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LINE COZERO-DIVISOR GRAPHS

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Let R be a commutative ring. The cozero-divisor graph of R denoted by $\Gamma'(R)$ is a graph with the vertex set $W^*(R)$, where $W^*(R)$ is the set of all non-zero and non-unit elements of R, and two distinct vertices x and y are adjacent if and only if $x \notin Ry$ and $y \notin Rx$. In this paper, we investigate when the cozero-divisor graph is a line graph. We completely present all commutative rings which their cozero-divisor graphs are line graphs. Also, we study when the cozero-divisor graph is the complement of a line graph.

1. Introduction

In 1988, Beck [12] introduced the concept of the zero-divisor graph. The zero-divisor graphs of commutative rings has been studied by several authors. We refer to the reader the papers [7, 8] and [9] for the properties of zero-divisor graphs. Also, the line zero divisor graphs was studied in [11]. For an arbitrary commutative ring R, the *cozero-divisor graph* $\Gamma'(R)$, as the dual notion of zero-divisor graphs, was introduced in [2]. Let $W^*(R)$ be the set of all non-zero and non-unit elements of R. The vertex set of $\Gamma'(R)$ is $W^*(R)$, and two distinct vertices x and y in $W^*(R)$ are adjacent if and only if $x \notin Ry$ and $y \notin Rx$, where Rz is the ideal generated by the element z in R. Many papers have been devoted to the study of cozero-divisor graphs, for instance see $[1-6]$. Motivated by

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the previous works on the zero divisor graph and cozero-divisor graph, in this paper we study line cozero-divisor graphs. Throughout this paper, all graphs are simple with no loops and multiple edges and R is a commutative ring with non-zero identity. We denote the set of all zero-divisor elements and the set of all unit elements of R by $\mathrm{Z}(R)$ and $U(R)$, respectively. If R has a unique maximal ideal $\mathfrak{m}$, then R is said to be a local ring and it is denoted by $(R,\mathfrak{m})$. Also, $\mathbb{F}_q$ denotes a finite field with q elements, for some positive integer q.

For basic definitions on graphs, one may refer to [14]. Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. If x is adjacent to y, then we write x—y or $\{x,y\} \in E(G)$. A graph G is *complete* if each pair of distinct vertices is joined by an edge. For a positive integer n, we use K_n to denote the complete graph with n vertices. Also, we say that G is *totally disconnected* if no two vertices of G are adjacent. Note that a graph whose vertex set is empty is an *empty graph*. The *complement* of G, denoted by $\overline{G}$ is a graph on the same vertices such that two distinct vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G. If $|V(G)| \geq 2$, then a *path* from x to y is a series of adjacent vertices $x — x_1 — x_2 — \cdots — x_n — y$. A *cycle* is a path that begins and ends at the same vertex in which no edge is repeated and all vertices other than the starting and ending vertex are distinct. We use P_n and C_n to denote the path and the cycle with n vertices, respectively. Suppose that H is a non-empty subset of $V(G)$. The subgraph of G whose vertex set is H and whose edge set is the set of those edges of G with both ends in H is called the subgraph of G *induced* by H. For every positive integer r, an r-partite graph is one whose vertex set can be partitioned into r subsets, or parts, in such a way that no edge has both ends in the same part. An r-partite graph is *complete r-partite* if any two vertices in different parts are adjacent. We denote the complete r-partite graph, with part sizes $n_1,\ldots,n_r$ by $K_{n_1,\ldots,n_r}$. For every $n \geq 2$, the *star graph* with n vertices is the complete bipartite graph with part sizes 1 and $n-1$. The *line graph* $L(G)$ is a graph such that each vertex of $L(G)$ represents an edge of G, and two vertices of $L(G)$ are adjacent if and only if their corresponding edges are incident in G.

Here is a brief summary of the present paper. In this paper, we investigate when the cozero-divisor graph is a line graph. Also, we study when the cozero-divisor graph is the complement of a line graph. In Sec. 2, we characterize all finite rings whose cozero-divisor graphs are line graphs. In Sec. 3, we characterize all finite non-local rings whose cozero-divisor graphs are complements of line graphs. Also, we prove that if $(R,\mathfrak{m})$ is a local ring with $\mathfrak{m} \neq 0$, $\Gamma'(R)$ is the complement of a line graph and $\{x,y\} \in E(\Gamma'(R))$, then $|Rx \cap Ry| \leq 2$. Finally, we determine a family of graphs can be occurred as the complement of line cozero-divisor graph of finite local rings.

2. When the Cozero-Divisor Graph is a Line Graph

In this section, we study when the graph $\Gamma'(R)$ is a line graph. We determine all finite commutative rings whose cozero-divisor graphs are line graphs. We will use one of the characterizations of line graphs which was proved in [13].

Theorem 2.1. *Let G be a graph. Then G is the line graph of some graph if and only if none of the nine graphs in Fig.* 1 *is an induced subgraph of G.*

Throughout the paper R is a finite commutative ring. By the structure theorem of Artinian rings [10, Theorem 8.7], there exists positive integer n such that $R \cong R_1 \times R_2 \times \cdots \times R_n$ and $(R_i, \mathfrak{m}_i)$ is a local ring for all $1 \leq i \leq n$. We use this theorem in the rest of the paper. Also, let e_i be the $1 \times n$ vector whose ith component is 1 and the other components are 0.

