+1,k} \cup V^2_{n-v+1,k}$, which yields that

$$(x_1, ..., x_{v-1}) \in (I_{1,k})^{v-1}, \quad x_v \in I_{2,k} \cup I_{3,k}, \quad (x_{v+1}, ..., x_n) \notin (I_{4,k})^{n-v},$$

which is a contradiction by (7) and the fact that $x_n \in I_{4,k}$. Consequently we obtain $x_1 \in I_{1,k}$ and $(x_2, ..., x_n) \in (I_{4,k})^{n-1}$. But $x \notin V_k$, then we have $x_1 \in I_{4,k}$, $x \in B(0, 2^{k-1})$ and this is a contradiction. Therefore

$$R_k \subset J_k \subset \tilde{R}_k, \quad k \in \mathbb{Z},$$

where $\tilde{R}_k = \{x \in \mathbb{R}^n : \frac{1}{\sqrt{n}} 2^{k-3} \le |x| \le \sqrt{n} 2^{k+4}\}$. Let $f \in \mathcal{D}(\mathbb{R}^n)$. We will prove the inequality (5). We write

$$\|f\|^r_{\dot{K}^{\alpha_1,r}_q(\mathbb{R}^n)} = \sum_{k=-\infty}^{\infty} 2^{k\alpha_1 r} \|f\chi_k\|^r_q.$$

Using Hölder's inequality we obtain

$$\|f\chi_k\|_q \le c 2^{k(\frac{n}{q}-n+1)} \|f\chi_k\|_{\frac{n}{n-1}}, \quad k \in \mathbb{Z},$$

where the constant $c > 0$ is independent of k. We have

$$\begin{aligned}
\int_{R_k} |f(x)|^{\frac{n}{n-1}} dx &\le \sum_{i=1}^{n-1} \int_{V_{i,k}} |f(x)|^{\frac{n}{n-1}} dx + \int_{V_k} |f(x)|^{\frac{n}{n-1}} dx \\
&\le \sum_{i=1}^{n-1} \int_{V^1_{i,k}} |f(x)|^{\frac{n}{n-1}} dx + \sum_{i=1}^{n-1} \int_{V^2_{i,k}} |f(x)|^{\frac{n}{n-1}} dx + \int_{V_k} |f(x)|^{\frac{n}{n-1}} dx \\
&= \sum_{i=1}^{n-1} J^1_{i,k} + \sum_{i=1}^{n-1} J^2_{i,k} + S_k..
\end{aligned}$$

We estimate $J^1_{i,k}, i \in \{1, 2, ..., n-1\}$. Let $\omega_1, \omega_2, \omega_3 \in \mathcal{D}(\mathbb{R})$ be such that

$$\omega_1(y) = 1 \quad \text{if} \quad |y| \le 2^{-M}, \quad \omega_2(y) = 1 \quad \text{if} \quad \frac{2^{-M-1}}{\sqrt{n}} \le |y| \le 2^{-M},$$

$$\omega_3(y) = 1 \quad \text{if} \quad |y| \le \frac{2^{-M}}{2\sqrt{n}},$$

$$\operatorname{supp}\omega_1 \subset \{y \in \mathbb{R} : |y| \le 2^{1-M}\}, \quad \operatorname{supp}\omega_2 \subset \{y \in \mathbb{R} : \frac{2^{-M-2}}{\sqrt{n}} \le |y| \le 2^{2-M}\}$$

and

$$\operatorname{supp}\omega_3 \subset \{y \in \mathbb{R} : |y| \leq \frac{2^{1-M}}{\sqrt{n}}\},$$

where $M > 1$ will be chosen later on. Let $x \in \mathbb{R}^n$. Define

$$f_k(x) = f(x)\Pi_{j=1}^{n-i}\omega_1(2^{-k-M}x_j)\omega_2(2^{-k-M}x_{n-i+1})\Pi_{j=n-i+2}^{n}\omega_3(2^{-k-M}x_j).$$

Obviously, if $x \in V_{i,k}^1$, then

$$f(x) = f_k(x).$$

Let $x \in V_{i,k}^1$. Taking into account the various conditions on the supports of ω_1, ω_2 and ω_3 we obtain

$$f(x) = \int_{-2^{k+1}}^{x_j} \frac{\partial f_k}{\partial x_j}(x_1,...,x_{j-1},y_j,x_{j+1},...,x_n)dy_j,$$

which yields that

$$|f(x)| \leq \int_{-2^{k+1}}^{x_j} \Big|\frac{\partial f_k}{\partial x_j}(x_1,...,x_{j-1},y_j,x_{j+1},...,x_n)\Big|dy_j$$

for any $j \in \{1,2,...,n-i\}$. In the same way we obtain

$$|f(x)| \leq \int_{\frac{1}{6\sqrt{n}}2^k}^{x_{n-i+1}} \Big|\frac{\partial f_k}{\partial x_{n-i+1}}(x_1,...,x_{n-i},y_{n-i+1},x_{n-i+1},...,x_n)\Big|dy_{n-i+1}$$

and

$$|f(x)| \leq \int_{-\frac{1}{\sqrt{n}}2^{k+1}}^{x_j} \Big|\frac{\partial f_k}{\partial x_j}(x_1,...,x_{n-i+1},...,x_{j-1},y_j,x_{j+1},...,x_n)\Big|dy_j$$

for any $j \in \{n-i+2,...,n\}$. Therefore for any $x \in V_{i,k}^1$, $|f(x)|^{\frac{n}{n-1}}$ is bounded by

$$\begin{aligned}
&\prod_{j=1}^{n-i}\Big(\int_{-2^{k+1}}^{2^k}\Big|\frac{\partial f_k}{\partial x_j}(x_1,...,x_{j-1},y_j,x_{j+1},...,x_n)\Big|dy_j\Big)^{\frac{1}{n-1}} \\
&\Big(\int_{\frac{1}{6\sqrt{n}}2^k}^{2^k}\Big|\frac{\partial f_k}{\partial x_{n-i+1}}(x_1,...,x_{n-i},y_{n-i+1},x_{n-i+2},...,x_n)\Big|dy_{n-i+1}\Big)^{\frac{1}{n-1}} \\
&\times\prod_{j=n-i+2}^{n}\Big(\int_{-\frac{1}{\sqrt{n}}2^{k+1}}^{\frac{2^{k-1}}{\sqrt{n}}}\Big|\frac{\partial f_k}{\partial x_j}(x_1,...,x_{n-i+1},...,x_{j-1},y_j,x_{j+1},...,x_n)\Big|dy_j\Big)^{\frac{1}{n-1}} \\
&=\prod_{j=1}^{n-i}\big(g(x_j')\big)^{\frac{1}{n-1}}\big(h(x_{n-i+1}')\big)^{\frac{1}{n-1}}\prod_{j=n-i+2}^{n}\big(w(x_j')\big)^{\frac{1}{n-1}},
\end{aligned}$$

where

$$x_j' = (x_1, ..., x_{j-1}, x_{j+1}, ..., x_n), \quad j \in \{1, ..., n\}$$

and

$$x_{n-i+1}' = (x_1, ..., x_{n-i}, x_{n-i+2}, ..., x_n).$$

Integrate with respect to x_1, over $I_{1,k}$ to obtain $\int_{I_{1,k}} |f(x)|^{\frac{n}{n-1}} dx_1$ is bounded by

$$\begin{aligned}
&\int_{I_{1,k}} \prod_{j=1}^{n-i} \left(g(x_j')\right)^{\frac{1}{n-1}} \left(h(x_{n-i+1}')\right)^{\frac{1}{n-1}} \prod_{j=n-i+2}^{n} \left(w(x_j')\right)^{\frac{1}{n-1}} dx_1 \\
&= \left(g(x_1')\right)^{\frac{1}{n-1}} \int_{I_{1,k}} \prod_{j=2}^{n-i} \left(g(x_j')\right)^{\frac{1}{n-1}} \left(h(x_{n-i+1}')\right)^{\frac{1}{n-1}} \prod_{j=n-i+2}^{n} \left(w(x_j')\right)^{\frac{1}{n-1}} dx_1,
\end{aligned}$$

which is bounded by, after using Hölder's inequality,

$$\begin{aligned}
&\left(g(x_1')\right)^{\frac{1}{n-1}} \prod_{j=2}^{n-i} \left(\int_{I_{1,k}} g(x_j') dx_1\right)^{\frac{1}{n-1}} \\
&\times \left(\int_{I_{1,k}} h(x_{n-i+1}') dx_1\right)^{\frac{1}{n-1}} \left(\prod_{j=n-i+2}^{n} \int_{I_{1,k}} w_j(x_j') dx_1\right)^{\frac{1}{n-1}}.
\end{aligned}$$

