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SINGULAR QUASILINEAR PROBLEMS WITH QUADRATIC GROWTH IN THE GRADIENT

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In this paper we consider the problem

$$\begin{cases} u \in H_0^1(\Omega), \\ -\mathrm{div}\,(A(x)Du) = H(x,u,Du) + \dfrac{a_0(x)}{|u|^\theta} + \chi_{\{u\neq 0\}}\, f(x) \quad \text{in} \quad \mathcal{D}'(\Omega), \end{cases}$$

where Ω is an open bounded set of $\mathbb{R}^N$ $(N \geq 3)$, $A(x)$ is a coercive matrix with coefficients in $L^\infty(\Omega)$, $H(x,s,\xi)$ is a Carathéodory function which satisfies for a given $\gamma > 0$ and some $c_0 \geq 0$

$$-c_0 A(x)\,\xi\xi \leq H(x,s,\xi)\,\mathrm{sign}(s) \leq \gamma A(x)\,\xi\xi \qquad \text{a.e. } x\in\Omega,\ \ \forall s\in\mathbb{R},\ \ \forall\xi\in\mathbb{R}^N.$$

The nonnegative term a_0 belongs to $L^{N/2}(\Omega)$, $\chi_{\{u\neq 0\}}$ is caracteristic function, f belongs to $L^{N/2}(\Omega)$ and $0 < \theta < 1$. For f and a_0 sufficiently small (and more precisely when f and a_0 satisfy the smallness condition (2.11)), we prove the existence of at least one solution u such that $e^{\delta|u|} - 1$ belongs to $H_0^1(\Omega)$ for some $\delta \geq \gamma$. Some a priori estimates are obtained.

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1. Introduction

In this paper, we consider the quasilinear problem

$$\begin{cases} u \in H_0^1(\Omega), \\ -\operatorname{div}(A(x)Du) = H(x,u,Du) + \dfrac{a_0(x)}{|u|^\theta} + \chi_{\{u\neq 0\}} f(x) \quad \text{in} \quad \mathcal{D}'(\Omega), \end{cases} \tag{1.1}$$

where Ω is a bounded open set of $\mathbb{R}^{\mathbb{N}}$, $N \geq 3$, $A(x)$ is a coercive matrix with bounded measurable coefficients. We assume that $a_0 \in L^{N/2}(\Omega)$, $a_0 \geq 0$, $0 < \theta \leq 1$, $\chi_{\{u\neq 0\}}$ is caracteristic function, $f \in L^{N/2}(\Omega)$ and $H(x,u,Du)$ is a Carathéodory function with quadratic growth in Du, more precisely

$$|H(x,s,\xi)| \leq c|\xi|^2,$$

for some positive constant c.

Under suitable smallness conditions on $||a_0||_{N/2}$ and $||f||_{N/2}$ we prove the existence of solution u of (2.1) which satisfies a regularity in the following sense. If we define w by

$$w = \delta^{-1}(e^{\delta|u|} - 1)\operatorname{sign}(u),$$

then w belongs to $H_0^1(\Omega)$ for every δ in a certain interval (γ, δ_0) which depends on a_0, f, the bound of H and the coercivity of the matrix A.

Compared to the results obtained in the latest papers, we prove in the present paper, as said above, the existence of (only) one solution of (2.1) in the case (2.5) (i.e. $a_0 \geq 0$) when a_0 and f satisfy the smallness condition (2.11), but our result is obtained in the general case of a nonlinearity $H(x,s,\xi)$ which satisfies only (2.3) with $f \in L^{N/2}(\Omega)$ and with a_0 to $L^{N/2}(\Omega)$.

We first review some recent results. The problem

$$\begin{cases} u \in H_0^1(\Omega), \\ -\operatorname{div}(A(x)Du) = H(x,u,Du) + a_0(x)u + f(x) \quad \text{in} \quad \mathcal{D}'(\Omega). \end{cases} \tag{1.2}$$

has been extensively studied by many authors, among which we will quote the works in a serie of papers [8], [9], [10] and [11], investigated the case where

$$a_0(x) \leq -\alpha_0 < 0. \tag{1.3}$$

Considering general nonlinear monotone operators, they proved an existence of at least one solution when a_0 satisfies (1.3), and when f belongs to $L^q(\Omega)$ $(q > \frac{N}{2})$. The solution is proven to verify some a priori estimates. The uniqueness of such a solution has been proved in [4] and [5] under some further structure assumptions.

The case where

$$a_0 = 0 \tag{1.4}$$

was considered in [2], [31], [20] and [23]. Among these papers the authors have considered nonlinear monotone operators and proved that if a_0 satisfies (1.4) and f belongs to $L^q(\Omega)$ $(q > \frac{N}{2})$ with $\|f\|_{L^q(\Omega)}$ sufficiently small then there exists at least one solution of (1.1) in $L^\infty(\Omega)$ and satisfies a priori estimates. The case where a_0 satisfies (1.4) and f only belongs to $L^{N/2}(\Omega)$ for $N \geq 3$ (and no longer to $L^q(\Omega)$ $(q > \frac{N}{2})$) was studied in [17] (and in [18] in the nonlinear monotone case), these authors proved that when $\|f\|_{L^{N/2}(\Omega)}$ is sufficiently small there exists at least one solution of (1.2) such that $e^{\delta|u|} - 1 \in H_0^1(\Omega)$ for some $\delta > \gamma$ and satisfies an a priori estimates. Similar results were obtained in the case where $f \in L^{N/2}(\Omega)$ in [16] for possibly unbounded domains when a_0 satisfies (1.3); in this case no smallness condition is required on f, however in [19] the authors discussed (also in the case of nonlinear monotone operators) when a_0 satisfies $a_0 \leq 0$ and f belongs to the Lorentz space $L^{N/2,\infty}(\Omega)$; in this case two smallness conditions should be fulfilled. Finally the case where a_0 satisfies $a_0 \leq 0$, let us quote the paper [33] the author investigated the asymptotic behaviour of the solution u of (1.2) when a_0 is a strictly positive constant sufficiently small, and proves that an ergodic constant appears at the limit $a_0 = 0$. Let us also mention the case where the nonlinearity $H(x,s,\xi)$ has the "good sign property", namely

$$-H(x,s,\xi)\,\mathrm{sign}\,(s) \geq 0. \tag{1.5}$$

The case where $a_0 \leq 0$ and f belongs to $H^{-1}(\Omega)$ was considered in [6] and [7], the authors proved the existence of at least one solution of (1.2) which belongs to $H_0^1(\Omega)$.

The case where

$$a_0 \geq 0, \quad a_0 \neq 0 \tag{1.6}$$

was considered in [3], [24] (and in [26] in the nonlinear monotone case), [25], [27] and [28]. The first attempt to study equations with a gradient term having quadratic growth, was carried out in [24] (see also [26] for extensions), the authors prove the existence of at least one solution of (1.2) when a_0 satisfies (1.6), and f belongs to $L^{N/2}(\Omega)$, with $\|f\|_{L^{N/2}(\Omega)}$ and $\|a_0\|_{L^q(\Omega)}$ sufficiently small.
In [27], the authors proved a similar result when a_0 satisfies (1.6) and f belongs to $L^q(\Omega)$ $(q > \frac{N}{2})$ with $\|f\|_{L^q(\Omega)}$ and $\|a_0\|_{L^q(\Omega)}$ sufficiently small. Moreover, they proved the existence of at least two solutions of (1.2) (which moreover belong to $L^\infty(\Omega)$, when $A(x) = Id$, $H(x) = \mu|\xi|^2$, $\mu > 0$, $f \in L^q(\Omega)$ $(q > \frac{N}{2})$, $f \geq 0$, and $a_0 \in L^q(\Omega)$ $(q > \frac{N}{2})$ with $\|f\|_{L^q(\Omega)}$ and $\|a_0\|_{L^q(\Omega)}$ sufficiently small).
In [3], the authors proved the existence of a continuum (u,λ) of solutions (with u which belongs to $L^\infty(\Omega)$) when $A(x) = Id$, $H(x,s,\xi) = \mu(x)\,|\xi|^2$, with

$\mu \in L^\infty(\Omega)$, $\mu(x) \geq \mu > 0$, $f \in L^q(\Omega)$ ($q > \frac{N}{2}$), $f \geq 0$, $f \neq 0$ and $a_0(x) = \lambda a_0^\star(x)$ with $a_0^\star \in L^q(\Omega)$, $a_0^\star \geq 0$ and $a_0^\star \neq 0$. In addition, under further conditions on f, these authors proved that this continuum is defined for $\lambda \in]-\infty, \lambda_0]$ with $\lambda_0 > 0$ and that there are at least two nonnegative solutions of (1.2) when $\lambda > 0$ is sufficiently small.