We first present the following lemma.

Lemma 2.2. *Let $R \cong R_1 \times R_2 \times \cdots \times R_n$ and let $(R_i, \mathfrak{m}_i)$ be a local ring for all $1 \leq i \leq n$. If $n \geq 4$, then $\Gamma'(R)$ is not a line graph.*

Proof. It is easy to see that $R(\sum_{i=4}^{n} e_i) \subsetneqq R(\sum_{i=3}^{n} e_i) \subsetneqq R(\sum_{i=2}^{n} e_i)$ and e_1 is adjacent to $\sum_{i=2}^{n} e_i, \sum_{i=3}^{n} e_i$ and $\sum_{i=4}^{n} e_i$. Hence the induced subgraph by the set $\{e_1, \sum_{i=2}^{n} e_i, \sum_{i=3}^{n} e_i, \sum_{i=4}^{n} e_i\}$ is isomorphic to $K_{1,3}$. Therefore by Theorem 2.1, $\Gamma'(R)$ is not a line graph. $\square$

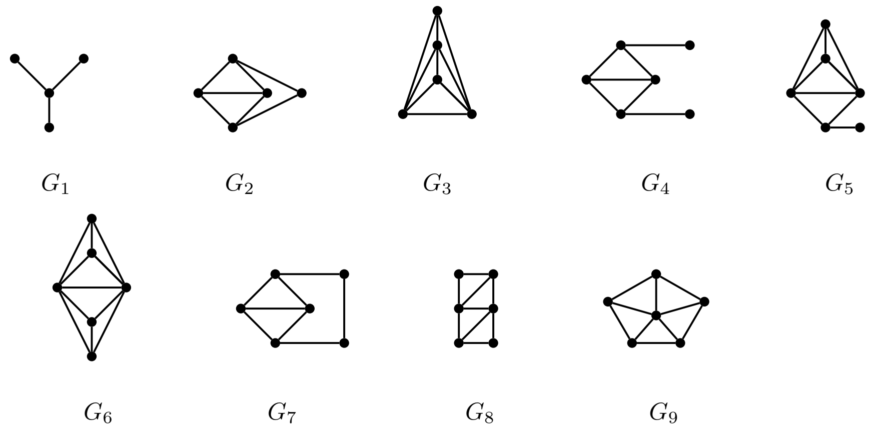


Fig. 1. Forbidden induced subgraphs of line graphs.

Lemma 2.3. *Let $R \cong R_1 \times R_2 \times R_3$ and let $(R_i, \mathfrak{m}_i)$ be a local ring for $i = 1, 2, 3$. Then $\Gamma'(R)$ is a line graph if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. Let $\Gamma'(R)$ be a line graph. If $|R_1| \geq 3$, then the induced subgraph by the set $\{e_2, e_3, e_1 + e_3, xe_1 + e_3\}$ is isomorphic to $K_{1,3}$, for every $x \in R_1 \setminus \{0, 1\}$

which is impossible. Hence $|R_1| = 2$ and similarly, $|R_2| = |R_3| = 2$. Therefore $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. We draw the graph $\Gamma'(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ in Fig. 2. One can easily see that the graph $\Gamma'(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ is the line graph of the graph $K_{2,3}$ which is drawn in Fig. 2. The proof of converse is clear. □

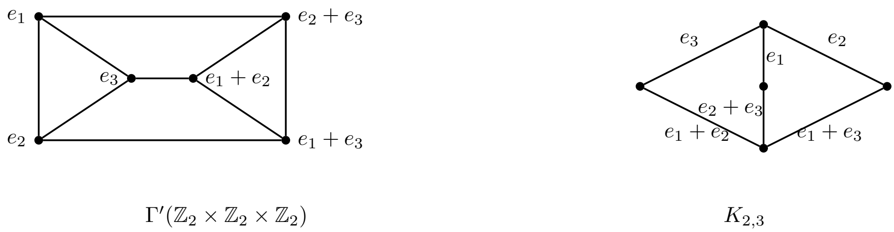


Fig. 2. $\Gamma'(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ is the line graph of $K_{2,3}$.

Lemma 2.4. *Let $R \cong R_1 \times R_2$ and let $(R_i, \mathfrak{m}_i)$ be a local ring for $i = 1,2$. Then $\Gamma'(R)$ is a line graph if and only if R is isomorphic to one of the rings $\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_3$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$.*

Proof. One side is obvious. For the other side assume that $\Gamma'(R)$ is a line graph. We know that $|\mathfrak{m}_i| \leq |U(R_i)|$, for $i = 1,2$. If $|\mathfrak{m}_1| \geq 2$, then we can put $a \in \mathfrak{m}_1^*$ and $u, v \in U(R_1)$. Then the induced subgraph on $\{ae_1, ue_1, ve_1, e_2\}$ is isomorphic to $K_{1,3}$, a contradiction. So, R_1 is a field. Similarly, R_2 is a field. Then $\Gamma'(R) = K_{|R_1|-1,|R_2|-1}$ and hence R is isomorphic to one of the rings $\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_3$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$. □

The next theorem, follows immediately from the above lemmas.