Integrate with respect to x_2, over $I_{1,k}$ and using Hölder's inequality to obtain that $\int_{(I_{1,k})^2} |f(x)|^{\frac{n}{n-1}} dx_1 dx_2$ is bounded by

$$\begin{aligned}
&\left(\int_{I_{1,k}} g(x_2') dx_1\right)^{\frac{1}{n-1}} \int_{I_{1,k}} \left(g(x_1')\right)^{\frac{1}{n-1}} \prod_{j=3}^{n-i} \left(\int_{I_{1,k}} g(x_j') dx_1\right)^{\frac{1}{n-1}} \\
&\times \left(\int_{I_{1,k}} h(x_{n-i+1}') dx_1\right)^{\frac{1}{n-1}} \left(\prod_{j=n-i+2}^{n} \int_{I_{1,k}} w(x_j') dx_1\right)^{\frac{1}{n-1}} dx_2 \\
&\leq \left(\int_{I_{1,k}} g(x_2') dx_1\right)^{\frac{1}{n-1}} \left(\int_{I_{1,k}} g(x_1') dx_2\right)^{\frac{1}{n-1}} \prod_{j=3}^{n-i} \left(\int_{(I_{1,k})^2} g(x_j') dx_1 dx_2\right)^{\frac{1}{n-1}} \\
&\times \left(\int_{(I_{1,k})^2} h(x_{n-i+1}') dx_1 dx_2\right)^{\frac{1}{n-1}} \left(\prod_{j=n-i+2}^{n} \int_{(I_{1,k})^2} w(x_j') dx_1 dx_2\right)^{\frac{1}{n-1}}.
\end{aligned}$$

Hence $\int_{(I_{1,k})^{n-i}} |f(x)|^{\frac{n}{n-1}} dx_1 \cdots dx_{n-i}$ is bounded by

$$\begin{aligned}
&\prod_{j=1}^{n-i} \left(\int_{(I_{1,k})^{n-i-1}} g(x_j') dx_1 \cdots dx_{j-1} dx_{j+1} \cdots dx_{n-i-1}\right)^{\frac{1}{n-1}} \\
&\times \left(\int_{(I_{1,k})^{n-i}} h(x_{n-i+1}') dx_1 dx_2 \cdots dx_{n-i}\right)^{\frac{1}{n-1}} \\
&\times \left(\prod_{j=n-i+2}^{n} \int_{(I_{1,k})^{n-i}} w(x_j') dx_1 dx_2 \cdots dx_{n-i}\right)^{\frac{1}{n-1}}.
\end{aligned}$$

In the same way $\int_{(I_{1,k})^{n-i}\times I_{2,k}} |f(x)|^{\frac{n}{n-1}} dx_1 \cdots dx_{n-i} dx_{n-i+1}$ is bounded by

$$\prod_{j=1}^{n-i}\Big(\int_{(I_{1,k})^{n-i-1}\times I_{2,k}} g(x_j')dx_1\cdots dx_{j-1}dx_{j+1}\cdots dx_{n-i+1}\Big)^{\frac{1}{n-1}}$$
$$\times\Big(\int_{(I_{1,k})^{n-i}} h(x_{n-i+1}')dx_1dx_2\cdots dx_{n-i}\Big)^{\frac{1}{n-1}}$$
$$\times\prod_{j=n-i+2}^{n}\Big(\int_{(I_{1,k})^{n-i}\times I_{2,k}} w(x_j')dx_1dx_2\cdots dx_{n-i+1}\Big)^{\frac{1}{n-1}}.$$

Consequently $\int_{V_{i,k}^1} |f(x)|^{\frac{n}{n-1}} dx_1 \cdots dx_n$ is bounded by

$$\prod_{j=1}^{n-i}\Big(\int_{(I_{1,k})^{n-i-1}\times I_{2,k}\times (I_{4,k})^{i-1}} g(x_j')dx_1\cdots dx_{j-1}dx_{j+1}\cdots dx_{n}\Big)^{\frac{1}{n-1}}$$
$$\times\Big(\int_{(I_{1,k})^{n-i}\times (I_{4,k})^{i-1}} h(x_{n-i+1}')dx_1dx_2\cdots dx_{n-i}\Big)^{\frac{1}{n-1}}$$
$$\times\prod_{j=n-i+2}^{n}\Big(\int_{(I_{1,k})^{n-i}\times I_{2,k}\times (I_{4,k})^{i-2}} w(x_j')dx_1dx_2\cdots\cdot dx_{j-1}dx_{j+1}\cdots dx_{n}\Big)^{\frac{1}{n-1}},$$

which is bounded by

$$\prod_{j=1}^{n}\Big(\int_{\tilde{R}_k}\Big|\frac{\partial f_k}{\partial x_j}(x)\Big|dx\Big)^{\frac{1}{n-1}}.$$

Observe, that

$$\Big|\frac{\partial f_k}{\partial x_j}\Big| \leq C2^{-(k+M)}\big|f\big| + \big|\nabla f\big|, \quad j\in\{1,2,...,n\},$$

where the positive constant C is independent of k. Consequently,

$$J_{i,k}^1 \leq \Big(C2^{-(k+M)}\big\|f\chi_{\tilde{R}_k}\big\|_{L^1(\mathbb{R}^n)} + \big\|(\nabla f)\chi_{\tilde{R}_k}\big\|_{L^1(\mathbb{R}^n)}\Big)^{\frac{n}{n-1}}$$

for any $k\in\mathbb{Z}$ and any $i\in\{1,2,...,n-1\}$. Using the fact that $\alpha_2+n-1=\alpha_1+\frac{n}{q}$ we deduce the following estimation

$$\Big(\sum_{k=-\infty}^{\infty} 2^{k(\alpha_1+\frac{n}{q}-n+1)r}\Big(\sum_{i=1}^{n-1} J_{i,k}^1\Big)^{\frac{(n-1)r}{n}}\Big)^{\frac{1}{r}}$$
$$\leq c_1\big\|f\big\|_{\dot{W}_{1,1}^{\alpha_2,r}(\mathbb{R}^n)} + c_2 2^{-M}\big\|f\big\|_{\dot{K}_1^{\alpha_2-1,r}(\mathbb{R}^n)},$$

where both c_1 and c_2 are independent of M. We estimate $V_{i,k}^2, i \in \{1,2,...,n-1\}$. We have

$$J_{i,k}^2 = \int_{(I_{1,k})^{n-i}\times I_{2,k}\times (I_{4,k})^{i-1}} \left|f_k(x_1,...,x_{n-i},-x_{n-i+1},x_{n-i+2},...,x_n)\right|^{\frac{n}{n-1}} dx_1dx_2\cdots dx_n$$

for any $k \in \mathbb{Z}$ and any $i \in \{1,2,...,n-1\}$. The estimate of $\sum_{i=1}^{n-1} J_{i,k}^2$ can be done in the same way as in $V_{i,k}^1$. The estimate of $S_{k.}$ can be done in the same way as in $\sum_{i=1}^{n-1} J_{i,k}^1$ and $\sum_{i=1}^{n-1} J_{i,k}^2$. Collecting these estimations in one formula we find that

$$\|f\|_{\dot{K}_q^{\alpha_1,r}(\mathbb{R}^n)} \le c_3\|f\|_{\dot{W}_{1,1}^{\alpha_2,r}(\mathbb{R}^n)} + c_4 2^{-M}\|f\|_{\dot{K}_1^{\alpha_2-1,r}(\mathbb{R}^n)},$$

where both c_3 and c_4 are independent of M. Using the fact that $\alpha_2+n-1 = \alpha_1+\frac{n}{q}$ and Hölder's inequality to obtain

$$\|f\|_{\dot{K}_1^{\alpha_2-1,r}(\mathbb{R}^n)} \le B\|f\|_{\dot{K}_q^{\alpha_1,r}(\mathbb{R}^n)}.$$

Choosing M such that $c_4B2^{-M} \le \frac{1}{2}$ we obtain the desired inequality. □

Remark 2.10. We mention here that our embedding covers the Caffarelli–Kohn–Nirenberg inequality because of (4) yields that $1 \le q \le \frac{n}{n-1}$.

Theorem 2.11. *Let* $1 \le q \le \frac{n}{\frac{n}{p}-1}, 0 < r \le \infty, \alpha_2 \ge \alpha_1$, *and*

$$\frac{n}{p} + \alpha_2 - 1 = \frac{n}{q} + \alpha_1 > 0.$$

Then

$$\|f\|_{\dot{K}_q^{\alpha_1,r}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{W}_{p,1}^{\alpha_2,r}(\mathbb{R}^n)}, \quad f \in \mathcal{D}(\mathbb{R}^n), \tag{8}$$

holds, where

$$\|f\|_{\dot{W}_{p,1}^{\alpha_2,r}(\mathbb{R}^n)} = \Big(\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 r}\|(\nabla f)\chi_k\|_p^r\Big)^{\frac{1}{r}}.$$

Proof. Let $f \in \mathcal{D}(\mathbb{R}^n)$ and $\frac{n}{\sigma} = \frac{n}{q} + n - \frac{n}{p}$. According to Theorem 2.9, since $1 \le \sigma \le \frac{n}{n-1}$, one has

$$\|g\|_{\dot{K}_\sigma^{\alpha_1,\tau}(\mathbb{R}^n)} \lesssim \sum_{j=1}^{n}\Big\|\frac{\partial g}{\partial x_j}\Big\|_{\dot{K}_1^{\alpha_2,\tau}(\mathbb{R}^n)}, \quad 0 < \tau \le \infty. \tag{9}$$