In the singular case, the problem (1.1) has been studied in many papers in the case where $A(x) = Id$, $H(x,.,.) = 0$, $f = 0$ and a_0 is smooth. Among these papers is a serie of papers [1], [12], [13], [14], [15] and [32].
In [21] and [22], the authors proved the existence of at least one nonnegative solution and a stability result for the following problem

$$\begin{cases} -\mathrm{div}\,(A(x)Du) = f(x)g(u) + l(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $A(x) \in L^\infty(\Omega)^{N\times N}$ is coercive matrix, $g : [0,+\infty) \to [0,+\infty)$ is continuous and $0 \leq g(s) \leq \frac{1}{s^\theta} + 1$, $\forall s > 0$, $0 < \theta \leq 1$; and f, $l \in L^r(\Omega)$ where r satisfies some conditions. In [13], the authors proved the existence, regularity and nonexistence results for problems whose model is

$$-\Delta u = \frac{f(x)}{u^\theta} \quad \text{in } \Omega,$$

with $u = 0$ on $\partial\Omega$, Ω is bounded open of $\mathbb{R}^N$, $\theta > 0$ and f is nonegative function on Ω and belongs to some Lebesgue spaces. For this, they have introduced an approximate problem by treating the singular term $\frac{1}{u^\theta}$ and construct an increasing sequence $(u_n)_{n\in\mathbb{N}}$ of solutions to nonsingular problem

$$\begin{cases} -\mathrm{div}\,(A(x)Du_n) = \dfrac{f_n(x)}{\left(u_n + \frac{1}{n}\right)^\theta} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f_n = \min(f(x), n)$. This sequence satisfies, for any $\omega \subset\subset \Omega$, and

$$u_n \geq u_{n-1} \geq \cdots \geq u_1 \geq C_\omega, \quad \forall x \in \omega.$$

The authors discussed in [1] the solution of the elliptic problem, with a gradient term and a singular nonlinearity

$$\begin{cases} -\Delta u = |\nabla u|^q + \dfrac{f}{g(u)} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded regular domain, $g : \mathbb{R}_+ \to \mathbb{R}$ is a continuous increasing function with additional hypotheses given, $1 < q \leq 2$ and f is a measurable nonnegative function and obtained optimal conditions on g, q which allow to get the existence positive solution for the largest possible class of datum f.

The plan of this paper is as follows: The precise statement of our result is given in Section 1 (Theorem 2.1), as well as the precise assumptions under which we are able to prove it, these conditions in particular include the smallness condition (2.11). Our method for proving Theorem 2.1 is based on an approximate result and by applying Shauder's fixed point theorem in a classical way. We will proceed by approximating the singular term $\frac{1}{|u|^\theta}$ and we will get some a priori estimates on the solutions of equivalent problem (4.2) (see also Proposition 4.1). The equivalence result is given in [18].
More precisely, the proof of Theorem 2.1 consists in carrying out a change of unknown function $w = \delta^{-1}(e^{\delta|u|-1}\operatorname{sign}(u)$, by transforming equation (3.4) into equation (3.8) in order to obtain the a priori estimate and the strong convergence of w_n. Another difficulty in the proof is to obtain the a priori estimate of the singular term in the set $\{u = 0\}$. For that, we use the method introduced in [21] and [22].

2. Main result

In this paper we consider the following quasilinear problem

$$\begin{cases} u \in H_0^1(\Omega), \\ -\operatorname{div}(A(x)Du) = H(x,u,Du) + \dfrac{a_0(x)}{|u|^\theta} + \chi_{\{u\neq 0\}} f(x) \quad \text{in } \mathcal{D}'(\Omega), \end{cases} \tag{2.1}$$

where Ω is an open bounded set of $\mathbb{R}^N$, $N \geq 3$, A is a coercive matrix with bounded measurable coefficients, i.e.

$$\begin{cases} A \in (L^\infty(\Omega))^{N\times N}, \\ \exists \alpha > 0, \quad A(x)\xi\xi \geq \alpha|\xi|^2 \quad \text{a.e. } x \in \Omega, \quad \forall \xi \in \mathbb{R}^N, \end{cases} \tag{2.2}$$

the nonlinearity $H(x,s,\xi)$ is a Carathéodory function with quadratic growth in ξ, and more precisely satisfies

$$\begin{cases} -c_0 A(x)\xi\xi \leq H(x,s,\xi)\operatorname{sign}(s) \leq \gamma A(x)\xi\xi, \\ \qquad\qquad\qquad\qquad \text{a.e. } x \in \Omega, \ \forall s \in \mathbb{R}, \ \forall \xi \in \mathbb{R}^N, \\ \text{where } \gamma > 0 \text{ and } c_0 \geq 0, \end{cases} \tag{2.3}$$

the function sign : $\mathbb{R} \to \mathbb{R}$ is defined by

$$\operatorname{sign}(s) = \begin{cases} +1 & \text{if } s > 0, \\ 0 & \text{if } s = 0, \\ -1 & \text{if } s < 0, \end{cases} \tag{2.4}$$

the coefficient a_0 satisfies

$$a_0 \in L^{N/2}(\Omega), \quad a_0 \geq 0, \quad a_0 \neq 0, \tag{2.5}$$

the exponent θ satisfies

$$0 < \theta \leq 1, \tag{2.6}$$

the caracteristic function is defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases} \tag{2.7}$$

and finally

$$f \in L^{N/2}(\Omega). \tag{2.8}$$

Since $N \geq 3$, let 2^* be the Sobolev's exponent defined by

$$\frac{1}{2^*} = \frac{1}{2} - \frac{1}{N},$$

and let C_N be the Sobolev's constant defined as the best constant such that

$$\|\varphi\|_{2^*} \leq C_N \|D\varphi\|_2, \quad \forall \varphi \in H_0^1(\Omega). \tag{2.9}$$

Since Ω is bounded, we equip the space $H_0^1(\Omega)$ with the norm

$$\|u\|_{H_0^1(\Omega)} = \|Du\|_{L^2(\Omega)^N}. \tag{2.10}$$

We finally assume that f and a_0 are sufficiently small (see Remark 2.3), and more precisely that

$$\|f\|_{N/2} \leq \frac{\alpha}{\gamma C_N^2} - \frac{\gamma^\theta}{\log^\theta(1+\gamma)} \|a_0\|_{N/2}, \tag{2.11}$$

Our main result is the following Theorem.

Theorem 2.1. *Assume that* (2.2), (2.3), (2.5), (2.6) and (2.8) *hold true. Assume moreover that the smallness condition* (2.11) *hold true.*

Then there exists at least one solution u of (2.1)*, which further satisfies*

$$\begin{cases} (e^{\delta|u|}-1) \in H_0^1(\Omega), \quad \forall \delta \geq \gamma \text{ such that} \\ |f\|_{N/2} \leq \dfrac{\alpha}{\delta C_N^2} - \dfrac{\delta^\theta}{\log^\theta(1+\delta)} \|a_0\|_{N/2}. \end{cases} \tag{2.12}$$

Remark 2.2. In the case where the function $H(x,s,\xi) = H(x,\xi)$ does not depend on s, assumption (2.3) is satisfied if and only if

$$|H(x,\xi)| \leq cA(x)\xi\xi,$$

for some $c > 0$.
Since A is a coercive matrix with bounded entries, the last condition is satisfied if and only if

$$|H(x,\xi)| \leq c|\xi|^2,$$

for some $c > 0$, which means that $H(x,\xi)$ has a quadratic growth with respect to ξ.

When $\gamma = 0$ in (2.3), the nonlinearity function $H(x,\xi)$ satisfies a sign condition and existence result can be proved for every $f \in H^{-1}(\Omega)$.

Remark 2.3. In this Remark, we consider that the open set Ω, the matrix A and the function H are fixed and we consider the functions a_0 and f as parameters.

Our set of assumptions on these parameters is made of the smallness condition (2.11).

Indeed, if, for example, a_0 is sufficiently small such that it satisfies

$$\frac{\alpha}{\gamma C_N^2} - \frac{\gamma^\theta}{\log^\theta(1+\gamma)}\|a_0\|_{N/2} > 0,$$

then the smallness condition (2.11) is satisfied if $\|f\|_{N/2}$ (and therefore $\|f\|_{H^{-1}(\Omega)}$, since $L^{N/2}(\Omega) \subset H^{-1}(\Omega)$ is sufficiently small).

Similarly, if, for example, f is sufficiently small such that

$$\|f\|_{N/2} \leq \frac{\alpha}{\gamma C_N^2},$$

then the smallness condition (2.11) is always satisfied if $\|a_0\|_{N/2}$ is sufficiently small.

Remark 2.4. The smallness condition (2.11) is in some sense sharp, implies that δ is bounded when the function a_0 satisfies the assumption (2.12), we have

$$\frac{\alpha}{\delta C_N^2} - \frac{\delta^\theta}{\log^\theta(1+\delta)}\|a_0\|_{N/2} \geq 0,$$

this implies

$$\frac{\alpha}{\delta C_N^2} \geq \frac{\delta^\theta}{\log^\theta(1+\delta)}\,\|a_0\|_{N/2} \geq a_0\|_{N/2}.$$

Finally

$$\gamma \leq \delta \leq \frac{\alpha}{C_N^2\|a_0\|_{_{N/2}}}.$$

Our proof of Theorem 2.1 is based on an approximate result which will be stated and proved in Section 3.

3. Proof of Theorem 2.1

The proof of Theorem 2.1 will be made in six steps.

Step 1: *Approximation and change of unknown function*

For $n \in \mathbb{N}$, consider two sequences a_n and f_n such that

$$a_n(x) = \begin{cases} a_0(x) & 0 \leq a_0 \leq n \\ \\ n & \text{otherwise,} \end{cases} \tag{3.1}$$

and

$$f_n(x) = T_n f(x), \tag{3.2}$$

where T_n is defined in (3.26).