Theorem 2.5. *Let R be a commutative non-local ring. Then $\Gamma'(R)$ is a line graph if and only if R is isomorphic to one of the rings $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_3$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$.*

For the last case of our discussion, we must assume that $n = 1$. So, R is a local ring. Let $\mathfrak{m}$ be the only maximal ideal of R. We note that if R is a field, then $W^*(R) = \emptyset$ which implies that $\Gamma'(R)$ is an empty graph and so it is the line graph of the graph K_1. So, we may assume that R is a local ring which is not a field. This implies that $\mathfrak{m} \neq 0$. Also, it is clear that if $\Gamma'(R)$ is totally disconnected with t vertices, for some positive integer t, then $\Gamma'(R)$ is the line graph of $\bigcup_{i=1}^{t} K_2$. In the rest of this section, we study the case that R is a local ring with non-zero maximal ideal and $E(\Gamma'(R)) \neq \emptyset$. Our starting point is the following lemma.

Lemma 2.6. *Let $(R, \mathfrak{m})$ be a local ring with $\mathfrak{m} \neq 0$ and let $\Gamma'(R)$ be a line graph. If $\{x, y\} \in E(\Gamma'(R))$, then $|Rx \cap Ry| \leq 2$.*

Proof. By contradiction, suppose that $0 \neq a, b \in Rx \cap Ry$. If $a \in U(R)y$, then we have $y \in Ra \subseteq Rx$, which is impossible. Therefore $a \in \mathfrak{m}y$. Similarly, $b \in \mathfrak{m}y$ and so $R(y+a) = R(y+b) = Ry$. Now, the set $\{x, y, y+a, y+b\}$ determines an induced subgraph of the type $K_{1,3}$. Therefore by Theorem 2.1, $\Gamma'(R)$ is not a line graph, a contradiction. Hence $|Rx \cap Ry| \leq 2$. □

Lemma 2.7. *Let $(R, \mathfrak{m})$ be a local ring with $\mathfrak{m} \neq 0$, $\Gamma'(R)$ be a line graph and let $\{x, y\} \in E(\Gamma'(R))$. If $Rx \cap Ry = \{0\}$, then the following hold:*
(i) $Rx = \{0, x\}$ or $Rx = \{0, x, -x\}$.
(ii) $Ry = \{0, y\}$ or $Ry = \{0, y, -y\}$.

Proof. (i) We prove that $|Rx| \leq 3$. By contradiction, assume that $|Rx| \geq 4$. Let $a, b \in Rx \setminus \{0, x\}$. There are three following cases:

Case 1. $a, b \in U(R)x$. Then $Rx = Ra = Rb$ and the set $\{y, x, a, b\}$ determines an induced subgraph of the type $K_{1,3}$. This is a contradiction, by Theorem 2.1.

Case 2. $a, b \in \mathfrak{m}x$. Then $Rx = R(x+a) = R(x+b)$ and the set $\{y, x, x+a, x+b\}$ determines an induced subgraph of the type $K_{1,3}$, which is a contradiction, by Theorem 2.1.

Case 3. $a \in U(R)x$ and $b \in \mathfrak{m}x$. Then $Rx = Ra$ and $Rb \subseteq Rx$. Since $Ra = Rx$ and $\{x, y\} \in E(\Gamma'(R))$, y is adjacent to a. If $y \in Rb$, then $y \in Rx$, which is impossible. On the other hand, if $b \in Ry$, then $b \in Rx \cap Ry = \{0\}$, a contradiction. Therefore y is adjacent to b. Now, the set $\{y, x, a, b\}$ determines an induced subgraph of the type $K_{1,3}$, a contradiction.

By the above cases, we deduce that $|Rx| = 2, 3$. Clearly, if $|Rx| = 2$, then $Rx = \{0, x\}$. Also, it is not hard to see that if $|Rx| = 3$, then $Rx = \{0, x, -x\}$. This completes the proof.

(ii) It is similar to the proof of part (i). □

Now, we are in a position to prove one of the main results.

Lemma 2.8. *Let $(R, \mathfrak{m})$ be a local ring, $\mathfrak{m} \neq 0$, $E(\Gamma'(R)) \neq \emptyset$ and for every $\{x, y\} \in E(\Gamma'(R))$, let $Rx \cap Ry = \{0\}$. Then $\Gamma'(R)$ is a line graph if and only if it is a complete graph.*

Proof. Suppose that $\Gamma'(R)$ is a line graph. Let $A = \{x \in V(\Gamma'(R)) | Rx = \{0, x\}\}$, $B = \{x \in V(\Gamma'(R)) | Rx = \{0, x, -x\}\}$ and let C be the set of all isolated vertices of $\Gamma'(R)$. We note that the induced subgraph of $\Gamma'(R)$ by the set A is a complete graph. Also, there exists $r \geq 0$ such that $|B| = 2r$. Because we have $x, -x \in B$ or $x, -x \notin B$, for every $0 \neq x \in \mathfrak{m}$. Moreover, if $r > 0$, then the induced subgraph of $\Gamma'(R)$ by the set B is complete r-partite graph and every part is equal to $\{x, -x\}$, for some $x \in B$. Furthermore, by Lemma 2.7, $V(\Gamma'(R)) = A \cup B \cup C$. We use these facts in the rest of the proof. Since $E(\Gamma'(R)) \neq \emptyset$, $A \cup B \neq \emptyset$. Consider two following cases:

Case 1. $A = \emptyset$. We note that $E(\Gamma'(R)) \neq \emptyset$. This yields that $|B| = 2r > 0$ and B has two elements say b_1 and b_2 such that $b_1 \neq -b_2$ and $\{b_1, b_2\} \in E(\Gamma'(R))$. We claim that $C = \emptyset$. By contradiction, suppose that $c \in C$. If $c \in Rb_1$, then $c = b_1$ or $c = -b_1$. Hence c is not an isolated vertex, which is a contradiction. Therefore $c \notin Rb_1$. Similarly, $c \notin Rb_2$. Since c is an isolated vertex, we find that $b_1, b_2 \in Rc$. Assume that $b_1 = r_1 c$ and $b_2 = r_2 c$, for some $r_1, r_2 \in R$. If $r_1 \in U(R)$, then $Rc = Rb_1$. This implies that c and b_2 are adjacent, which is impossible. Hence $r_1 \in \mathfrak{m}$ and similarly, $r_2 \in \mathfrak{m}$. Since b_1 and b_2 are adjacent, we deduce that r_1 and r_2 are adjacent. Therefore $r_1, r_2 \in B$. Moreover, we conclude that $r_1 \in \{b_1, -b_1\}$ and $r_2 \in \{b_2, -b_2\}$. It follows that $c = 0$, a contradiction. Therefore $C = \emptyset$ and the claim is proved. This implies that $\Gamma'(R)$ is a complete r-partite graph, because $|B| = 2r$. Also, as we mentioned before, every part of $\Gamma'(R)$ is equal to $\{b, -b\}$, for some $b \in B$. If $|B| \geq 8$, then there exists $b_1, b_2, b_3, b_4 \in B$ such that $b_i \neq -b_j$, for every $i \neq j$. Now, the induced subgraph by the set $\{b_1, b_2, b_3, -b_3, b_4\}$ is isomorphic to G_3 (see Fig. 3), a contradiction. Hence $|B| = 4, 6$ and so $\Gamma'(R) = K_{2,2}$ or $\Gamma'(R) = K_{2,2,2}$. By [4, Lemma 2], we conclude that $\Gamma'(R) \neq K_{2,2}$. Therefore $\Gamma'(R) = K_{2,2,2}$. It follows that $\Gamma'(R)$ is a complete 3-partite graph. By [6, Corollary 3], $\Gamma'(R)$ is a triangle, which is impossible.

Case 2. $A \neq \emptyset$. Let $a_1 \in A$. First, we prove that $C = \emptyset$. By contradiction, suppose that $C \neq \emptyset$. We know that $Ra_1 = \{0, a_1\}$. This yields that $a_1 \in Rc$, for every $c \in C$. Also, if $B \neq \emptyset$, then $b \in Rc$, for every $b \in B$ and every $c \in C$. Since $\mathfrak{m}$ is finite, we find that there exists $c_0 \in C$ such that $\mathfrak{m} = Rc_0$. On the other hand, by [2, Theorem 2.7], we conclude that $\Gamma'(R)$ is totally disconnected, a contradiction. Therefore $C = \emptyset$.

Now, we prove that $B = \emptyset$. By contradiction, assume that $|B| = 2r > 0$ and $B = \{b_1, \ldots, b_{2r}\}$. Since $a_1 + b_1$ is a vertex of $\Gamma'(R)$, $a_1 + b_1 \in V(\Gamma'(R)) = A \cup B$. If $a_1 + b_1 \in A$, then $R(a_1 + b_1) = \{0, a_1 + b_1\}$ and so $a_1 + b_1 = -(a_1 + b_1) = a_1 - b_1$. This yields that $b_1 = -b_1$, a contradiction. Therefore $a_1 + b_1 \in B$. With no loss of generality, we may assume that $a_1 + b_1 = b_2$. Then $a_1 = b_2 - b_1$. Since $2b_1 \neq 0, b_1$, we have $2b_1 = -b_1$. Hence $3b_1 = 0$. Similarly, $3b_2 = 0$. This implies that $3a_1 = 3(b_2 - b_1) = 0$. On the other hand, we have $2a_1 \in Ra_1 = \{0, a_1\}$ which shows that $2a_1 = 0$. Hence $a_1 = 0$, a contradiction. Thus $B = \emptyset$ and $V(\Gamma'(R)) = A$. Therefore $\Gamma'(R)$ is a complete graph.