Let $g = |f|^{\frac{q}{\sigma}}$. It is easily seen that

$$\|f\|_{\dot{K}_q^{\alpha_1,r}(\mathbb{R}^n)} = \|g\|^{\frac{\sigma}{q}}_{\dot{K}_\sigma^{\frac{q}{\sigma}\alpha_1,r\frac{\sigma}{q}}(\mathbb{R}^n)}.$$

Let

$$\widetilde{\alpha_2} = \alpha_2 - \alpha_1 + \frac{q}{\sigma}\alpha_1 \quad \text{and} \quad \widetilde{r} = r\frac{\sigma}{q}.$$

From the inequality (9), we deduce

$$\begin{aligned}\big\|g\big\|_{\dot{K}_\sigma^{\frac{q}{\sigma}\alpha_1,\widetilde{r}}(\mathbb{R}^n)} &\lesssim \sum_{j=1}^{n}\Big\|\frac{\partial g}{\partial x_j}\Big\|_{\dot{K}_1^{\widetilde{\alpha_2},\widetilde{r}}(\mathbb{R}^n)} \\ &\lesssim \sum_{j=1}^{n}\Big\|s|f|^{s-1}\frac{\partial f}{\partial x_j}\Big\|_{\dot{K}_1^{\widetilde{\alpha_2},\widetilde{r}}(\mathbb{R}^n)},\end{aligned}$$

with $s = \frac{q}{\sigma}$. By combining this estimate with

$$\widetilde{\alpha_2} = \alpha_1\frac{q}{p'} + \alpha_2 \quad \text{and} \quad \frac{1}{\widetilde{r}} = \frac{1}{r}\frac{q}{p'} + \frac{1}{r}$$

we see that

$$\begin{aligned}\Big\|s|f|^{s-1}\frac{\partial f}{\partial x_j}\Big\|_{\dot{K}_1^{\widetilde{\alpha_2},\widetilde{r}}(\mathbb{R}^n)} &\lesssim \big\|s|f|^{s-1}\big\|_{\dot{K}_{p'}^{\alpha_1\frac{q}{p'},\frac{rp'}{q}}(\mathbb{R}^n)}\Big\|\frac{\partial f}{\partial x_j}\Big\|_{\dot{K}_p^{\alpha_2,r}(\mathbb{R}^n)} \\ &= c\big\|f\big\|_{\dot{K}_q^{\alpha_1,r}(\mathbb{R}^n)}^{\frac{q}{p'}}\Big\|\frac{\partial f}{\partial x_j}\Big\|_{\dot{K}_p^{\alpha_2,r}(\mathbb{R}^n)},\end{aligned}$$

where we have used the Hölder inequality. Therefore

$$\big\|f\big\|_{\dot{K}_q^{\alpha_1,r}(\mathbb{R}^n)} \lesssim \big\|f\big\|_{\dot{K}_q^{\alpha_1,r}(\mathbb{R}^n)}^{\frac{\sigma}{p'}}\sum_{j=1}^{n}\Big\|\frac{\partial f}{\partial x_j}\Big\|_{\dot{K}_p^{\alpha_2,r}(\mathbb{R}^n)}^{\frac{\sigma}{q}}$$

and get finally

$$\big\|f\big\|_{\dot{K}_q^{\alpha_1,r}(\mathbb{R}^n)} \lesssim \sum_{j=1}^{n}\Big\|\frac{\partial f}{\partial x_j}\Big\|_{\dot{K}_p^{\alpha_2,r}(\mathbb{R}^n)}.$$

Hence the proof is complete. □

Remark 2.12. Again our embedding covers the Caffarelli–Kohn–Nirenberg inequality because of (4) yields that $1 \leq q \leq \frac{n}{\frac{n}{p}-1}$. Let $1 \leq p \leq q < \infty$ and $\frac{n}{q} - \frac{n}{p} = \alpha_2 - 1 - \alpha_1$. By (8) we easily obtain that

$$\begin{aligned}\Big(\int_{\mathbb{R}^n}|x|^{\alpha_1 q}|f(x)|^q dx\Big)^{\frac{1}{q}} &\lesssim \Big(\sum_{k=-\infty}^{\infty}2^{k\alpha_2 q}\big\|(\nabla f)\chi_k\big\|_p^q\Big)^{\frac{1}{q}} \\ &\lesssim \Big(\int_{\mathbb{R}^n}|x|^{\alpha_2 p}|\nabla f(x)|^p dx\Big)^{\frac{1}{p}},\end{aligned}$$

whenever the right-hand side is finite. In particular,

$$\|f\|_q \lesssim \Big(\sum_{k=-\infty}^{\infty} \|(\nabla f)\chi_k\|_p^q\Big)^{\frac{1}{q}} \lesssim \|\nabla f\|_p,$$

where $1 \le p < q < \infty$ and $1 - \frac{n}{p} = -\frac{n}{q}$, whenever the right-hand side is finite, which is the Sobolev's inequality.

In reality, the inequality of Caffarelli–Kohn–Nirenberg inequality says that

$$\big\||x|^{\alpha_1} f\big\|_q \le c\big\||x|^{\alpha_2} f\big\|_p^{\theta}\big\||x|^{\alpha_3}\nabla f\big\|_u^{1-\theta}, \quad f \in \mathcal{D}(\mathbb{R}^n), \tag{10}$$

where $p,u \ge 1, q > 0, 0 \le \theta \le 1$,

$$\frac{n}{p} + \alpha_2 > 0, \quad \frac{n}{u} + \alpha_3 > 0, \quad \frac{n}{q} + \alpha_1 > 0,$$

$$\frac{n}{q} + \alpha_1 = \theta\big(\frac{n}{p} + \alpha_2 - 1\big) + \big(\frac{n}{u} + \alpha_3\big)(1-\theta), \quad \alpha_1 = \theta\sigma + (1-\theta)\alpha_3,$$

$$\sigma \le \alpha_2 \quad \text{if} \quad \theta > 0$$

and

$$\alpha_2 \le \sigma + 1 \quad \text{if} \quad \theta > 0 \quad \text{and} \quad \frac{n}{q} + \alpha_1 = \frac{n}{p} + \alpha_2 - 1.$$

Our aim is to extend this result to Herz spaces. We begin by the following special case.

Theorem 2.13. *Let $u \ge 1, q, v, r, s > 0, 0 \le \theta \le 1$,*

$$n + \alpha_2 > 0, \quad \frac{n}{u} + \alpha_3 > 0, \quad \frac{n}{q} + \alpha_1 > 0, \quad \sigma \le \alpha_2 \le \sigma + 1,$$

$$\alpha_1 = \theta\sigma + (1-\theta)\alpha_3, \quad \frac{n}{q} + \alpha_1 = \theta\big(n + \alpha_2 - 1\big) + \big(\frac{n}{u} + \alpha_3\big)(1-\theta)$$

and

$$\frac{1}{r} = \frac{\theta}{s} + \frac{1-\theta}{v}.$$

Then

$$\|f\|_{\dot{K}_q^{\alpha_1,r}(\mathbb{R}^n)} \le c\|\nabla f\|_{\dot{K}_1^{\alpha_2,s}(\mathbb{R}^n)}^{\theta}\|f\|_{\dot{K}_u^{\alpha_3,v}(\mathbb{R}^n)}^{1-\theta}, \quad f \in \mathcal{D}(\mathbb{R}^n),$$

Proof. Obviously, we need only to study the case $0 < \theta < 1$. Let $h = \frac{n}{n-1+\alpha_2-\sigma}$. Therefore

$$\frac{1}{q} = \frac{\theta}{h} + \frac{1-\theta}{u}.$$

Using Hölder's inequality we obtain

$$2^{k\alpha_1}\big\|f\chi_k\big\|_q \leq c\big(2^{k\sigma}\big\|f\chi_k\big\|_h\big)^{\theta}\big(2^{k\alpha_3}\big\|f\chi_k\big\|_u\big)^{1-\theta}, \quad k\in\mathbb{Z}.$$

Therefore

$$\big\|f\big\|_{\dot{K}_q^{\alpha_1,r}(\mathbb{R}^n)} \leq c\big\|f\big\|^{\theta}_{\dot{K}_h^{\sigma,s}(\mathbb{R}^n)}\big\|f\big\|^{1-\theta}_{\dot{K}_u^{\alpha_3,v}(\mathbb{R}^n)}.$$

Observe that

$$\frac{n}{h}+\sigma = n-1+\alpha_2, \quad 1\leq h\leq \frac{n}{n-1}.$$

Hence by Theorem 2.9,

$$\big\|f\big\|_{\dot{K}_h^{\sigma,s}(\mathbb{R}^n)} \leq c\big\|\nabla f\big\|_{\dot{K}_1^{\alpha_2,s}(\mathbb{R}^n)}.$$

The proof is complete. □

Now we formulate our main theorem.