For $n \in \mathbb{N}$, define $H_n(x,s,\xi)$ by

$$H_n(x,s,\xi) = \frac{H(x,s,\xi)}{1+\frac{1}{n}|H(x,s,\xi)|}. \tag{3.3}$$

Observe that $H_n(x,s,\xi)$ satisfies $|H_n(x,s,\xi)| \leq H(x,s,\xi)$ as well as (2.3).

Since $a_n(x)$, $|f_n(x)|$ and $H_n(x,s,\xi)$ are bounded, a classical result of J. Leray and J.-L. Lions [29, 30] asserts that the following approximate problem (3.4) has at least one solution.

$$\begin{cases} u_n \in H_0^1(\Omega), \\ \\ -\mathrm{div}\,(A(x)Du_n) = H_n(x,u_n,Du_n) + \dfrac{a_n(x)}{(|u_n|+\frac{1}{n})^\theta} + \chi_{\{u_n\neq 0\}}\, f_n(x) \ \text{ in } \mathcal{D}'(\Omega). \end{cases} \tag{3.4}$$

Observe that u_n belongs to $L^\infty(\Omega)$ for each n given since $a_n(x) \in L^\infty(\Omega)$, $f_n(x) \in L^\infty(\Omega)$, and $H_n(x,u_n,Du_n) \in L^\infty(\Omega)$.

Let $\delta > 0$ be fixed satisfies

$$\begin{cases} \gamma \leq \delta \quad \text{such that} \\ \|f\|_{N/2} \leq \dfrac{\alpha}{\delta C_N^2} - \dfrac{\delta^\theta}{\log^\theta(1+\delta)} \|a_0\|_{N/2}. \end{cases} \tag{3.5}$$

Define

$$w_n = \phi(u_n), \tag{3.6}$$

where

$$\varphi(s) = \delta^{-1}(e^{\delta|s|} - 1)\ \mathrm{sign}(s), \quad \forall s \in \mathbb{R}.$$

Observe that $\varphi \in C^1(\mathbb{R})$ with $\varphi(0) = 0$, we have at least formally

$$\begin{cases} w_n \in H_0^1(\Omega) \cap L^\infty(\Omega), \\ e^{\delta u_n} = 1 + \delta|w_n|,\ \ Dw_n = e^{\delta|u_n|} Du_n,\ \ \mathrm{sign}(u_n) = \mathrm{sign}(w_n), \end{cases}$$

and

$$\begin{cases} -\mathrm{div}(A(x)Dw_n) \\ \qquad = -\delta e^{\delta|u_n|}\,\mathrm{sign}(u_n) A(x) Du_n Du_n \\ \qquad\qquad -e^{\delta|u_n|}\,\mathrm{div}(A(x)Du_n)\mathrm{sign}(u_n) \quad \text{in } \mathcal{D}'(\Omega), \end{cases} \tag{3.7}$$

where $-e^{\delta|u_n|}\mathrm{div}(A(x)Du_n)$ is the distrubution defined by

$$_{\mathcal{D}'(\Omega)}\left\langle -e^{\delta|u_n|}\mathrm{div}(A(x)Du_n), \varphi \right\rangle_{C_0^\infty(\Omega)} = \int_\Omega A(x) Du_n D(\varphi e^{\delta|u_n|})\, dx,$$

for any $\varphi \in C_0^\infty(\Omega)$ or $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

Since $e^{\delta|u_n|} = 1 + \delta|w_n|$, we deduce that w_n is, at least formally, a solution (see Proposition 4.1) of the problem

$$\begin{cases} -\mathrm{div}(A(x)Dw_n) = -K_\delta(x, w_n, Dw_n)\,\mathrm{sign}(w_n) \\ \qquad +(1+\delta|w_n|)\chi_{\{w_n \neq 0\}} f_n + \dfrac{1+\delta|w_n|}{(\delta^{-1}\log(1+\delta|w_n|) + \frac{1}{n})^\theta}\, a_n \quad \text{in} \quad \mathcal{D}'(\Omega), \\ w_n = 0, \quad \text{on} \quad \partial\Omega, \end{cases} \tag{3.8}$$

where the function $K_\delta : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is defined by the following formulas

$$K_\delta(x, s, \xi) = \frac{1 - \theta_n(x)}{1 + \delta|s|}\, \delta A(x) \xi\, \xi, \tag{3.9}$$

with

$$\theta_n(x) = \begin{cases} \dfrac{H_n(x,u_n,Du_n)}{\delta\,\mathrm{sign}(u_n)A(x)Dw_nDw_n} & \text{if} \quad \mathrm{sign}(u_n)A(x)Du_nDu_n \neq 0, \\ \\ 0 & \text{otherwise,} \end{cases} \tag{3.10}$$

from the condition (2.3) on H and $0 < \gamma \leq \delta$, we have

$$-\frac{c_0}{\delta} < \theta_n \leq 1, \quad \text{a.e.} \quad x \in \Omega. \tag{3.11}$$

When $\gamma \leq \delta$, this computation in particular implies that

$$K_\delta(x,s,\xi) \geq 0 \quad \text{a.e. } x \in \Omega, \quad \forall s \in \mathbb{R}, \quad \forall \xi \in \mathbb{R}^N. \tag{3.12}$$

Step 2: *A priori estimate*

Since that the right hand side of (3.8) belongs to $L^1(\Omega)$, we can use w_n which belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$, as a test function in (3.8).
Taking into the fact that $K_\delta(x,s,\xi) \geq 0$ (see (3.12)) and using Hölder's inequality with $\dfrac{1}{2^\star} + \dfrac{1}{2^\star} + \dfrac{2}{N} = 1$, we have

$$\begin{cases} \displaystyle\int_\Omega A(x)Dw_nDw_n\,dx \\ \\ \displaystyle\quad \leq \int_\Omega (1+\delta|w_n|)w_n\,f_n(x)\,dx \\ \\ \displaystyle\quad + \int_\Omega \frac{(1+\delta|w_n|)w_n}{\left(\delta^{-1}\log(1+\delta|w_n|)+\frac{1}{n}\right)^\theta}\,a_n(x)\,dx \end{cases} \tag{3.13}$$

By the coercivity condition (2.2), we get

$$\begin{cases} \displaystyle\alpha\int_\Omega |Dw_n|^2\,dx \leq \int_\Omega (1+\delta|w_n|)w_n\,f_n(x)\,dx \\ \\ \displaystyle\qquad\qquad + \int_\Omega \frac{(1+\delta|w_n|)w_n}{\left(\delta^{-1}\log(1+\delta|w_n|)+\frac{1}{n}\right)^\theta}\,a_n(x)\,dx. \end{cases} \tag{3.14}$$

Using the chains of Hölder's and Sobolev's inequalities (2.9) this implies that

$$\begin{cases} \displaystyle\int_\Omega (1+\delta|w_n|)w_n\,f_n(x)\,dx \\ \\ \displaystyle\qquad = \int_\Omega |w_n|\,f_n(x)\,dx + \delta\int_\Omega |w_n|^2\,f_n(x)\,dx \\ \\ \qquad \leq \|f\|_{H^{-1}(\Omega)}\|Dw_n\|_2 + \delta\,C_N^2\,\|f\|_{N/2}\,\|Dw_n\|_2^2. \end{cases} \tag{3.15}$$

Splitting Ω into $\Omega = \{|w_n| \le 1\} \cup \{|w_n| > 1\}$ and writing the last term of the right-hand side of (3.13) as

$$\left\{\begin{aligned} &\int_\Omega \frac{(1+\delta|w_n|)\,w_n}{\left(\delta^{-1}\log(1+\delta|w_n|)+\frac{1}{n}\right)^\theta}\,a_n(x)\,dx = \\ &\qquad \int_{\{|w_n|\le 1\}} \frac{(1+\delta|w_n|)\,w_n}{\left(\delta^{-1}\log(1+\delta|w_n|)+\frac{1}{n}\right)^\theta}\,a_n(x)\,dx \\ &\qquad + \int_{\{|w_n|>1\}} \frac{(1+\delta|w_n|)\,w_n}{\left(\delta^{-1}\log(1+\delta|w_n|)+\frac{1}{n}\right)^\theta}\,a_n(x)\,dx. \end{aligned}\right. \tag{3.16}$$