From the above cases, we conclude that if $\Gamma'(R)$ is a line graph, then it is a complete graph. Clearly, if $\Gamma'(R) = K_t$, for some positive integer t, then it is the line graph of $K_{1,t}$. This completes the proof. $\square$

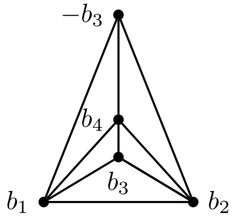


Fig. 3

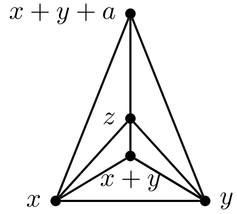


Fig. 4

A graph is said to be *planar* if it can be drawn in the plane so that its edges intersect only at their ends. A *subdivision* of a graph is a graph obtained by replacing edges of this graph with pairwise internally-disjoint paths. A remarkably simple characterization of planar graphs was given by Kuratowski in 1930, that says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$ [14].

Let $(R,\mathfrak{m})$ be a local ring with $\mathfrak{m} \neq 0$, $|Rx \cap Ry| = 2$, for some $\{x,y\} \in E(\Gamma'(R))$ and let $\Gamma'(R)$ be a line graph. In the following theorem, first we prove that $\Gamma'(R)$ is planar. Then by using [1, Proposition 2.7], we characterize all local rings whose cozero-divisor graphs are line graphs.

Lemma 2.9. *Let $(R,\mathfrak{m})$ be a local ring with $\mathfrak{m} \neq 0$. If there exists $\{x,y\} \in E(\Gamma'(R))$ such that $|Rx \cap Ry| = 2$, then $\Gamma'(R)$ is a line graph if and only if R is isomorphic to one of the following rings:*
$\mathbb{Z}_2[x,y]/(x^2-y^2,xy)$, $\mathbb{Z}_2[x,y]/(x^2,y^2)$, $\mathbb{Z}_4[x,y]/(x^2-2,xy,y^2-2,2x)$,
$\mathbb{Z}_4[x,y]/(x^2,xy-2,y^2)$, $\mathbb{Z}_4[x]/(x^2)$, $\mathbb{Z}_4[x]/(x^2-2x)$, $\mathbb{Z}_8[x]/(2x,x^2-4)$.

Proof. First assume that $(R,\mathfrak{m})$ is a local ring, $\Gamma'(R)$ is a line graph, $\{x,y\} \in E(\Gamma'(R))$ and $Rx \cap Ry = \{0,a\}$. We note that every element of the set $Rx \setminus \{0,a\}$ is adjacent to every element of the set $Ry \setminus \{0,a\}$. Since $\Gamma'(R)$ is a line graph and $K_{1,3}$ is not an induced subgraph of $\Gamma'(R)$, we find that $Rx = \{0,a,x,x+a\}$ and $Ry = \{0,a,y,y+a\}$. Since $x \notin Ry$ and $y \notin Rx$, we conclude that $x+y \notin Rx \cup Ry$. If $x \in R(x+y)$, then $x = r(x+y)$, for some $r \in \mathfrak{m}$. Hence $(1-r)x = ry$. This yields that $x = (1-r)^{-1}ry \in Ry$, which is impossible. Therefore $x \notin R(x+y)$. Similarly, $y \notin R(x+y)$. Thus $x+y$ is adjacent to both x and y. If $x+y$ is adjacent to a, then the set $\{x+y,x,x+a,a\}$ implies that $\Gamma'(R)$ has a $K_{1,3}$ as an induced subgraph, a contradiction. Therefore $a \in R(x+y)$. By the same argument as we saw before, $R(x+y) = \{0,a,x+y,x+y+a\}$. If $\Gamma'(R)$ has other vertex say z, then with no loss of generality, we may assume that there are the following cases:

Case 1. z is adjacent to x,y and $x+y$. Then the induced subgraph by the set $\{x,y,x+y,x+y+a,z\}$ is isomorphic to G_3 (see Fig. 4), a contradiction.

Case 2. z is adjacent to x and z is not adjacent to $x+y$. Then $x+y \in Rz$ and $Rz = R(x+y+z) = R(a+z)$. The set $\{x, z, x+y+z, a+z\}$ determines an induced subgraph of the type $K_{1,3}$, which is contradiction.

Case 3. z is adjacent to $x+y$ and z is not adjacent to x. Then $x \in Rz$ and $Rz = R(x+z) = R(a+z)$. The set $\{x+y, z, x+z, a+z\}$ implies that $\Gamma'(R)$ has a $K_{1,3}$ as an induced subgraph, which is contradiction.

Case 4. z is not adjacent to x, y and $x+y$. Since x and z are not adjacent and $z \in \mathfrak{m} \setminus (Rx \cup Ry \cup R(x+y))$, $x \in Rz$. This yields that $x = x_1 z$, for some $x_1 \in \mathfrak{m}$. Similarly, $y = y_1 z$, for some $y_1 \in \mathfrak{m}$. We note that x_1 and y_1 are adjacent and $Rx_1 = R(x+x_1) = R(a+x_1)$. It follows that the induced subgraph by the set $\{y_1, x_1, x+x_1, a+x_1\}$ is isomorphic to $K_{1,3}$, a contradiction.
According to the above cases, we find that $\mathfrak{m} = \{0, a, x, y, x+y, x+a, y+a, x+y+a\}$ and $\Gamma'(R) = K_{2,2,2} \cup K_1$. Since $\Gamma'(R)$ is isomorphic to $K_{2,2,2} \cup K_1$, it is the line graph of $K_4 \cup K_1$. It is not hard to see that there exists a prime integer p and positive integers t, l, k such that $Char(R) = p^t$, $|\mathfrak{m}| = p^l$, $|R| = p^k$ and $Char(R/\mathfrak{m}) = p$. Since $|\mathfrak{m}| = 2^3$, we deduce that $p = 2$ and so $Char(R/\mathfrak{m}) = 2$. Also, we know that $\mathfrak{m}$ is not principal and $\Gamma'(R)$ is planar. In [1], the authors proved that the local rings of order 2^k for which their maximal ideal is not principal, their cozero-divisor graph is planar and $\Gamma'(R)$ is isomorphic to $K_{2,2,2} \cup K_1$ are the following rings:

$\mathbb{Z}_2[x,y]/(x^2-y^2, xy)$, $\mathbb{Z}_2[x,y]/(x^2, y^2)$, $\mathbb{Z}_4[x,y]/(x^2-2, xy, y^2-2, 2x)$,
$\mathbb{Z}_4[x,y]/(x^2, xy-2, y^2)$, $\mathbb{Z}_4[x]/(x^2)$, $\mathbb{Z}_4[x]/(x^2-2x)$, $\mathbb{Z}_8[x]/(2x, x^2-4)$.

In view of proof of [1, Proposition 2.7], we deduce that R is isomorphic to one of the above rings (see [1, Figure. 1]). The proof of other side is clear. □

The following theorem can be obtained directly from Lemmas 2.8 and 2.9.

Theorem 2.10. *Let R be a commutative local ring. Then $\Gamma'(R)$ is a line graph if and only if $\Gamma'(R)$ is totally disconnected, $\Gamma'(R)$ is complete graph or R is isomorphic to one of the rings $\mathbb{F}_q$, $\mathbb{Z}_2[x,y]/(x^2-y^2, xy)$, $\mathbb{Z}_2[x,y]/(x^2, y^2)$, $\mathbb{Z}_4[x,y]/(x^2-2, xy, y^2-2, 2x)$, $\mathbb{Z}_4[x,y]/(x^2, xy-2, y^2)$,*
$\mathbb{Z}_4[x]/(x^2)$, $\mathbb{Z}_4[x]/(x^2-2x)$ and $\mathbb{Z}_8[x]/(2x, x^2-4)$.

Finally, in the following theorem, we characterize all commutative rings such that their cozero- divisor graphs are line graphs.

Theorem 2.11. *Let R be a commutative ring. Then $\Gamma'(R)$ is a line graph if and only if $\Gamma'(R)$ is totally disconnected, $\Gamma'(R)$ is complete graph or R is isomorphic to one of the following rings:*

$\mathbb{F}_q$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2[x,y]/(x^2-y^2, xy)$,
$\mathbb{Z}_2[x,y]/(x^2, y^2)$, $\mathbb{Z}_4[x,y]/(x^2-2, xy, y^2-2, 2x)$, $\mathbb{Z}_4[x,y]/(x^2, xy-2, y^2)$,
$\mathbb{Z}_4[x]/(x^2)$, $\mathbb{Z}_4[x]/(x^2-2x)$, $\mathbb{Z}_8[x]/(2x, x^2-4)$.

3. When the Cozero-Divisor Graph is the Complement of a Line Graph

In this section, we investigate when the graph $\Gamma'(R)$ is the complement of a line graph. We use the following version of Theorem 2.1.

Theorem 3.1. *A graph G is the complement of a line graph if and only if none of the nine graphs $\overline{G_i}$ of Fig. 5 is an induced subgraph of G.*

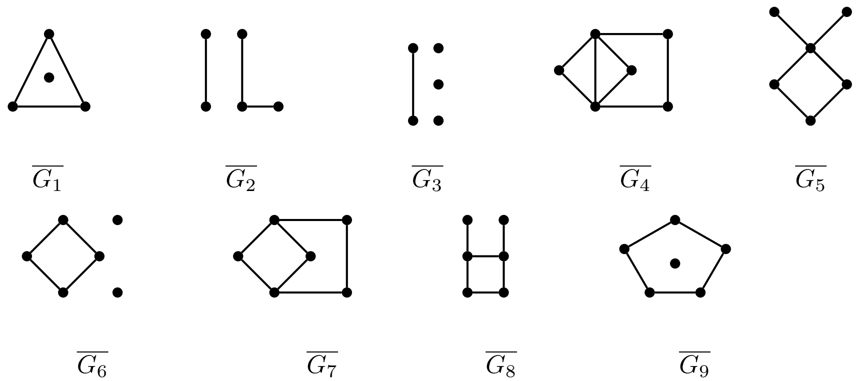
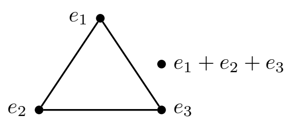
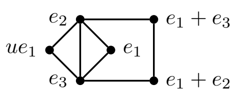


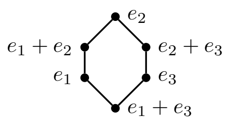
Fig. 5. Forbidden induced subgraphs of complement of line graphs.