Theorem 2.14. *Let* $p,u\geq 1, q,r,v,s>0, 0\leq\theta\leq 1$,

$$\frac{n}{p}+\alpha_2>0, \quad \frac{n}{u}+\alpha_3>0, \quad \frac{n}{q}+\alpha_1>0, \quad \sigma\leq\alpha_2\leq\sigma+1,$$

$$\alpha_1=\theta\sigma+(1-\theta)\alpha_3, \quad \frac{n}{q}+\alpha_1=\theta\big(\frac{n}{p}+\alpha_2-1\big)+\big(\frac{n}{u}+\alpha_3\big)(1-\theta)$$

and

$$\frac{1}{r}=\frac{\theta}{s}+\frac{1-\theta}{v}.$$

Then

$$\big\|f\big\|_{\dot{K}_q^{\alpha_1,r}(\mathbb{R}^n)} \leq c\big\|\nabla f\big\|^{\theta}_{\dot{K}_p^{\alpha_2,s}(\mathbb{R}^n)}\big\|f\big\|^{1-\theta}_{\dot{K}_u^{\alpha_3,v}(\mathbb{R}^n)}, \quad f\in\mathcal{D}(\mathbb{R}^n), \tag{11}$$

Proof. We have $\frac{1}{q}=\frac{\theta}{\tau}+\frac{1-\theta}{u}$, where $\tau=\frac{n}{\frac{n}{p}-1+\alpha_2-\sigma}$. Using Hölder's inequality we obtain

$$\big\|f\big\|_{\dot{K}_q^{\alpha_1,r}(\mathbb{R}^n)} \leq c\big\|f\big\|^{\theta}_{\dot{K}_\tau^{\sigma,s}(\mathbb{R}^n)}\big\|f\big\|^{1-\theta}_{\dot{K}_u^{\alpha_3,v}(\mathbb{R}^n)}.$$

Observe that

$$\frac{n}{\tau}-\frac{n}{p}=\alpha_2-1-\sigma\leq 0.$$

According to Theorem 2.11, since $1\leq\tau\leq\frac{n}{\frac{n}{p}-1}$, one has

$$\big\|f\big\|_{\dot{K}_\tau^{\sigma,s}(\mathbb{R}^n)} \leq c\big\|\nabla f\big\|_{\dot{K}_p^{\alpha_2,s}(\mathbb{R}^n)},$$

which completes the proof.. □

Remark 2.15. More Caffarelli–Kohn–Nirenberg inequalities in function spaces are given in [6]. From (11) we easily obtain

$$\|f\|_{\dot{K}_q^{\alpha_1,q}(\mathbb{R}^n)} \le c\|\nabla f\|^{\theta}_{\dot{K}_p^{\alpha_2,\tau}(\mathbb{R}^n)}\|f\|^{1-\theta}_{\dot{K}_u^{\alpha_3,u}(\mathbb{R}^n)}, \quad f \in \mathcal{D}(\mathbb{R}^n),$$

but $\tau \ge p$, then we obtain

$$\|f\|_{\dot{K}_q^{\alpha_1,q}(\mathbb{R}^n)} \le c\|\nabla f\|^{\theta}_{\dot{K}_p^{\alpha_2,p}(\mathbb{R}^n)}\|f\|^{1-\theta}_{\dot{K}_u^{\alpha_3,u}(\mathbb{R}^n)}, \quad f \in \mathcal{D}(\mathbb{R}^n),$$

which is the classical Caffarelli–Kohn–Nirenberg inequality, see (10).

3. Herz-type Sobolev spaces

In this section we prove the basic properties of Herz-type Sobolev spaces in analogy to the classical Sobolev spaces.

Definition 3.1. Let $\Omega \subset \mathbb{R}^n$ be open, $(\alpha,p,q) \in V_{\alpha,p,q}$ and $m \in \mathbb{N}_0$. We define the Herz-type Sobolev space $\dot{K}^{\alpha,q}_{p,m}(\Omega)$ as the set of functions $f \in \dot{K}_p^{\alpha,q}(\Omega)$ with weak derivatives $D^\beta f \in \dot{K}_p^{\alpha,q}(\Omega)$ for $|\beta| \le m$. We define the norm of $\dot{K}^{\alpha,q}_{p,m}(\Omega)$ by

$$\|f\|_{\dot{K}^{\alpha,q}_{p,m}(\Omega)} = \Big(\sum_{k=-\infty}^{\infty} 2^{k\alpha q}\Big(\sum_{|\beta|\le m}\|(D^\beta f)\chi_{R_k\cap\Omega}\|_p^p\Big)^{\frac{q}{p}}\Big)^{1/q}$$

if $1 \le p,q < \infty$ and

$$\|f\|_{\dot{K}^{\alpha,\infty}_{p,m}(\Omega)} = \sup_{k\in\mathbb{Z}} 2^{k\alpha}\Big(\sum_{|\beta|\le m}\|(D^\beta f)\chi_{R_k\cap\Omega}\|_p^p\Big)^{\frac{1}{p}}.$$

Remark 3.2. One recognizes immediately that if $p=q$ and $\alpha=0$, then we have $\dot{K}^{0,p}_{p,m}(\Omega) = W_p^m(\Omega)$.

As in classical Sobolev spaces, see [2, Theorem 3.3], we have the following statements:

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^n$ be open and $(\alpha,p,q) \in V_{\alpha,p,q}$. For each $m \in \mathbb{N}_0$, the Herz-type Sobolev space $\dot{K}^{\alpha,q}_{p,m}(\Omega)$ is a Banach space.*

Exactly in the same way as in the classical Sobolev spaces, see [2], but we use Theorem 2.8 we immediately arrive at the following result.

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^n$ be open, $m \in \mathbb{N}_0$ and $(\alpha,p,q) \in V_{\alpha,p,q}$ with $1 < p < \infty$, $1 \le q < \infty$ and $-\frac{n}{p} < \alpha < n - \frac{n}{p}$. Let Ω' be an open subset of Ω such that $\overline{\Omega'}$ is a compact subset of Ω. Let J_ε be as above and $f \in \dot{K}^{\alpha,q}_{p,m}(\Omega)$. Then*

$$\lim_{\varepsilon\to 0_+}\|J_\varepsilon * f - f\|_{\dot{K}^{\alpha,q}_{p,m}(\Omega')} = 0.$$

Similarly as in [2, Theorems 3.6 and 3.17] with the help of Theorem 2.7 we have the following statements:

Theorem 3.5. *Let $\Omega \subset \mathbb{R}^n$ be open, $m \in \mathbb{N}_0$ and $(\alpha, p, q) \in V_{\alpha,p,q}$ with $1 < p < \infty, 1 \leq q < \infty$ and $-\frac{n}{p} < \alpha < n - \frac{n}{p}$. $\dot{K}_{p,m}^{\alpha,q}(\Omega)$ is separable and $C^\infty(\Omega) \cap \dot{K}_{p,m}^{\alpha,q}(\Omega)$ is dense in $\dot{K}_{p,m}^{\alpha,q}(\Omega)$.*

3.1. Embeddings

In this subsection we present some embeddings of the spaces introduced above.

Definition 3.6. Let $v \in \mathbb{R}^n \backslash \{0\}$ and for each $x \neq 0$ let $\angle(x, v)$ be the angle between the position vector x and v. Let κ satisfying $0 < \kappa < \pi$. The set

$$C = \{x \in \mathbb{R}^n : x = 0 \text{ or } 0 < |x| \leq \rho, \angle(x, v) \leq \kappa/2\}$$

is called a finite cone of height ρ, axis direction v and aperture angle κ with vertex at the origin.

Remark 3.7. Let C be a finite cone with vertex at the origin. Note that $x + C = \{x + y : y \in C\}$ is a finite cone with vertex at x but the same dimensions and axis direction as C and is obtained by parallel translation of C.

We are now in a position to state the definition of domain satisfying the cone condition

Definition 3.8. Let $\Omega \subset \mathbb{R}^n$ be open. Ω satisfies the cone condition if there exists a finite cone C such that each $x \in \Omega$ is the vertex of a finite cone C_x contained in Ω and congruent to C.

Remark 3.9. In Definition 3.8 the cone C_x is not obtained from C by parallel translation, but simply by rigid motion.

The following statement can be found in [2, Lemma 4.15], that plays an essential for us.

Lemma 3.10. *Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the cone condition. Then we can find a positive constant K depending on m, n, and the dimensions ρ and κ of the cone C specified the cone condition for Ω such that for every $f \in C^\infty(\Omega)$, every $x \in \Omega$, and every r satisfying $0 < r \leq \rho$, we have*

$$|f(x)| \leq K\Big(\sum_{|\beta| \leq m-1} r^{|\beta|-n} \int_{C_{x,r}} |D^\beta f(y)| dy + \sum_{|\beta|=m-1} \int_{C_{x,r}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} dy\Big),$$

where $C_{x,r} = \{y \in C_x : y \in B(x, r)\}$.

Let $0 < \lambda < n$. The Riesz potential operator $\mathcal{I}_\lambda$ is defined by

$$\mathcal{I}_\lambda f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\lambda}} dy.$$

Let p^* be the Sobolev exponent defined by $\frac{1}{p^*} = \frac{1}{p} - \frac{\lambda}{n}$. The following statement plays a crucial role in our embeddings results, see [10].

Theorem 3.11. *Let* $0 < \lambda < n$, $0 < q_0 \le q_1 \le \infty$ *and* $1 < p < p^* < \frac{n}{\lambda}$. *If*

$$\lambda - \frac{n}{p} < \alpha < n - \frac{n}{p},$$

then $\mathcal{I}_\lambda$ *is bounded from* $\dot{K}_p^{\alpha,q_0}(\mathbb{R}^n)$ *into* $\dot{K}_{p^*}^{\alpha,q_1}(\mathbb{R}^n)$.