Since $||a_n||_{N/2}$ converges to $||a_0||_{N/2}$, $||f_n||_{N/2}$ to $||f||_{N/2}$ and using that the function $F(x) = \dfrac{x}{\log(1+x)}$ is increasing in $\mathbb{R}_+^\star$, we have

$$\left\{\begin{aligned} &\int_{\{|w_n|\le 1\}} \frac{(1+\delta|w_n|)\,w_n}{\left(\delta^{-1}\log(1+\delta|w_n|)+\frac{1}{n}\right)^\theta}\,a_n(x)\,dx \\ &\quad \le (1+\delta)\int_{|w_n|\le 1} a_n(x)\,|w_n|^{1-\theta}\left(\frac{\delta|w_n|}{\log(1+\delta|w_n|)}\right)^\theta dx \\ &\quad \le \frac{(1+\delta)\delta^\theta}{\log^\theta(1+\delta)}\int_{|w_n|\le 1} a_n(x)\,|w_n|^{1-\theta}\,dx \\ &\quad \le (1+\delta)C_\delta(\theta)\,C_N^{1-\theta}\,|\Omega|^{\frac{1+\theta}{2^\star}}\,\|a_0\|_{N/2}\,\|Dw_n\|_2^{1-\theta}, \end{aligned}\right. \tag{3.17}$$

where $\qquad C_\delta(\theta) = \dfrac{\delta^\theta}{\log^\theta(1+\delta)}$

and

$$\left\{\begin{aligned} &\int_{|w_n|\ge 1} \frac{(1+\delta|w_n|)\,w_n}{\left(\delta^{-1}\log(1+\delta|w_n|)+\frac{1}{n}\right)^\theta}\,a_n(x)\,dx \\ &\le C_\delta(\theta)\int_{|w_n|\ge 1}(1+\delta|w_n|)\,|w_n|\,a_n(x)\,dx \\ &\le C_\delta(\theta)\left(C_N\,|\Omega|^{1/2^\star}\|a_0\|_{N/2}\,\|Dw_n\|_2 + \delta\,C_N^2\|a_0\|_{N/2}\,\|Dw_n\|_2^2\right). \end{aligned}\right. \tag{3.18}$$

From (3.14), (3.15), (3.16), (3.17) and (3.18) we have

$$\begin{cases} \alpha \|Dw_n\|_2^2 \le (1+\delta) C_\delta(\theta)\, C_N^{1-\theta}\, |\Omega|^{\frac{1+\theta}{2^\star}}\, \|a_0\|_{N/2} \|Dw_n\|_2^{1-\theta} \\ \qquad + C_\delta(\theta)\, C_N\, |\Omega|^{1/2^\star}\, \|a_0\|_{N/2} \|Dw_n\|_2 \\ \qquad + \delta\, C_\delta(\theta)\, C_N^2\, \|a_0\|_{N/2} \|Dw_n\|_2^2 \\ \qquad + \|f\|_{H^{-1}(\Omega)} \|Dw_n\|_2 + \delta\, C_N^2 \|f\|_{N/2}\, \|Dw_n\|_2^2 \quad \text{if } w_n \neq 0, \end{cases} \tag{3.19}$$

dividing by $\|Dw_n\|_2^{1-\theta}$, this implies that (note that the result remains true in the case where $w_n = 0$)

$$\begin{cases} \left(\alpha - \delta\, C_\delta(\theta)\, C_N^2 \|a_0\|_{N/2} - \delta\, C_N^2\, \|f\|_{N/2}\right)\, \|Dw_n\|_2^{1+\theta} \\ \qquad \le \left(C_\delta(\theta)\, C_N\, |\Omega|^{1/2^\star}\, \|a_0\|_{N/2} + \|f\|_{H^{-1}(\Omega)} \right) \|Dw_n\|_2^{\theta} \\ \qquad + (1+\delta) C_\delta(\theta)\, C_N^{1-\theta}\, |\Omega|^{\frac{1+\theta}{2^\star}}\, \|a_0\|_{N/2}. \end{cases} \tag{3.20}$$

In view of the definition of (4.6) of the function Φ_δ (see also Figure 1), we have proved if w_n is any solution of (3.4), one has

$$\Phi_\delta(\|Dw_n\|_2) \le 0, \quad \text{if } \gamma \le \delta, \tag{3.21}$$

this implies that

$$\|Dw_n\| \le Z_\delta \;\; (\text{does not depend to } n). \tag{3.22}$$

where the constant $Z_\delta > 0$ satisfies

$$\Phi_\delta(Z_\delta) = 0. \tag{3.23}$$

Since $u_n = \delta^{-1}(\log(1+\delta |w_n|))\,\mathrm{sign}(w_n)$, and from (3.22) implies that

$$u_n \quad \text{is bounded in } H_0^1(\Omega). \tag{3.24}$$

Step 3: *Proof of regularity result*

Extracting a subsequence, still denoted n, we have, for some $u \in H_0^1(\Omega)$ and $w \in H_0^1(\Omega)$

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{weakly in } H_0^1(\Omega), \;\text{ a.e. in } \Omega, \\ w_n &\rightharpoonup w \quad \text{weakly in } H_0^1(\Omega), \;\text{ a.e. in } \Omega, \end{aligned}$$

where

$$w = \varphi(u) = \delta^{-1}(e^{\delta|u|-1})\,\mathrm{sign}(u).$$

Observe that u and w do not belong to $L^\infty(\Omega)$ in general.

If we consider another δ, say δ', which also satisfies

$$\gamma \le \delta' \quad \text{such that} \quad \|f\|_{N/2} \le \frac{\alpha}{\delta' C_N^2} - \left(\frac{\delta'}{\log(1+\delta')}\right)^\theta \|a_0\|_{N/2}. \tag{3.25}$$

The above a priori estimate (3.22) again shows that w'_n defined by

$$w'_n = \delta'^{-1}(e^{\delta'|u_n|-1})\,\mathrm{sign}(u_n),$$

is bounded in $H_0^1(\Omega)$, this proves that u is such that

$$(e^{\delta'|u|-1})\,\mathrm{sign}(u) \in H_0^1(\Omega), \quad \forall \delta' \text{ such that } \quad \gamma \le \delta' \text{ satisfies (3.25)},$$

that is (2.12).

Step 4: *An estimate for* $\displaystyle\int_{|w_n|>k} |Dw_n|^2$

Let us define, for $k \ge 0$, the fuction $T_k : \mathbb{R} \to \mathbb{R}$ is the usual truncation at height k defined by

$$T_k(s) = \begin{cases} -k & \text{if } s \le -k \\ s & \text{if } -k \le s \le +k \\ +k & \text{if } +k \le s, \end{cases} \tag{3.26}$$

and we define $G_k : \mathbb{R} \to \mathbb{R}$ as the remainder of the truncation at height k, namely

$$G_k(s) = s - T_k(s), \quad \forall s \in \mathbb{R}, \tag{3.27}$$

in other terms

$$G_k(s) = \begin{cases} s+k & \text{if } s \le -k \\ 0 & \text{if } -k \le s \le +k \\ s-k & \text{if } s \ge +k, \end{cases} \tag{3.28}$$

Since $G_k(w_n) \in H_0^1(\Omega)$, the use of $G_k(w_n)$ as test function in (3.8) is licit. This gives

$$\begin{cases} \displaystyle\int_\Omega A(x) Dw_n DG_k(w_n)\,dx + \int_\Omega K_\delta(x, w_n, Dw_n)\,\mathrm{sign}(w_n)\,G_k(w_n)\,dx \\ \displaystyle\qquad = \int_\Omega \frac{(1+\delta|w_n|)\,G_k(w_n)}{(\delta^{-1}\log(1+\delta|w_n|)+\frac{1}{n})^\theta}\,a_n(x)\,dx \\ \displaystyle\qquad\quad + \int_\Omega (1+\delta|w_n|) G_k(w_n)\,\chi_{\{w_n\neq 0\}}\,f_n(x)\,dx. \end{cases} \tag{3.29}$$

Using the coercivity (2.2) of the matrix A, we have for the first term of (3.29)

$$\left\{\begin{aligned} \int_\Omega A(x)Dw_n DG_k(w_n)\,dx &= \int_\Omega A(x)DG_k(w_n)DG_k(w_n)\,dx \\ &\ge \alpha\int_\Omega |DG_k(w_n)|^2\,dx. \end{aligned}\right. \tag{3.30}$$

On the other hand, since

$$\operatorname{sign}(s)\,G_k(s)\ge 0,\quad \forall s\in\mathbb{R}, \tag{3.31}$$

and since $K_\delta(x,s,\xi)\ge 0$, in view of (3.12), this implies

$$\int_\Omega K_\delta(x,w_n,Dw_n)\operatorname{sign}(w_n)\,G_k(w_n)\,dx\ge 0, \tag{3.32}$$

Let k be fixed, we have

$$\frac{(1+\delta|w_n|)G_k(w_n)}{\left(\delta^{-1}\log(1+\delta|w_n|)+\frac{1}{n}\right)^\theta}\,a_n \to \frac{(1+\delta|w|)G_k(w)}{(\delta^{-1}\log(1+\delta|w|))^\theta}\,a_0 \quad \text{a.e. in } \Omega,$$

and

$$(1+\delta|w_n|)\,G_k(w_n)\chi_{\{w_n\neq 0\}}\,f_n \to (1+\delta|w|)\,G_k(w)\chi_{\{w\neq 0\}}\,f \quad \text{a.e. in } \Omega.$$

On the other hand the following functions
$\dfrac{(1+\delta|w_n|)\,G_k(w_n)}{\left(\delta^{-1}\log(1+\delta|w_n|)+\frac{1}{n}\right)^\theta}\,a_n$ and $(1+\delta|w_n|)\,G_k(w)\,f_n$ are equiintegrable.