Lemma 3.2. *Let $R \cong R_1 \times R_2 \times \cdots \times R_n$ and let $(R_i, \mathfrak{m}_i)$ be a local ring for all $1 \leq i \leq n$. If $\Gamma'(R)$ is the complement of a line graph, then $n \leq 3$.*

Proof. By contradiction, suppose that $n \geq 4$. Then the graph $\Gamma'(R)$ has an induced subgraph which is isomorphic to $\overline{G_1}$ (see Fig. 6). This is a contradiction. Hence $n \leq 3$. □







. Fig. 6 Fig. 7 Fig. 8 $\overline{\Gamma'(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)}$

Lemma 3.3. *Let $R \cong R_1 \times R_2 \times R_3$ and let $(R_i, \mathfrak{m}_i)$ be a local ring for $i = 1, 2, 3$. Then $\Gamma'(R)$ is the complement of a line graph if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. Let $\Gamma'(R)$ be the complement of a line graph. We prove that $|U(R_1)| = 1$. By contradiction, suppose that $1 \neq u \in U(R_1)$. Then the induced subgraph by the set $\{e_1, e_2, e_3, ue_1, e_1 + e_2, e_1 + e_3\}$ is isomorphic to $\overline{G_4}$ (see Fig. 7), a contradiction. Therefore $|U(R_1)| = 1$. This yields that $R_1 \cong \mathbb{Z}_2$. Similarly, $R_2 \cong R_3 \cong \mathbb{Z}_2$ and so $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. The graph $\Gamma'(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ was drawn in Fig. 2. It is not hard to see that $\overline{\Gamma'(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)} = C_6$, and so $\Gamma'(R)$ is the complement of the line graph of the graph C_6 (see Fig. 8). This completes the proof. □

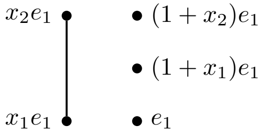


Fig. 9

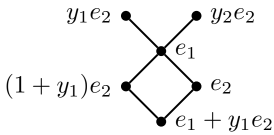


Fig. 10

Lemma 3.4. *Let $R \cong R_1 \times R_2$ and let $(R_i, \mathfrak{m}_i)$ be a local ring for $i = 1,2$. Then $\Gamma'(R)$ is the complement of a line graph if and only if R is isomorphic to one of the rings $\mathbb{F}_{q_1} \times \mathbb{F}_{q_2}, \mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$.*

Proof. Let $\Gamma'(R)$ be the complement of a line graph. First, we claim that $\Gamma'(R_1)$ is totally disconnected or R_1 is a field. If $\{x_1, x_2\} \in E(\Gamma'(R_1))$, then the induced subgraph by the set $\{e_1, x_1e_1, x_2e_1, (1+x_1)e_1, (1+x_2)e_1\}$ is isomorphic to $\overline{G_3}$ (see Fig. 9), which is a contradiction. Therefore $\Gamma'(R_1)$ has not any edge. This implies that $\Gamma'(R_1)$ is totally disconnected or R_1 is a field and the claim is proved. Similarly, $\Gamma'(R_2)$ is totally disconnected or R_2 is a field. We divide the proof in to three following cases:

Case 1. R_1 and R_2 are fields. Let $R_1 = \mathbb{F}_{q_1}$ and $R_2 = \mathbb{F}_{q_2}$, for some positive integers q_1 and q_2. Let $A = \{xe_1 | 0 \neq x \in \mathbb{F}_{q_1}\}$ and let $B = \{ye_2 | 0 \neq y \in \mathbb{F}_{q_2}\}$. Clearly, $V(\Gamma'(R)) = A \cup B$ and $\Gamma'(R)$ is a complete bipartite graph with parts A and B. It follows that $\Gamma'(R) = K_{q_1-1,q_2-1}$ and it is the complement of the line graph of the union of two stars K_{1,q_1-1} and K_{1,q_2-1}.

Case 2. R_1 is a field and $\Gamma'(R_2)$ is totally disconnected. We prove that $|\mathfrak{m}_2| = 2$. Assume, on the contrary, $0 \neq y_1, y_2 \in \mathfrak{m}_2$. With no loss of generality, we may assume that $y_2 \in Ry_1$. Then the induced subgraph by the set $\{e_1, e_2, y_1e_2, y_2e_2, e_1 + y_1e_2, (1+y_1)e_2\}$ is isomorphic to $\overline{G_5}$ (see Fig. 10), which is a contradiction. Therefore $|\mathfrak{m}_2| = 2$. Let $\mathfrak{m}_2 = \{0, y_1\}$. We note that $\mathfrak{m}_2 = Z(R_2)$ and by [7, Remark 1], we find that $|R_2| \leq |\mathfrak{m}_2|^2$ and so $R_2 \cong \mathbb{Z}_4$ or $R_2 \cong \mathbb{Z}_2[x]/(x^2)$. If $x \in R_1 \setminus \{0,1\}$, then the induced subgraph by the set $\{e_1, xe_1, e_2, y_1e_2, e_1 + y_1e_2, xe_1 + y_1e_2\}$ is isomorphic to $\overline{G_5}$ (see Fig. 11), which is a contradiction. Therefore $R_1 \cong \mathbb{Z}_2$ and so R is isomorphic to one of the rings $\mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$. Clearly, $\Gamma'(\mathbb{Z}_2 \times \mathbb{Z}_4) \cong \Gamma'(\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2))$. The graph $\Gamma'(\mathbb{Z}_2 \times \mathbb{Z}_4)$ was drawn in Fig. 13. It is not hard to see that it is the complement of the line graph of the graph H (see Fig. 13).