Now we state the first embeddings theorem.

Theorem 3.12. *Let* $\Omega \subset \mathbb{R}^n$ *be a domain satisfying the cone condition,* $0 \in \Omega$ *and* $m \in \mathbb{N}_0$. *Let* $1 < p < \infty, 1 \le r < \infty, \alpha_2 \ge \alpha_1, m - \frac{n}{p} < \alpha_2 < n - \frac{n}{p}$,

$$m - \alpha_2 + \alpha_1 > 0 \quad \text{and} \quad \frac{n}{q} = \frac{n}{p} - m + \alpha_2 - \alpha_1 > 0. \tag{12}$$

Then

$$\dot{K}_{p,m}^{\alpha_2,r}(\Omega) \hookrightarrow \dot{K}_q^{\alpha_1,r}(\Omega)$$

holds.

Proof. We use Theorem 3.5 and we will do the proof in two steps. Let $f \in C^\infty(\Omega) \cap \dot{K}_{p,m}^{\alpha_2,r}(\Omega)$.

Step 1. $\alpha_1 = \alpha_2$. From Lemma 3.10,

$$|f(x)| \lesssim \sum_{|\beta| \le m} \mathcal{I}_m((D^\beta f)\chi_\Omega)(x), \quad x \in \Omega.$$

Using Theorem 3.11 we obtain

$$\|f\|_{\dot{K}_q^{\alpha_1,r}(\Omega)} \lesssim \sum_{|\beta| \le m} \|(D^\beta f)\chi_\Omega\|_{\dot{K}_p^{\alpha_1,r}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{K}_{p,m}^{\alpha_1,r}(\Omega)}.$$

Step 2. $\alpha_2 > \alpha_1$. We write

$$\begin{aligned} \|f\|^r_{\dot{K}_q^{\alpha_1,r}(\Omega)} &= \sum_{k=-\infty}^{\infty} 2^{k\alpha_1 r} \|f\chi_{R_k \cap \Omega}\|_q^r \\ &= \sum_{k=-\infty}^{-1} 2^{k\alpha_1 r} \|f\chi_{R_k \cap \Omega}\|_q^r + \sum_{k=0}^{\infty} 2^{k\alpha_1 r} \|f\chi_{R_k \cap \Omega}\|_q^r \\ &= I_1 + I_2. \end{aligned}$$

Estimate of I_1. Let ρ be as in Lemma 3.10. We decompose I_1 as follows: $I_1 = I_3 + I_4$, where

$$I_3 = \sum_{k\leq -1, \rho\leq 2^{k-2}} 2^{k\alpha_1 r}\big\|f\chi_{R_k\cap\Omega}\big\|_q^r \quad \text{and} \quad I_4 = \sum_{k\leq -1, \rho> 2^{k-2}} 2^{k\alpha_1 r}\big\|f\chi_{R_k\cap\Omega}\big\|_q^r.$$

Let $x \in R_k \cap \Omega$, $k \in \mathbb{Z}$. We estimate I_3. Since $x \in R_k \cap \Omega$ and $\rho \leq 2^{k-2}$, we get $C_{x,\rho} \subset \tilde{R}_k = \{z : 2^{k-2} \leq |z| \leq 2^{k+1}\}$. From Lemma 3.10, we easily obtain

$$|f(x)| \lesssim \sum_{|\beta|\leq m-1} \rho^{|\beta|-n}\int_{C_{x,\rho}} |D^\beta f(y)|\chi_{\tilde{R}_k}(y)dy + \sum_{|\beta|=m}\int_{C_{x,\rho}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}}\chi_{\tilde{R}_k}(y)dy,$$

which is bounded by, because of $\alpha_2 > \alpha_1$ and $m < n$,

$$\begin{aligned}
&c2^{k(\alpha_2-\alpha_1)}\sum_{|\beta|\leq m-1} \rho^{|\beta|-n}\int_{C_{x,\rho}} \frac{|D^\beta f(y)|}{|x-y|^{n-m+\alpha_2-\alpha_1}}\chi_{\tilde{R}_k}(y)dy \\
&+c2^{k(\alpha_2-\alpha_1)}\sum_{|\beta|=m}\int_{C_{x,\rho}} \frac{|D^\beta f(y)|}{|x-y|^{n-m+\alpha_2-\alpha_1}}\chi_{\tilde{R}_k}(y)dy \\
&\lesssim 2^{k(\alpha_2-\alpha_1)}\sum_{|\beta|\leq m-1} \mathcal{I}_{m-\alpha_2+\alpha_1}((D^\beta f)\chi_{\Omega\cap\tilde{R}_k})(x),
\end{aligned}$$

where the positive constant c is independent of k. Thanks to Theorem 3.11 there exists some constant c such that

$$I_3 \lesssim \sum_{|\beta|\leq m} \big\|\mathcal{I}_{m-\alpha_2+\alpha_1}((D^\beta f)\chi_{\Omega\cap\tilde{R}_k})\big\|^r_{\dot{K}_q^{\alpha_2,r}(\mathbb{R}^n)} \leq c\big\|f\big\|^r_{\dot{K}_{p,m}^{\alpha_2,r}(\Omega)}.$$

Now we estimate I_4. We set

$$J_{1,k}(x) = \sum_{|\beta|\leq m-1} \rho^{|\beta|-n}\int_{C_{x,\rho}} |D^\beta f(y)|dy$$

and

$$J_{2,k}(x) = \sum_{|\beta|=m}\int_{C_{x,\rho}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}}dy.$$

To estimate the first term we use the fact that $m < n$ and $\rho > 2^{k-2}$ which leads to

$$\begin{aligned}
J_{1,k}(x) &\lesssim 2^{(m-n)k}\sum_{|\beta|\leq m-1}\int_{C_{x,2^{k-2}}} |D^\beta f(y)|dy \\
&+\sum_{|\beta|\leq m-1} \rho^{|\beta|-n}\int_{2^{k-2}\leq|x-y|\leq\rho} |D^\beta f(y)|\chi_\Omega(y)dy \\
&= c(J^1_{1,k}(x) + J^2_{1,k}(x)).
\end{aligned}$$

Let us estimate each term separately. By assumption (12) and Hölder's inequality it is easy to see that

$$J^1_{1,k}(x) \lesssim \sum_{|\beta|\le m-1} 2^{(m-n)k} \int_{C_{x,2^{k-2}}} |D^\beta f(y)|\chi_{\tilde{R}_k}(y)\chi_\Omega(y)dy$$
$$\lesssim 2^{k(\alpha_2-\alpha_1-\frac{n}{q})} \sum_{|\beta|\le m-1} \left\|(D^\beta f)\chi_{\tilde{R}_k\cap\Omega}\right\|_p.$$

Therefore

$$\left\|(J^1_{1,k})\chi_{R_k\cap\Omega}\right\|_q \lesssim 2^{k(\alpha_2-\alpha_1)} \sum_{|\beta|\le m-1} \left\|(D^\beta f)\chi_{\tilde{R}_k\cap\Omega}\right\|_p$$

for any $k\le -1$ such that $\rho > 2^{k-2}$. Rewriting $J^2_{1,k}$ as follows: $J^2_{1,k} = J^2_{1,k,1} + J^2_{1,k,2}$, where

$$J^2_{1,k,1}(x) = \sum_{|\beta|\le m-1} \rho^{|\beta|-n} \int_{2^{k-2}\le|x-y|\le 2^{k+2}} |D^\beta f(y)|\chi_\Omega(y)dy$$

and

$$J^2_{1,k,2}(x) = \sum_{|\beta|\le m-1} \rho^{|\beta|-n} \int_{2^{k+2}\le|x-y|\le \rho} |D^\beta f(y)|\chi_\Omega(y)dy.$$

$J^2_{1,k,1}(x)$ can be estimated from above by

$$c2^{(\alpha_2-\alpha_1)k} \sum_{|\beta|\le m-1} \mathcal{I}_{m-\alpha_2+\alpha_1}((D^\beta f)\chi_\Omega)(x)$$

for any $k\le -1$ such that $\rho > 2^{k-2}$. Now we consider the second term. We have

$$J^2_{1,k,2}(x) \lesssim \sum_{|\beta|\le m-1} \int_{2^{k+2}\le|x-y|\le\rho} \frac{|D^\beta f(y)|\chi_\Omega(y)}{|x-y|^{n-m}}dy,$$

which can be estimated by

$$c\sum_{|\beta|\le m-1}\sum_{i=k+2}^{j} \int_{2^i\le|x-y|\le 2^{i+1}} \frac{|D^\beta f(y)|\chi_\Omega(y)}{|x-y|^{n-m}}\chi_{\tilde{R}_i}(y)dy$$
$$\lesssim \sum_{|\beta|\le m-1}\sum_{i=k+2}^{j} 2^{(m-\frac{n}{p}-\alpha_2)i}2^{i\alpha_2}\left\|(D^\beta f)\chi_{\tilde{R}_i\cap\Omega}\right\|_p,$$

where $2^{j-1}\le\rho<2^j$, $j\in\mathbb{Z}$ and we used Hölder's inequality. By assumption (12) we obtain

$$2^{k\alpha_1}\left\|(J^2_{1,k,2})\chi_{R_k\cap\Omega}\right\|_q \lesssim 2^{(\frac{n}{p}-m+\alpha_2)k} \sum_{|\beta|\le m-1}\sum_{i=k+2}^{j} 2^{(m-\frac{n}{p}-\alpha_2)i}2^{i\alpha_2}\left\|(D^\beta f)\chi_{\tilde{R}_i\cap\Omega}\right\|_p$$

for any $k \leq -1$ such that $\rho > 2^{k-2}$.