Indeed, from (3.17) and (3.18), since a_n strongly converges in $L^q(\Omega)$, and for every Borel set $E\subset\Omega$, we have

$$\left\{\begin{aligned} &\int_E \frac{(1+\delta|w_n|)G_k(w_n)}{\left(\delta^{-1}\log(1+\delta|w_n|)+\frac{1}{n}\right)^\theta}\,a_n\,dx \\ &\le \int_E \frac{(1+\delta|w_n|)|w_n|}{\left(\delta^{-1}\log(1+\delta|w_n|)+\frac{1}{n}\right)^\theta}\,a_n\,dx \\ &\quad \le (1+\delta)\,C_\delta(\theta)\,C_N^{1-\theta}\,|\Omega|^{\left(\frac{1}{q'}-\frac{1-\theta}{2^\star}\right)}\,\|Dw_n\|_2^{1-\theta}\left(\int_E |a_n|^q\,dx\right)^{1/q} \\ &\quad +C_\delta(\theta)\left(C_N\,|\Omega|^{1/2^\star}\,\|Dw_n\|_2\left(\int_E |a_n|^{N/2}\,dx\right)^{2/N}\right. \\ &\qquad\qquad\qquad \left. +\delta\,C_N^2\|Dw_n\|_2^2\left(\int_E |a_n|^{N/2}\,dx\right)^{2/N}\right) \\ &\quad \le c\left(\int_E |a_n|^q\,dx\right)^{1/q} + c'\left(\int_E |a_n|^{N/2}\,dx\right)^{2/N} \end{aligned}\right. \tag{3.33}$$

Thus Vitali's Theorem implies that

$$\frac{(1+\delta|w_n|)\,G_k(w_n)}{\big(\delta^{-1}\log(1+\delta|w_n|)+\frac{1}{n}\big)^{\theta}}\,a_n \to \frac{(1+\delta|w|)\,G_k(w)}{(\delta^{-1}\log(1+\delta|w_n|)^{\theta}}\,a_0 \quad \text{in } L^1(\Omega),$$

and the functions $(1+\delta|w_n|)\,G_k(w_n)\,f_n$ are too equiintegrable, since f_n strongly converges in $L^{N/2}(\Omega)$, and for Borel set $E\subset\Omega$, we have

$$\left\{\begin{aligned} &\int_E (1+\delta|w_n|)\,|G_k(w_n)|\,\chi_{\{w_n\neq 0\}}\,|f_n|\,dx \le \int_E (1+\delta|w_n|)\,|w_n|\,|f_n|\,dx\\ &\qquad\le \|(1+\delta|w_n|)\|_{2^\star}\|w_n\|_{2^\star}\left(\int_E |f_n|^{N/2}\,dx\right)^{2/N}\\ &\qquad\le c\left(\int_E |f_n|^{N/2}\,dx\right)^{2/N}.\end{aligned}\right. \tag{3.34}$$

By Vitali's theorem implies that

$$(1+\delta|w_n|)\,G_k(w_n)\,\chi_{\{w_n\neq 0\}}\,f_n \to (1+\delta|w|)\,G_k(w)\,\chi_{\{w\neq 0\}}\,f \quad \text{in } L^1(\Omega).$$

Using the strong convergence of (3.1) and (3.2) of a_n and f_n in $L^{N/2}(\Omega)$, the almost everywhere of w_n, the bound of $L^{2^\star}(\Omega)$ of w_n and Vitali's theorem, we have for every k fixed and for n tends to infinity.

$$\frac{(1+\delta|w_n|)\,G_k(w_n)}{\big(\delta^{-1}\log(1+\delta|w_n|)+\frac{1}{n}\big)^{\theta}}\,a_n \to \frac{(1+\delta|w|)\,G_k(w)}{(\delta^{-1}\log(1+\delta|w|))^{\theta}}\,a_0 \quad \text{strongly in } L^1(\Omega), \tag{3.35}$$

and

$$(1+\delta|w_n|)G_k(w_n)\,\chi_{\{w_n\neq 0\}}\,f_n \to (1+\delta|w|)G_k(w)\,\chi_{\{w\neq 0\}}\,f \quad \text{strongly in } L^1(\Omega). \tag{3.36}$$

Passing to the limit in (3.29), for any k fixed, we obtain

$$\left\{\begin{aligned} &\alpha \limsup_{n\to+\infty}\int_\Omega |DG_k(w_n)|^2\,dx\\ &\quad\le \int_\Omega \frac{(1+\delta|w|)G_k(w)}{(\delta^{-1}\log(1+\delta|w|))^{\theta}}\,a_0\,dx + \int_\Omega (1+\delta|w|)G_k(w)\,\chi_{\{w\neq 0\}}\,f\,dx\end{aligned}\right. \tag{3.37}$$

Since $|G_k(w)|\le|w|$ and $G_k(w)=0$ in the set $\{|w|\le k\}$ the right-hand side of (3.37) is bounded in $L^1(\Omega)$ and from above

$$\int_{|w|>k}\left(\frac{(1+\delta|w|)|w|}{(\delta^{-1}\log(1+\delta|w|))^{\theta}}\,a_0 + (1+\delta|w|)|w|\,\chi_{\{w\neq 0\}}\,f\right)dx,$$

which tends to zero when k tends to infinity.

We deduce that

$$\alpha \limsup_{n\to+\infty}\int_\Omega |DG_k(w_n)|^2\,dx \to 0 \quad \text{as } k\to+\infty. \tag{3.38}$$

Step 5: *Strong convergence of $DT_k(w_n)$ in $L^2(\Omega))^N$*

In this step, we will fix $k > 0$ and prove that

$$DT_k(w_n) \to DT_k(w) \quad \text{strongly in } (L^2(\Omega))^N, \quad \text{as } n \to +\infty, \quad \text{for } k \text{ fixed.} \tag{3.39}$$

Let k be fixed, we define

$$z_n = T_k(w_n) - T_k(w), \tag{3.40}$$

and we choose an increasing, C^1 function $\psi : \mathbb{R} \to \mathbb{R}$ such that

$$\psi(0) = 0, \quad \psi'(s) - (c_0 + \delta)|\psi(s)| \geq 1/2, \quad \forall s \in \mathbb{R}, \tag{3.41}$$

where c_0 is the constant which appears in the left-hand side of assumption (2.3) on H and there exist such functions ψ : for example $\psi(s) = se^{\lambda s^2}$ with $\lambda = (c_0 + \delta)^2/4$.

Since $z_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$, and since $\psi(0) = 0$, the function $\psi(z_n)$ belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$. The use of $\psi(z_n)$ as test function in (3.8) is licit. This gives

$$\begin{cases} \displaystyle\int_\Omega A(x) Dw_n Dz_n \psi'(z_n)\, dx + \int_\Omega K_\delta(x, w_n, Dw_n)\, \mathrm{sign}(w_n)\, \psi(z_n)\, dx = \\[2ex] \displaystyle\qquad + \int_\Omega \frac{(1 + \delta|w_n|)\, \psi(z_n)}{(\delta^{-1}\log(1 + \delta|w_n|) + \frac{1}{n})^\theta}\, a_n(x)\, dx \\[2ex] \displaystyle\qquad + \int_\Omega (1 + \delta|w_n|) \psi(z_n)\, \chi_{\{w_n \neq 0\}}\, f_n(x)\, dx. \end{cases} \tag{3.42}$$

Since

$$Dw_n = DT_k(w_n) + DG_k(w_n) = Dz_n + DT_k(w) + DG_k(w_n), \tag{3.43}$$

the first term of the left-hand side of (3.42) reads as

$$\begin{cases} \displaystyle\int_\Omega A(x) Dw_n Dz_n \psi'(z_n)\, dx = \int_\Omega A(x) Dz_n Dz_n \psi'(z_n)\, dx \\[2ex] \displaystyle\qquad + \int_\Omega A(x) DT_k(w) Dz_n \psi'(z_n)\, dx \\[2ex] \displaystyle\qquad + \int_\Omega A(x) DG_k(w_n) Dz_n \psi'(z_n)\, dx. \end{cases} \tag{3.44}$$

On the other hand, splitting Ω into $\Omega = \{|w_n| > k\} \cup \{|w_n| \leq k\}$, the second term of the left-hand side of (3.42) reads as

$$\begin{cases} \displaystyle\int_\Omega K_\delta(x, w_n, Dw_n)\, \mathrm{sign}(w_n) \psi(z_n)\, dx = \\[2ex] \displaystyle\qquad \int_{\{|w_n| > k\}} K_\delta(x, w_n, Dw_n)\, \mathrm{sign}(w_n) \psi(z_n)\, dx \\[2ex] \displaystyle\qquad + \int_{\{|w_n| \leq k\}} K_\delta(x, w_n, Dw_n)\, \mathrm{sign}(w_n) \psi(z_n)\, dx, \end{cases} \tag{3.45}$$

the first term of the right-hand side of (3.45), we claim that

$$\int_{\{|w_n|>k\}} K_\delta(x,w_n,Dw_n)\,\mathrm{sign}(w_n)\,\psi(z_n)\,dx \geq 0, \tag{3.46}$$

indeed in $\{|w_n|>k\}$, the integrand is nonnegative since on the first hand the function $K_\delta(x,w_n,Dw_n)\geq 0$ in view of (3.12) and $\delta\geq\gamma$, and since on the other hand one has

$$\mathrm{sign}(w_n)\,\psi(z_n)\geq 0 \quad \text{in } \{|w_n|>k\}, \tag{3.47}$$

indeed in $\{|w_n|>k\}$, one has $z_n = T_k(w_n)-T_k(w) = k\,\mathrm{sign}(w_n)-T_k(w)$, and therefore $\mathrm{sign}(z_n)=\mathrm{sign}(w_n)$; this implies

$$\mathrm{sign}(w_n)\,\psi(z_n) = \mathrm{sign}(z_n)\,\psi(z_n) = |\psi(z_n)| \quad \text{in } \{|w_n|>k\}, \tag{3.48}$$

which proves (3.46).