We estimate $J_{2,k}$. We write

$$J_{2,k}(x) = \sum_{|\beta|=m} \int_{C_{x,2^{k-2}}} \frac{|D^{\beta} f(y)|}{|x-y|^{n-m}} dy + \sum_{|\alpha|=m} \int_{B_{x,\rho,k}} \frac{|D^{\beta} f(y)|}{|x-y|^{n-m}} dy, \qquad (13)$$

where $B_{x,\rho,k} = C_x \cap \{y : 2^{k-2} \leq |x-y| < \rho\}$. The first term is bounded by

$$\begin{aligned} & c2^{k(\alpha_2-\alpha_1)} \sum_{|\beta|\leq m} \int_{C_{x,2^{k-2}}} \frac{|D^{\beta} f(y)|\chi_{\Omega}(y)}{|x-y|^{n-m-\alpha_1+\alpha_2}} dy \\ & \lesssim 2^{k(\alpha_2-\alpha_1)} \sum_{|\beta|\leq m} \mathcal{I}_{m-\alpha_2+\alpha_1}((D^{\beta} f)\chi_{\Omega})(x). \end{aligned}$$

Rewriting the second term of (13) as follows: $J_{2,k,1} + J_{2,k,2}$; where

$$J_{2,k,1}(x) = \sum_{|\beta|=m} \int_{2^{k-2}\leq|x-y|\leq 2^{k+2}} \frac{|D^{\beta} f(y)|}{|x-y|^{n-m}} dy$$

and

$$J_{2,k,2}(x) = \sum_{|\beta|=m} \int_{2^{k+2}\leq|x-y|\leq\rho} \frac{|D^{\beta} f(y)|}{|x-y|^{n-m}} dy.$$

Observe that

$$J_{2,k,1}(x) \lesssim 2^{k(\alpha_2-\alpha_1)} \sum_{|\beta|=m} \mathcal{I}_{m-\alpha_2+\alpha_1}((D^{\beta} f)\chi_{\Omega})(x).$$

As in the estimation of $J_{1,k,2}^2$, we obtain

$$\begin{aligned} & 2^{k\alpha_1} \big\| (J_{2,k,2})\chi_{R_k\cap\Omega} \big\|_q \\ & \lesssim 2^{(\frac{n}{p}-m+\alpha_2)k} \sum_{|\beta|\leq m-1} \sum_{i=k+2}^{j} 2^{(m-\frac{n}{p}-\alpha_2)i} 2^{i\alpha_2} \big\| (D^{\beta} f)\chi_{\tilde{R}_i\cap\Omega} \big\|_p. \end{aligned}$$

Using the fact that $\alpha_2 > m - \frac{n}{p}$, we obtain by Lemma 2.3 that $I_4 \leq c\big\|f\big\|^r_{\dot{K}^{\alpha_2,r}_{p,m}(\Omega)}$.

Estimate of I_2. Since $\alpha_2 > \alpha_1$, we obtain that

$$I_2 \leq \sup_{k\in\mathbb{N}_0} 2^{kr\alpha_2} \big\| f\chi_{\Omega} \big\|^r_q \lesssim \big\| f \big\|^r_{\dot{K}^{\alpha_2,\infty}_q(\Omega)}.$$

Again from Lemma 3.10,

$$|f(x)| \lesssim \sum_{|\beta|\leq m} \mathcal{I}_{m-\alpha_2+\alpha_1}((D^{\beta} f)\chi_{\Omega})(x), \quad x \in \Omega.$$

Using again Theorem 3.11 it follows as above that

$$\begin{aligned} I_2 &\lesssim \sum_{|\beta|\le m} \left\| \mathcal{I}_{m-\alpha_2+\alpha_1}((D^\beta f)\chi_\Omega) \right\|^r_{\dot{K}_q^{\alpha_2,r}(\mathbb{R}^n)} \\ &\lesssim \sum_{|\beta|\le m} \left\| (D^\beta f)\chi_\Omega \right\|^r_{\dot{K}_p^{\alpha_2,r}(\mathbb{R}^n)} \\ &\lesssim \|f\|^r_{\dot{K}_{p,m}^{\alpha_2,r}(\Omega)}, \end{aligned}$$

since $m - \frac{n}{p} < \alpha_2 < n - \frac{n}{p}$. The proof is complete. □

Remark 3.13. We mention that Theorem 3.12 covers the Sobolev inequality. In addition

$$W_p^m(\Omega, |\cdot|^{\alpha_2 p}) \hookrightarrow \dot{K}_q^{\alpha_1,p}(\Omega) \hookrightarrow L^q(\Omega, |\cdot|^{\alpha_1 q}),$$

under the same assumptions of Theorem 3.12 with $r = p$. In particular

$$W_p^m(\Omega) \hookrightarrow \dot{K}_q^{0,p}(\Omega) \hookrightarrow L^q(\Omega),$$

holds if $1 < p < \infty, 0 < m < \frac{n}{p}$ and

$$\frac{n}{q} = \frac{n}{p} - m.$$

Theorem 3.14. *Let domain $\Omega \subset \mathbb{R}^n$ satisfy the cone condition , $0 \in \Omega$ and $m \in \mathbb{N}$. Let $1 < p < \infty, 1 \le r < \infty, \alpha_2 \ge \alpha_1$, $\alpha_1 + \frac{n}{p} > 0$ and*

$$\max\left(\frac{n}{p} + \alpha_2, \frac{n}{p} + \alpha_2 - \alpha_1\right) < m < n.$$

Then

$$\dot{K}_{p,m}^{\alpha_2,r}(\Omega) \hookrightarrow \dot{K}_p^{\alpha_1,r}(\Omega)$$

holds.

Proof. We use Theorem 3.5. Let $f \in C^\infty(\Omega) \cap \dot{K}_{p,m}^{\alpha_2,r}(\Omega)$. We write

$$\begin{aligned} \|f\|^r_{\dot{K}_p^{\alpha_1,r}(\Omega)} &= \sum_{k=-\infty}^{\infty} 2^{k\alpha_1 r} \left\| f\chi_{R_k\cap\Omega} \right\|_p^r \\ &= \sum_{2^{k+2}>\rho} 2^{k\alpha_1 r} \left\| f\chi_{R_k\cap\Omega} \right\|_p^r + \sum_{2^{k+2}\le\rho} 2^{k\alpha_1 r} \left\| f\chi_{R_k\cap\Omega} \right\|_p^r \\ &= I_1 + I_2. \end{aligned}$$

Let us estimate I_1. Let $t>0$ be such that $\frac{n}{m}<t<\min(p,\frac{n}{\max(0,\alpha_2+\frac{n}{p})})$. By Hölder's inequality, we obtain

$$\begin{aligned}|f(x)| &\lesssim \sum_{|\beta|\le m-1}\rho^{|\beta|-n}\int_{C_{x,\rho}}|D^\beta f(y)|dy+\sum_{|\beta|=m}\int_{C_{x,\rho}}\frac{|D^\beta f(y)|}{|x-y|^{n-m}}dy\\ &\lesssim \sum_{|\beta|\le m}\mathcal{M}_t((D^\beta f)\chi_\Omega)(x)\end{aligned}$$

for any $x\in R_k\cap\Omega$. Therefore

$$\begin{aligned}I_1 &\lesssim \sum_{|\beta|\le m}\sum_{2^{k+2}>\rho}2^{k\alpha_1 r}\big\|\mathcal{M}_t((D^\beta f)\chi_\Omega)\chi_{R_k}\big\|_p^r\\ &\lesssim \sum_{|\beta|\le m}\big\|\mathcal{M}_t((D^\beta f)\chi_\Omega)\big\|_{\dot{K}_p^{\alpha_2,r}(\mathbb{R}^n)}^r\\ &\lesssim \|f\|_{\dot{K}_{p,m}^{\alpha_2,r}(\Omega)}^r,\end{aligned}$$

by Lemma 2.2.