The second term of the right-hand side of (3.45), in view of (3.12) and $\delta\geq\gamma$, we obtain

$$|K_\delta(x,w_n,Dw_n)\,\mathrm{sign}(w_n)\,\psi(z_n)| \leq (c_0+\delta)\,|\psi(z_n)|A(x)Dw_nDw_n. \tag{3.49}$$

Since in view of (3.43) one has

$$Dw_n = Dz_n + DT_k(w^\star) \ \text{ in } \ \{|w_n|\leq k\},$$

and implies that

$$\begin{cases} \displaystyle\int_{\{|w_k|\leq k\}} (K_\delta(x,w_n,Dw_n))\,\mathrm{sign}(w_n)\,\psi(z_n)\,dx \\ \displaystyle\geq -\int_{\{|w_n|\leq k\}} (c_0+\delta)|\psi(z_n)|A(x)Dw_nDw_n\,dx \\ \displaystyle= -\int_{\{|w_n|\leq k\}} (c_0+\delta)|\psi(z_n)|A(x)(Dz_n+DT_k(w))(Dz_n+DT_k(w))\,dx \\ \displaystyle\geq -\int_\Omega (c_0+\delta)|\psi(z_n)|A(x)(Dz_n+DT_k(w))(Dz_n+DT_k(w))\,dx \\ \displaystyle\geq -\int_\Omega (c_0+\delta)|\psi(z_n)|A(x)Dz_nDz_n\,dx \\ \displaystyle-\int_\Omega (c_0+\delta)|\psi(z_n)| \\ \qquad (A(x)DT_k(w)Dz_n + A(x)Dz_n\,DT_k(w) + A(x)DT_k(w)DT_k(w))\,dx. \end{cases} \tag{3.50}$$

From (3.42), (3.44), (3.45), (3.46) and (3.50) we deduce that

$$
\begin{cases}
\displaystyle\int_\Omega A(x)Dz_nDz_n\,\left(\psi'(z_n)-(c_0+\delta)|\psi(z_n)|\right)\,dx \\
\displaystyle\leq -\int_\Omega A(x)DT_k(w)Dz_n\,\psi'(z_n)dx \\
\displaystyle-\int_\Omega A(x)DG_k(w_n)Dz_n\,\psi'(z_n)\,dx \\
\displaystyle+\int_\Omega (c_0+\delta)|\psi(z_n)| \\
\qquad (A(x)DT_k(w)Dz_n+A(x)Dz_nDT_k(w)+A(x)DT_k(w)DT_k(w))\,dx \\
\displaystyle+\int_\Omega\left(\frac{(1+\delta|w_n|}{\left(\delta^{-1}(\log(1+\delta|w_n|+\frac{1}{n}\right)^\theta}\,a_n(x)+(1+\delta|w_n|)\,\chi_{\{w_n\neq0\}}\,f_n(x)\right)\,\psi(z_n)\,dx.
\end{cases}
\tag{3.51}
$$

We claim that each term of the right-hand side of (3.51) tends to zero as n tends to infinity.

Since $\psi'(z_n)-(c_0+\delta)\,|\psi(z_n)|\geq 1/2$ by (3.41), and the matrix A is coercive (see (2.2)), this will imply that

$$z_n\to 0\quad \text{in } H_0^1(\Omega)\quad \text{strongly},$$

or in other terms (see the definition (3.40) of z_n) that

$$T_k(w_n)\to T_k(w)\quad \text{in } H_0^1(\Omega) \text{ strongly}\quad \text{as}\quad n\to+\infty.$$

In order to prove the claim let us recall that of the definition (3.40) of z_n one has

$$z_n\rightharpoonup 0\quad \text{in}\quad H_0^1(\Omega)\quad \text{weakly},\quad L^\infty(\Omega)\quad \text{weakly star and}\quad \text{a.e. in } \Omega\quad \text{as } n\to+\infty.$$

Since $\psi(0)=0$, this implies that $\psi(z_n)$ tends to zero almost everywhere in Ω and in $L^\infty(\Omega)$ weakly star as n tends to infinity, which is in turn implies that

$$Dz_n\,\psi'(z_n)=D\psi(z_n)\rightharpoonup 0\quad \text{in}\quad L^2(\Omega)^N\quad \text{weakly}\quad \text{as}\quad n\to+\infty.$$

This implies that the first term of the right-hand side of (3.51) tends to zero as n tends to infinity.

For the second term of the right-hand side of of (3.51) we observe that

$$A(x)DG_k(w_n)Dz_n=A(x)DG_k(w_n)(DT_k(w_n)-DT_k(w))=-A(x)DG_k(w_n)DT_k(w),$$

and that by Lebesgue's dominated convergence theorem

$$DT_k(w)\,\psi'(z_n) \to DT_k(w)\,\psi'(0) \quad \text{in} \quad L^2(\Omega)^N \quad \text{strongly} \quad \text{as} \quad n \to +\infty,$$

while $DG_k(w_n)$ tends to $DG_k(w)$ weakly in $L^2(\Omega)^N$.
Since almost everywhere one has $A(x)DG_k(w)DT_k(w) = 0$, the second term of the right-hand side of (3.51) tends to zero.

For the third term of the right-hand side of (3.51), we observe that

$$(c_0 + \delta)|\psi(z_n)|A(x)DT_k(w) \to 0 \quad \text{in } L^2(\Omega)^N \quad \text{strongly} \quad \text{as} \quad \to +\infty$$

by Lebesgue's dominated convergence Theorem, since $\psi(z_n)$ is bounded in $L^\infty(\Omega)$ and since $\psi(z_n)$ tends almost everywhere to zero because $\psi(0) = 0$. Since Dz_n is bounded in $L^2(\Omega)^N$, this implies that the first part of this third term tends to zero. A similar proof holds true for the two others parts of this third term.

Finally the fourth term of the right-hand side of (3.51) tends to zero, since the integrand converges almost everywhere to zero and is equiintegrable (see (3.15), (3.17) and (3.18)).

This proves that z_n tend to zero strongly in $H_0^1(\Omega)$, namely

$$DT_k(w_n) \to DT_k(w) \quad \text{strongly in } (L^2(\Omega))^N, \text{ as } n \to +\infty, \text{ for } k \text{ fixed}$$

Since we have

$$w_n - w = T_k(w_n) + G_k(w_n) - T_k(w) - G_k(w),$$

and using (3.38) and (3.39) we have

$$Dw_n \to Dw \quad \text{in } (L^2(\Omega))^N \text{ strongly as } n \to +\infty, \tag{3.52}$$

Thus w_n tends to w strongly in $H_0^1(\Omega)$. Since we have

$$u_n = \delta^{-1} \log(1 + \delta|w_n|),$$

it follows that

$$u_n \to u \quad \text{strongly in } H_0^1(\Omega).$$

Step 6: *Control of* $\displaystyle\int_{|u_n|\le\mu} \frac{a_n(x)}{(|u_n| + \frac{1}{n})^\theta}\,\varphi$ *when* μ *is small*

In this step we prove that

$$\lim_n \int_\Omega \frac{a_n(x)}{\left(|u_n| + \frac{1}{n}\right)^\theta}\,\varphi\,dx = \int_\Omega \frac{a_0(x)}{|u|^\theta}\,\varphi\,dx,$$

for all $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

First we observe that

$$\begin{cases} \displaystyle\int_\Omega \frac{a_n(x)}{\left(|u_n|+\frac{1}{n}\right)^\theta}\,\varphi\,dx = \int_\Omega A(x)Du_nD\varphi\,dx \\ \qquad\qquad\qquad\qquad \displaystyle -\int_\Omega H_n(u_n,Du_n)\,\varphi\,dx - \int_\Omega \chi_{\{u_n\neq 0\}} f_n\varphi\,dx. \end{cases} \tag{3.53}$$

Taking into account the boundness of the matrix A, using the Young's inequality and the Sobolev's inequality, we get

$$\begin{cases} \displaystyle\int_\Omega \frac{a_n(x)}{\left(|u_n|+\frac{1}{n}\right)^\theta}\,\varphi\,dx \\ \qquad \le \|A\|_\infty(\|Du_n\|_2^2+\|D\varphi\|_2^2) \\ \qquad\qquad +(c_0+\gamma)\|\varphi\|_\infty\,\|Du_n\|^2 + C_N\,|\Omega|^{1/2^\star}\|f\|_{N/2}\|D\varphi\|_2 \\ \qquad \le \|A\|_\infty(\|Du_n\|_2^2+\|D\varphi\|_2^2) \\ \qquad\qquad \displaystyle +(c_0+\gamma)\|\varphi\|_\infty\,\|Du_n\|_2^2 + \frac{C_N^2|\Omega|^{2/2^\star}}{2}\|f\|_{N/2}^2 + \frac{1}{2}\|D\varphi\|_2^2 \\ \qquad \le c + (c_\varphi\|Du_n\|_2^2 + c'\|D\varphi\|_2^2), \end{cases} \tag{3.54}$$

where c, c_φ and c' are the positive constants.