Now we estimate I_2. We employ the same notation as in Theorem 3.12. We have

$$J_{1,k}(x)\lesssim \sum_{|\beta|\le m-1}\rho^{|\beta|-n}\int_{C_{x,\rho}}\frac{|D^\beta f(y)|}{|x-y|^{n-m}}dy,\quad x\in R_k\cap\Omega.$$

Therefore we need only to estimate $J_{2,k}$. We write

$$J_{2,k}(x)=\sum_{|\beta|=m}\int_{C_{x,2^{k-2}}}\frac{|D^\beta f(y)|}{|x-y|^{n-m}}dy+\sum_{|\beta|=m}\int_{B_{x,\rho,k}}\frac{|D^\beta f(y)|}{|x-y|^{n-m}}dy, \qquad (14)$$

where $B_{x,\rho,k}=C_x\cap\{y:2^{k-2}\le|x-y|<\rho\}$. Let $t>0$ be such that $m-\frac{n}{t}+\alpha_1-\alpha_2>0$ and $t<p$. By Hölder's inequality the first integral of (14) is bounded by,

$$\begin{aligned}&c2^{k(\alpha_2-\alpha_1)}\int_{C_{x,2^{k-2}}}\frac{|D^\beta f(y)|\chi_{\tilde{R}_k\cap\Omega}(y)}{|x-y|^{n-m-\alpha_1+\alpha_2}}dy\\ &\lesssim 2^{k(\alpha_2-\alpha_1)}2^{(m-\frac{n}{t}+\alpha_1-\alpha_2)k}\mathcal{M}_t((D^\beta f)\chi_{\tilde{R}_k\cap\Omega})(x),\quad |\beta|=m.\end{aligned}$$

The boundedness of the maximal function on $L^{\frac{p}{t}}(\mathbb{R}^n)$ yield that

$$\big\|\mathcal{M}_t((D^\beta f)\chi_{\tilde{R}_k\cap\Omega})\big\|_p\lesssim\big\|(D^\beta f)\chi_{\tilde{R}_k\cap\Omega}\big\|_p,\quad |\beta|=m.$$

Now

$$\begin{aligned}\int_{B_{x,2^{k+2},k}}\frac{|D^\beta f(y)|}{|x-y|^{n-m}}dy &\lesssim 2^{(m-n)k}\int_{B_{x,2^{k+2},k}}|D^\beta f(y)|\chi_\Omega(y)dy\\ &\lesssim 2^{mk}\mathcal{M}((D^\beta f)\chi_\Omega)(x)\\ &\lesssim 2^{(m+\alpha_1-\alpha_2)k}2^{(\alpha_2-\alpha_1)k}\mathcal{M}((D^\beta f)\chi_\Omega)(x),\quad |\beta|=m.\end{aligned}$$

Let $j \in \mathbb{Z}$ be such that $2^{j-1} \le \rho < 2^j$. As in Theorem 3.12 we obtain

$$\int_{B_{x,\rho,k+4}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} dy \lesssim \sum_{i=k+2}^{j} 2^{(m-\frac{n}{p}-\alpha_2)i} 2^{\alpha_2 i} \left\| (D^\beta f)\chi_{R_i \cap \Omega} \right\|_p$$
$$\lesssim \|f\|_{\dot{K}^{\alpha_2,r}_{p,m}(\Omega)}, \quad |\beta| = m.$$

The desired estimate follows by Lemma 2.2 and the fact that $\alpha_1 + \frac{n}{p} > 0$. The proof is complete. □

Theorem 3.15. *Let domain $\Omega \subset \mathbb{R}^n$ satisfy the cone condition, $0 \in \Omega$ and $m \in \mathbb{N}_0$. Let $1 < p < \infty, 1 \le r < \infty$ and $\frac{n}{p} + \alpha_2 < m < n$. Assume that $\alpha_2 \ge \alpha_1 > 0$. Then*

$$\dot{K}^{\alpha_2,r}_{p,m}(\Omega) \hookrightarrow \dot{K}^{\alpha_1,r}_{\infty}(\Omega)$$

holds.

Proof. Let $f \in C^\infty(\Omega) \cap \dot{K}^{\alpha_2,r}_{p,m}(\Omega)$. We write

$$\|f\|^r_{\dot{K}^{\alpha_1,r}_{\infty}(\Omega)} = \sum_{k=-\infty}^{\infty} 2^{k\alpha_1 r} \|f\chi_{R_k \cap \Omega}\|^r_\infty$$
$$= \sum_{2^{k-2} > \rho} 2^{k\alpha_1 r} \|f\chi_{R_k \cap \Omega}\|^r_\infty + \sum_{2^{k-2} \le \rho} 2^{k\alpha_1 r} \|f\chi_{R_k \cap \Omega}\|^r_\infty$$
$$= S_1 + S_2.$$

Estimate of S_1. From Lemma 3.10 and Hölder's inequality, because of $m > \frac{n}{p}$, we obtain

$$|f(x)| \lesssim \sum_{|\beta| \le m} \left\| (D^\beta f)\chi_{\tilde{R}_k \cap \Omega} \right\|_p$$

for any $x \in R_k \cap \Omega$, since $C_{x,\rho} \subset \tilde{R}_k$. Hence

$$S_1 \lesssim \sum_{|\beta| \le m} \sum_{2^{k-2} > \rho} 2^{k(\alpha_1 - \alpha_2)r} 2^{k\alpha_2 r} \left\| (D^\beta f)\chi_{\tilde{R}_k \cap \Omega} \right\|^r_p$$
$$\lesssim \|f\|^r_{\dot{K}^{\alpha_2,r}_{p,m}(\Omega)},$$

because of $\alpha_2 \ge \alpha_1$.

Estimate of S_2. We have

$$|f(x)| \lesssim \sum_{|\beta| \le m} \int_{C_{x,\rho}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} dy$$
$$= \sum_{|\beta| \le m} \int_{C_{x,2^{k-2}}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} dy + \sum_{|\beta| \le m} \int_{B_{x,\rho,k}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} dy$$
$$= P_{1,k}(x) + P_{2,k}(x),$$

where $B_{x,\rho,k} = C_x \cap \{y : 2^{k-2} \leq |x-y| < \rho\}$. Using again Hölder's inequality we obtain

$$P_{1,k}(x) \leq \sum_{|\beta|\leq m} \int_{C_{x,2^{k-2}}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} \chi_{\tilde{R}_k\cap\Omega}(y)dy \lesssim 2^{k(m-\frac{n}{p})} \sum_{|\beta|\leq m} \big\|(D^\beta f)\chi_{\tilde{R}_k\cap\Omega}\big\|_p,$$

because of $m > \frac{n}{p}$. Therefore

$$\sum_{2^{k-2}\leq\rho} 2^{k\alpha_1 r} \sup_{x\in R_k\cap\Omega} (P_{1,k}(x))^r \lesssim \sum_{|\beta|\leq m} \sum_{2^{k-2}\leq\rho} 2^{k(m-\frac{n}{p}+\alpha_1-\alpha_2)} 2^{k\alpha_2 r} \big\|(D^\beta f)\chi_{\tilde{R}_k\cap\Omega}\big\|_p^r$$
$$\lesssim \big\|f\big\|^r_{\dot{K}^{\alpha_2,r}_{p,m}(\Omega)},$$

since $m - \frac{n}{p} + \alpha_1 - \alpha_2 > 0$. Now we estimate $P_{2,k}$. We write $P_{2,k} = T_{1,k} + T_{2,k} + T_{3,k}$, where

$$T_{1,k}(x) = \sum_{|\beta|\leq m} \int_{B_{x,\rho,k}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} \chi_{|\cdot|\leq\frac{|x|}{2}}(y)dy,$$

$$T_{2,k}(x) = \sum_{|\beta|\leq m} \int_{B_{x,\rho,k}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} \chi_{\frac{|x|}{2}\leq|\cdot|\leq 2|x|}(y)dy$$

and

$$T_{3,k}(x) = \sum_{|\beta|\leq m} \int_{B_{x,\rho,k}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} \chi_{|\cdot|>2|x|}(y)dy$$

Let us consider the first term. Using the fact that $|x-y| \geq |y|$ if $|y| \leq \frac{|x|}{2}$ and Hölder's inequality to obtain

$$T_{1,k}(x) \lesssim \sum_{|\beta|\leq m} \int_{|y|\leq 2^k} \frac{|D^\beta f(y)|\chi_\Omega(y)}{|y|^{n-m}} dy$$
$$= c \sum_{|\beta|\leq m} \sum_{i=-\infty}^{k} 2^{i(m-\frac{n}{p.}-\alpha_2)} 2^{i\alpha_2} \big\|(D^\beta f)\chi_{R_i\cap\Omega}\big\|_p$$
$$= c 2^{k(m-\frac{n}{p.}-\alpha_2)} \sum_{|\alpha|\leq m} \sum_{i=-\infty}^{k} 2^{(i-k)(m-\frac{n}{p.}-\alpha_2)} 2^{i\alpha_2} \big\|(D^\beta f)\chi_{R_i\cap\Omega}\big\|_p$$
$$\lesssim 2^{k(m-\frac{n}{p.}-\alpha_2)} \big\|f\big\|_{\dot{K}^{\alpha_2,r}_{p,m}(\Omega)},$$

since $m - \frac{n}{p.} - \alpha_2 > 0$. This leads to

$$\sum_{2^{k-2}\leq\rho} 2^{k\alpha_1 r} \sup_{x\in R_k\cap\Omega} (T_{1,k}(x))^r \lesssim \big\|f\big\|^r_{\dot{K}^{\alpha_2,r}_{p,m}(\Omega)} \sum_{2^{k-2}\leq\rho} 2^{k(m-\frac{n}{p.}-\alpha_2+\alpha_1)r}$$
$$\lesssim \big\|f\big\|^r_{\dot{K}^{\alpha_2,r}_{p,m}(\Omega)}.$$

Now we easily obtain

$$T_{2,k}(x) \lesssim \sum_{|\alpha|\le m} 2^{k(m-n)} \int_\Omega |D^\beta f(y)| \chi_{\tilde{R}_k}(y)dy$$
$$\lesssim 2^{k(m-\frac{n}{p})} \sum_{|\alpha|\le m} \left\|(D^\beta f)\chi_{\tilde{R}_k\cap\Omega}\right\|_p$$

by Hölder's inequality. Therefore

$$\sum_{2^{k-2}\le\rho} 2^{k\alpha_1 r} \sup_{x\in R_k\cap\Omega} (T_{2,k}(x))^r \lesssim \sum_{|\beta|\le m} \sum_{2^{k-2}\le\rho} 2^{k(m-\frac{n}{p.}-\alpha_2+\alpha_1)} 2^{k\alpha_2 r} \left\|(D^\beta f)\chi_{\tilde{R}_k\cap\Omega}\right\|_p^r$$
$$\lesssim \|f\|^r_{\dot{K}^{\alpha_2,r}_{p,m}(\Omega)}.$$

Let us estimate $T_{3,k}$. We have $|x-y| \ge \frac{|y|}{2}$, if $|y| > 2|x|$. Then

$$T_{3,k}(x) \lesssim \sum_{|\beta|\le m} \int_{2^k\le|y|\le 2\rho} \frac{|D^\beta f(y)|}{|y|^{n-m}} \chi_\Omega(y)dy$$
$$\lesssim \sum_{|\beta|\le m} \sum_{i=k}^{j+1} 2^{(m-\frac{n}{p}-\alpha_2)i} 2^{i\alpha_2} \left\|(D^\beta f)\chi_{R_i\cap\Omega}\right\|_p$$
$$\lesssim \|f\|^r_{\dot{K}^{\alpha_2,r}_{p,m}(\Omega)},$$

where $2^{j-1} \le \rho < 2^j, j \in \mathbb{Z}$. Using the fact that $\alpha_1 > 0$ we obtain

$$\sum_{2^{k-2}\le\rho} 2^{k\alpha_1 r} \sup_{x\in R_k\cap\Omega} (T_{3,k}(x))^r \lesssim \|f\|^r_{\dot{K}^{\alpha_2,r}_{p,m}(\Omega)}.$$

The proof is complete. □

Collecting the results obtained in Theorems 3.14 and 3.15 we have the following statement.

Theorem 3.16. *Let domain* $\Omega \subset \mathbb{R}^n$ *satisfy the cone condition,* $0 \in \Omega$ *and* $m \in \mathbb{N}_0$. *Let* $1 < p < q < \infty, 1 \le r < \infty, \alpha_2 \ge \alpha_1 > 0$ *and*

$$\max\left(\frac{n}{p}+\alpha_2, \frac{n}{p}+\alpha_2-\alpha_1\right) < m < n.$$

Then

$$\dot{K}^{\alpha_2,r}_{p,m}(\Omega) \hookrightarrow \dot{K}^{\alpha_1,r}_{q}(\Omega)$$

holds.

Proof. Let $f \in \dot{K}^{\alpha_2,r}_{p,m}(\Omega)$ and $\theta = \frac{p}{q}$. We have

$$\|f\|_{\dot{K}^{\alpha_1,r}_q(\Omega)} \leq \|f\|^{\theta}_{\dot{K}^{\alpha_1,r}_p(\Omega)} \|f\|^{1-\theta}_{\dot{K}^{\alpha_1,r}_\infty(\Omega)} \lesssim \|f\|_{\dot{K}^{\alpha_2,r}_{p,m}(\Omega)},$$

by Theorems 3.14 and 3.15. The proof is complete. □

In the previous results we have not treated the case $q < p$. The next theorem gives a positive answer.

Theorem 3.17. *Let domain $\Omega \subset \mathbb{R}^n$ satisfy the cone condition, $0 \in \Omega$ and $m \in \mathbb{N}_0$. Let $1 < q < p < \infty, 1 \leq r < \infty, \alpha_2 + \frac{n}{p} \geq \alpha_1 + \frac{n}{q} > 0$ and*

$$\max\left(\frac{n}{p}, \frac{n}{p} + \alpha_2, \frac{n}{p} - \frac{n}{q} + \alpha_2 - \alpha_1\right) < m < n.$$

Then

$$\dot{K}^{\alpha_2,r}_{p,m}(\Omega) \hookrightarrow \dot{K}^{\alpha_1,r}_q(\Omega)$$

holds.

Proof. We use Theorem 3.5. Let $f \in C^\infty(\Omega) \cap \dot{K}^{\alpha_2,r}_{p,m}(\Omega)$. We employ the same notation as in Theorem 3.14. Let us estimate I_1. Let $t > 0$ be such that $1 < \frac{n}{m} < t < \min(p, \frac{n}{\max(0,\alpha_2+\frac{n}{p})})$. We have

$$\begin{aligned} |f(x)| &\lesssim \sum_{|\beta|\leq m-1} \rho^{|\beta|-n} \int_{C_{x,\rho}} |D^\beta f(y)| dy + \sum_{|\beta|=m} \int_{C_{x,\rho}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} dy \\ &\lesssim \sum_{|\beta|\leq m} \mathcal{M}_t((D^\beta f)\chi_{\tilde{R}_k\cap\Omega})(x) \end{aligned}$$

for any $x \in R_k \cap \Omega$. Hölder's inequality together with the boundedness of the maximal function on $L^{\frac{p}{t}}(\mathbb{R}^n)$ leads to

$$\begin{aligned} I_1 &\lesssim \sum_{|\beta|\leq m} \sum_{2^{k+2}>\rho} 2^{(\alpha_1+\frac{n}{q}-\frac{n}{p}-\alpha_2)kr} 2^{k\alpha_2 r} \big\|\mathcal{M}_t((D^\beta f)\chi_{\tilde{R}_k\cap\Omega})\big\|^r_p \\ &\lesssim \|f\|^r_{\dot{K}^{\alpha_2,r}_{p,m}(\Omega)}, \end{aligned}$$

since $\alpha_2 + \frac{n}{p} \geq \alpha_1 + \frac{n}{q}$.

To estimate I_2 we need only to estimate $J_{2,k}$. Recall that

$$J_{2,k}(x) = \sum_{|\beta|=m} \int_{C_{x,2^{k-2}}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} dy + \sum_{|\beta|=m} \int_{B_{x,\rho,k}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} dy,$$

where $B_{x,\rho,k} = C_x \cap \{y : 2^{k-2} \le |x-y| < \rho\}$. By Hölder's inequality the first integral is bounded by

$$c \sum_{|\beta|=m} 2^{k(m-\frac{n}{p})} \big\|(D^\beta f)\chi_{\tilde{R}_k\cap\Omega}\big\|_p,$$

where the positive constant c is independent of k. Now

$$\begin{aligned}\int_{B_{x,2^{k+2},k}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} dy &\lesssim 2^{(m-n)k} \int_{B_{x,2^{k+2},k}} |D^\beta f(y)|\chi_\Omega(y) dy \\ &\lesssim 2^{mk} \mathcal{M}((D^\beta f)\chi_\Omega)(x), \quad |\beta| = m.\end{aligned}$$

Let $j \in \mathbb{Z}$ be such that $2^{j-1} \le \rho < 2^j$. As in Theorem 3.12 we obtain

$$\begin{aligned}\int_{B_{x,\rho,k+4}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} dy &\lesssim \sum_{i=k+2}^{j} 2^{(m-\frac{n}{p}-\alpha_2)i} 2^{\alpha_2 i} \big\|(D^\beta f)\chi_{R_i\cap\Omega}\big\|_p \\ &\lesssim \big\|f\big\|_{\dot{K}^{\alpha_2,r}_{p;m}(\Omega)}, \quad |\beta| = m.\end{aligned}$$

Using Hölder's inequality and Lemma 2.2, I_2 can be estimated from above by

$$\begin{aligned}&c\big\|f\big\|^r_{\dot{K}^{\alpha_2,r}_{p;m}(\Omega)} \sum_{2^{k+2}\le\rho} 2^{k(\alpha_1+\frac{n}{q})r} + \\ &\sum_{|\beta|\le m} \sum_{2^{k+2}\le\rho} 2^{k(m-\frac{n}{p}+\alpha_1-\alpha_2+\frac{n}{q})r} 2^{k\alpha_2 r} \Big(\big\|(D^\beta f)\chi_{\tilde{R}_k\cap\Omega}\big\|^r_p + \big\|\mathcal{M}((D^\beta f)\chi_\Omega)\chi_{R_k}\big\|^r_p \Big) \\ &\lesssim \big\|f\big\|^r_{\dot{K}^{\alpha_2,r}_{p;m}(\Omega)},\end{aligned}$$

since $\alpha_1 + \frac{n}{q} > 0$ and $m - \frac{n}{p} + \alpha_1 - \alpha_2 + \frac{n}{q} > 0$. The proof is complete. □

Acknowledgements

We would like to thank the referee for many valuable comments and suggestions.

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