From now on, we consider a nonnegative $\varphi \in H_0^1(\Omega)\cap L^\infty(\Omega)$, applying Fatou's Lemma to the left-hand side of (3.54), we have.

$$\int_\Omega \frac{a_0(x)}{|u|^\theta}\,\varphi\,dx \le C_\varphi,$$

where C_φ does not depend to n. Hence $0 \le \dfrac{a_0(x)}{|u|^\theta}\,\varphi \in L^1(\Omega)$, for any nonnegative $\varphi \in H_0^1(\Omega)\cap L^\infty(\Omega)$.

As consequence, $\dfrac{1}{|s|^\theta}$ is unbounded as s tends to 0, we deduce that

$$\{u=0\} \subset \{a_0=0\},$$

up to set of zero Lebesgue measure.

From now on, we consider a nonnegative function $\varphi \in H_0^1(\Omega)\cap L^\infty(\Omega)$, and choising it was test function in the weak formulation, we have

$$\begin{cases} \displaystyle\int_\Omega A(x)\,Du_nD\varphi\,dx \\ \qquad \displaystyle = \int_\Omega H_n(u_n,Du_n)\,\varphi\,dx + \int_\Omega \frac{a_n(x)}{(|u_n|+\frac{1}{n})^\theta}\,\varphi\,dx + \int_\Omega \chi_{\{u_n\neq 0\}}\,f_n\,\varphi\,dx, \end{cases} \tag{3.55}$$

we want to pass to the limit in the second right-hand side of (3.55) as n tends to infinity. For $\mu > 0$ fixed, we consider the second right-hand side of (3.55)

$$\int_\Omega \frac{a_n(x)}{(|u_n|+\frac{1}{n})^\theta}\,\varphi\,dx = \int_{|u_n|\le\mu} \frac{a_n(x)}{(|u_n|+\frac{1}{n})^\theta}\,\varphi\,dx + \int_{|u_n|>\mu} \frac{a_n(x)}{(|u_n|+\frac{1}{n})^\theta}\,\varphi\,dx \tag{3.56}$$

Applying Lemma 1.1 of [34], we have that for $\mu > 0$, $V_\mu(u_n)$ belongs to $H_0^1(\Omega)$, where $V_\mu :]-\infty,+\infty[\to\ [0,+\infty[$ is defined by

$$V_\mu(s) = \begin{cases} 0 & s < -2\mu \\ 2+\dfrac{s}{\mu} & -2\mu \le s < -\mu \\ 1 & -\mu \le s \le \mu \\ 2-\dfrac{s}{\mu} & \mu < s < 2\mu \\ 0 & s \ge 2\mu. \end{cases}$$

Since $V_\mu(u_n) \in H_0^1(\Omega)$, the use of $(V_\mu(u_n)\,\varphi)$ as test function in (3.4) is licit. This gives

$$\begin{cases} \displaystyle\int_{|u_n|\le\mu} \frac{a_n(x)}{(|u_n|+\frac{1}{n})^\theta}\,\varphi\,dx \le \int_\Omega A(x)\,Du_n\,D(V_\mu(u_n)\,\varphi)dx \\ \qquad\qquad \displaystyle -\int_\Omega H_n(u_n,Du_n)V_\mu(u_n)\,\varphi\,dx - \int_\Omega \chi_{\{u_n\neq 0\}}\,f_n\,V_\mu(u_n)\,\varphi\,dx \end{cases} \tag{3.57}$$

The first term of the right-hand side of (3.57) can be written

$$\int_\Omega A(x)\,Du_n\,D(V_\mu(u_n)\varphi)\,dx = \int_\Omega A(x)\,Du_n\,D\varphi\,V_\mu(u_n)\,dx. \tag{3.58}$$

Indeed, splitting Ω into $\Omega = \{|u_n| \le \mu\} \cup \{|u_n| > \mu\}$

$$\begin{cases} \displaystyle\int_\Omega A(x)\,Du_n Du_n\,V'_\mu(u_n)\,\varphi\,dx = \\ \qquad\qquad \displaystyle -\frac{1}{\mu}\int_{\{u_n\ge 0\}} A(x)\,Du_n\,Du_n\,\varphi\,dx \\ \qquad\qquad \displaystyle +\frac{1}{\mu}\int_{\{u_n<0\}} A(x)\,Du_n\,Du_n\,\varphi\,dx \end{cases} \tag{3.59}$$

$$\begin{cases} \displaystyle\frac{1}{\mu}\int_{\{u_n<0\}} A(x)\,Du_n\,Du_n\,\varphi\,dx \\ \qquad\qquad \displaystyle = \frac{1}{\mu}\int_{\{-u_n>0\}} A(x)\,D(-u_n)\,D(-u_n)\,\varphi\,dx \\ \qquad\qquad \displaystyle = \frac{1}{\mu}\int_{\{u_n>0\}} A(x)\,Du_n\,Du_n\,\varphi\,dx. \end{cases} \tag{3.60}$$

Finally, we have

$$\int_\Omega A(x)\,Du_n Du_n\,V'_\mu(u_n)\,\varphi\,dx = 0 \tag{3.61}$$

Since $D\varphi\,V_\mu(u_n)$ converges to $D\varphi\,V_\mu(u)$ strongly in $L^2(\Omega)^N$ while $A(x)Du_n$ converges to $A(x)Du$ weakly in $L^2(\Omega)^N$, we obtain

$$\lim_n \int_\Omega A(x)\,Du_n\;D\varphi\,V_\mu(u_n)\,dx = \int_\Omega A(x)\,Du\;D\varphi\,V_\mu(u)\,dx. \tag{3.62}$$

In the second term of the right-hand side of (3.57), we observe that $\varphi V_\mu(u_n)$ is bounded in $L^\infty(\Omega)$ and

$$H_n(u_n,Du_n)\;\varphi V_\mu(u_n) \le \|\varphi\|_\infty(c_0+\gamma)|Du_n|^2,$$

which implies that the functions $H_n(u_n,Du_n)\;\varphi V_\mu(u_n)$ are equiintegrable since Du_n strongly converges to Du in $L^2(\Omega)^N$, we have

$$\lim_n \int_\Omega H_n(u_n,Du_n)\;\varphi V_\mu(u_n)\,dx = \int_\Omega H(u,Du)\;\varphi V_\mu(u)\,dx. \tag{3.63}$$

In the third term of the right-hand side of (3.57), the functions $f_n\,\varphi\,V_\mu(u_n)$ are equiintegrable, since f_n strongly converges in $L^{N/2}(\Omega)$ and $V_\mu(u_n)$ converges to $V_\mu(u)$ strongly in $L^{2^\star}(\Omega)$. Thus Vitali's theorem implies that

$$\lim_n \int_\Omega \chi_{\{u_n\neq 0\}}\,f_n\,\varphi V_\mu(u_n)\,dx = \int_\Omega \chi_{\{u\neq 0\}}\,f\;\varphi V_\mu(u)\,dx. \tag{3.64}$$

Together with (3.57), the three limits (3.62), (3.63) and (3.64) imply that

$$\begin{cases} \displaystyle\lim_n \int_{|u_n|\le\mu} \frac{a_n(x)}{\left(|u_n|+\frac{1}{n}\right)^\theta}\,\varphi\,dx \le \int_\Omega A(x)\,Du D\varphi\;V_\mu(u)\,dx \\ \displaystyle\qquad\qquad + \int_\Omega H(u,Du)\;\varphi V_\mu(u)\,dx + \int_\Omega \chi_{\{u\neq 0\}}\,f\varphi V_\mu(u)\,dx \end{cases} \tag{3.65}$$

Since $V_\mu(u)$ converges to $\chi_{\{u=0\}}$ a.e. in Ω, as $\mu\to 0$ and since $u\in H_0^1(\Omega)$, then

$$\big(A(x)\,Du D\varphi + H(u,Du)\,\varphi + \chi_{\{u\neq 0\}}\,f\,\varphi\big)\;V_\mu(u)\to 0 \quad \text{a.e. in } \Omega, \quad \text{as } \mu\to 0. \tag{3.66}$$

Applying the Lebesgue's dominated convergence Theorem on the right-hand side of (3.65), we obtain that

$$\lim_{\mu\to 0}\,\lim_n \int_{\{|u_n|\le\mu\}} \frac{a_n(x)}{\left(|u_n|+\frac{1}{n}\right)^\theta}\,\varphi\,dx = 0. \tag{3.67}$$

Finally, let us pass to limit in n for $\mu>0$ fixed in the second term of the right-hand side of (3.56)

$$\int_{\{|u_n|>\mu\}} \frac{a_n(x)}{\left(|u_n|+\frac{1}{n}\right)^\theta}\,\varphi\,dx = \int_\Omega \frac{a_n(x)}{\left(|u_n|+\frac{1}{n}\right)^\theta}\,\chi_{\{|u_n|>\mu\}}\,\varphi\,dx.$$

Using that u_n converges to u a.e. on Ω, we have

$$\frac{a_n(x)}{\left(|u_n|+\frac{1}{n}\right)^{\theta}}\,\varphi \to \frac{a_0(x)}{|u|^{\theta}}\,\varphi \quad \text{a.e. on } \Omega,$$

and

$$\chi_{\{|u_n|>\mu\}} \underset{n}{\to} \chi_{\{|u|>\mu\}} \quad \text{on } \{x\in\Omega\,:\,u(x)\neq\mu\},$$

defining the set $\mathcal{C}$ by

$$\mathcal{C}=\{\mu>0, \quad \text{meas}\{x\in\Omega\,:\,u(x)=\mu\}>0\},$$

and choising $\mu\notin\mathcal{C}$, Lebesgue's dominated convergence theorem implies that

$$\lim_n \int_{\{|u_n|>\mu\}} \frac{a_n(x)}{\left(|u_n|+\frac{1}{n}\right)^{\theta}}\,\varphi\,dx = \int_{\{|u|>\mu\}} \frac{a_0(x)}{|u|^{\theta}}\,\varphi\,dx, \quad \forall\mu\notin\mathcal{C}. \tag{3.68}$$

As the set $\mathcal{C}$ is at most countable, choising μ such that $\mu\notin\mathcal{C}$ and using the fact that

$$\chi_{\{|u|>\mu\}} \to \chi_{\{|u|>0\}} \quad \text{as } \mu\to 0,$$

the fact $\dfrac{a_0(x)}{|u|^{\theta}}\,\varphi$ belongs to $L^1(\Omega)$.
Finally, we have proved that

$$\int_{\{|u|>\mu\}} \frac{a_0(x)}{|u|^{\theta}}\,\varphi\,dx \to \int_{\{|u|>0\}} \frac{a_0(x)}{|u|^{\theta}}\,\varphi\,dx = \int_{\Omega} \frac{a_0(x)}{|u|^{\theta}}\,\varphi\,dx \quad \text{as } \mu\to 0.$$

Using (3.67) and (3.68), we deduce

$$\lim_n \int_{\Omega} \frac{a_n(x)}{\left(|u_n|+\frac{1}{n}\right)^{\theta}}\,\varphi\,dx = \int_{\Omega} \frac{a_0(x)}{|u|^{\theta}}\,\varphi\,dx, \quad \forall\varphi\in H_0^1(\Omega)\cap L^{\infty}(\Omega),\ \varphi\geq 0. \tag{3.69}$$

Moreover, decomposing any $\varphi=\varphi^+-\varphi^-$ and observing that (3.69) is linear in φ, we deduce that (3.69) holds for every $\varphi\in H_0^1(\Omega)\cap L^{\infty}(\Omega)$.

Remark 3.1. Since $u_n\in H_0^1(\Omega)$, one has for every $\mu>0$ fixed

$$\{u_n=0\}\subset\{|u_n|\leq\mu\},$$

this implies that

$$\int_{\{u_n=0\}} \frac{a_n(x)}{\left(|u_n|+\frac{1}{n}\right)^{\theta}}\,\varphi\,dx = 0 \quad \text{for every } \varphi\in H_0^1(\Omega)\cap L^{\infty}(\Omega).$$

As $u_n\to u$ strongly in $H_0^1(\Omega)$, it is then easy to pass to the limit in the approximate equation (3.4). This proves that u is a solution of (2.1). The proof of Theorem 2.1 is then complete.

4. Appendix

In this Appendix, we give an equivalent result of the approximate problem and the definition of the constant Z_δ which appears in Theorem 2.1 (see Lemma 4.2).

4.1. An equivalence result

Proposition 4.1. *Assume that* (2.2), (2.3), (2.5), (2.6), (2.8), (3.3), (3.1), (3.2) *hold true, and let* $\delta > 0$ *be fixed. Let the function* K_δ *be defined in* (3.9)*.*
If u_n *is any solution of* (2.1) *which satisfies*

$$(e^{\delta|u_n|} - 1) \in H_0^1(\Omega), \tag{4.1}$$

then the function w_n *defined by* (3.6)*, namely*

$$w_n = \delta^{-1}(e^{\delta|u_n|} - 1)\,\mathrm{sign}(w_n),$$

satisfies

$$\begin{cases} w_n \in H_0^1(\Omega), \\ -\mathrm{div}(A(x)Dw_n) + K_\delta(x, w_n, Dw_n)\,\mathrm{sign}(w_n) = \\ \quad (1+\delta|w_n|)\chi_{\{w_n\neq 0\}} f_n + \dfrac{1+\delta|w_n|}{(\delta^{-1}\log(1+\delta|w_n| + \frac{1}{n})^\theta} a_n(x) \qquad \text{in } \mathcal{D}'(\Omega). \end{cases} \tag{4.2}$$

Conversely, if w_n *is any solution of* (4.2)*, then the function* u_n *defined by*

$$u_n = \delta^{-1}\log(1+\delta|w_n|)\mathrm{sign}(w_n), \tag{4.3}$$

is a solution of (3.4) *which satisfies* (4.3)*.*

Proof. Define the function $\hat{f}_n$ by

$$\hat{f}_n(x) = \chi_{\{u_n\neq 0\}} f_n(x) + \frac{a_n(x)}{(|u_n| + \frac{1}{n})^\theta}.$$

In view of (3.6), one has:

$$\begin{cases} (1+\delta|w_n|)\chi_{\{w_n\neq 0\}} f_n(x) + \dfrac{(1+\delta|w_n|)a_n(x)}{\left(\delta^{-1}\log(1+\delta|w_n|) + \frac{1}{n}\right)^\theta} \\ \qquad = (1+\delta|w_n|)\left(\chi_{\{w_n\neq 0\}} f_n(x) + \dfrac{a_n(x)}{\left(\delta^{-1}\log(1+\delta|w_n|) + \frac{1}{n}\right)^\theta}\right) \\ \qquad = (1+\delta|w_n|)\left(\chi_{\{u_n\neq 0\}} f_n(x) + \dfrac{a_n(x)}{\left(|u_n| + \frac{1}{n}\right)^\theta}\right). \end{cases} \tag{4.4}$$

Then Proposition 4.1 becomes an immediate application of Proposition 1.8 of [17], once observes that

$$\hat{f}_n \in L^\infty(\Omega). \tag{4.5}$$

$\square$

4.2. Definition of Z_δ

The goal of this Subsection is to define the constant Z_δ which appear in Theorem 2.1. We will prove the following result.

Lemma 4.2. *For $\delta \geq 0$, let $\Phi_\delta : \mathbb{R}^+ \to \mathbb{R}$ (see Figure 1) be the function defined by*

$$\begin{cases} \Phi_\delta(X) = \left(\alpha - \dfrac{\delta^{1+\theta}}{\log^\theta(1+\delta)} C_N^2 \|a_0\|_{N/2} - \delta\, C_N^2\, \|f\|_{N/2}\right) X^{1+\theta} \\ \qquad\qquad - \left(\dfrac{\delta^\theta}{\log^\theta(1+\delta)} C_N\, |\Omega|^{1/2^\star}\, \|a_0\|_{N/2} + \|f\|_{H^{-1}(\Omega)}\right) X^\theta \\ \qquad\qquad - \dfrac{(1+\delta)\delta^\theta}{\log^\theta(1+\delta)} C_N^{1-\theta}\, |\Omega|^{\frac{1+\theta}{2^\star}}\, \|a_0\|_{N/2}, \end{cases} \tag{4.6}$$

where θ satisfies (2.6)*, namely* $0 < \theta < 1$. *and where C_N is the best constant in the Sobolev's inequality* (2.9).

Then, for $\delta \geq \gamma$, there exists a unique number Z_δ such that

$$\Phi_\delta(Z_\delta) = 0, \quad \text{and} \quad \forall X \leq Z_\delta \ :\ \Phi_\delta(X) \leq 0. \tag{4.7}$$

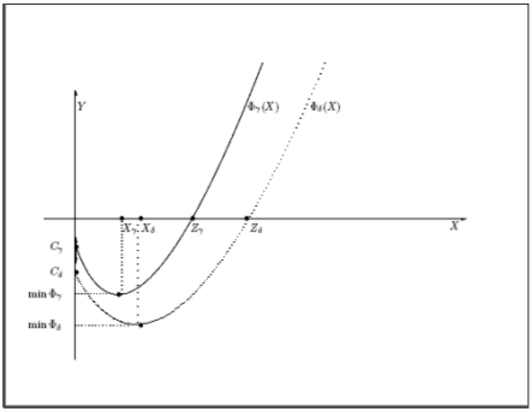

Figure 1: The graphs of the functions $\Phi_\delta(X)$ and $\Phi_\gamma(X)$.

Proof. Let us now study the family of functions $\Phi_\delta(X) : \mathbb{R}^+ \to \mathbb{R}$ defined by (4.6), from the smallness condition relative to δ (see 2.11), implies that

$$\alpha - \frac{\delta^{1+\theta}}{\log^\theta(1+\delta)} C_N^2 \|a_0\|_{N/2} - \delta C_N^2\, \|f\|_{N/2} \geq 0$$

Each function Φ_δ look like the restriction to $\mathbb{R}^+$ of a "convex parabola", when $0 < \gamma \leq \delta$ this "convex parabola" has a unique minimizer in X_δ of the function Φ_δ, and the minimum of Φ_δ, namely $\Phi_\delta(X_\delta)$ is negative and using the intermediate value theorem, then there exists Z_δ such that $\Phi_\delta(Z_\delta) = 0$. $\square$

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