Case 3. $\Gamma'(R_1)$ and $\Gamma'(R_2)$ are totally disconnected. Since R_1 and R_2 are not fields, $|\mathfrak{m}_1|, |\mathfrak{m}_2| \geq 2$. Let $0 \neq x_1 \in \mathfrak{m}_1$ and $0 \neq y_1 \in \mathfrak{m}_2$. The induced subgraph by the set $\{e_2, x_1e_1 + y_1e_2, y_1e_2, x_1e_1 + (1+y_1)e_2, x_1e_1 + e_2\}$ is isomorphic to $\overline{G_3}$ (see Fig. 12), which is a contradiction.

From the above cases, we find that if $\Gamma'(R)$ is the complement of a line graph, then R is isomorphic to one of the rings $\mathbb{F}_{q_1} \times \mathbb{F}_{q_2}, \mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$. The proof of converse is clear. $\square$

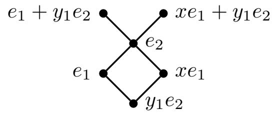


Fig. 11

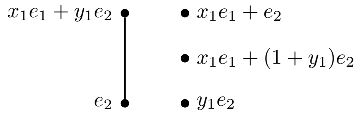


Fig. 12

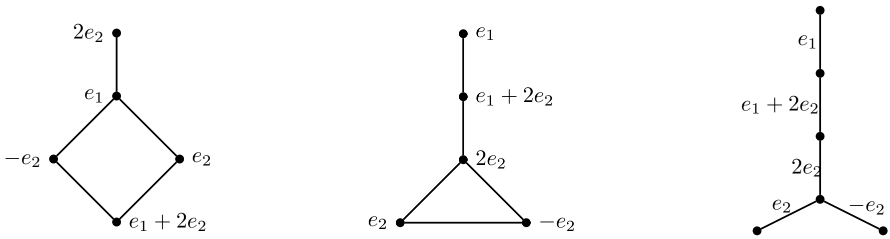


Fig. 13. $\Gamma'(\mathbb{Z}_2 \times \mathbb{Z}_4) \cong \Gamma'(\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)) = \overline{L(H)}$.

Now, we have the following conclusion which completely characterizes all finite commutative non-local rings R whose cozero-divisor graphs are the complement of line graphs.

Theorem 3.5. *Let R be a commutative non-local ring. Then $\Gamma'(R)$ is the complement of a line graph if and only if R is isomorphic to one of the rings $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{F}_{q_1} \times \mathbb{F}_{q_2}, \mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$.*

The only remaining case is that R is a local ring. As we mentioned in the previous section, if R is a field, then $\Gamma'(R)$ is an empty graph. It follows that $\Gamma'(R)$ is the complement of the line graph of the graph K_1. So, we may assume that R is a local ring with $\mathfrak{m} \neq 0$. In the following results, we characterize a family of graphs can be occurred as the complement of line cozero-divisor graph of local rings.

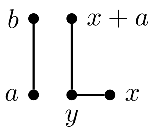


Fig. 14

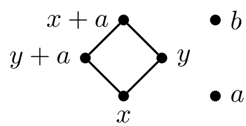


Fig. 15

Lemma 3.6. *Let $(R,\mathfrak{m})$ be a local ring and $\mathfrak{m} \neq 0$. If $\Gamma'(R)$ is the complement of a line graph and $\{x,y\} \in E(\Gamma'(R))$, then $|Rx \cap Ry| \leq 2$.*

Proof. By contradiction, assume that $0 \neq a,b \in Rx \cap Ry$. There are two following cases:

Case 1. a and b are adjacent. Then the induced subgraph by the set $\{a,b,x,y,x+a\}$ is isomorphic to $\overline{G_2}$ (see Fig. 14), a contradiction.

Case 2. a and b are not adjacent. Then the induced subgraph by the set $\{a,b,x,y,x+a,y+a\}$ is isomorphic to $\overline{G_6}$ (see Fig. 15), a contradiction. □

We close this paper by the following theorem.

Theorem 3.7. *Let $(R,\mathfrak{m})$ be a local ring with $\mathfrak{m} \neq 0$ and let $\Gamma'(R)$ be the complement of a line graph. Then $Rx \cap Ry = \{0\}$, for every $\{x,y\} \in E(\Gamma'(R))$ if and only if $\Gamma'(R)$ is a complete r-partite graph, for some positive integer r.*

Proof. Assume that $\Gamma'(R)$ is the complement of a line graph and $Rx \cap Ry = \{0\}$, for every $\{x,y\} \in E(\Gamma'(R))$. Since R is finite, $A = \{Rx | 0 \neq x \in \mathfrak{m}\}$ with the inclusion relation has maximal element. Let $\{Rx_1,\dots,Rx_r\}$ be the set of all maximal elements of A, for some positive integer r. We show that $\Gamma'(R)$ is a complete r-partite graph with parts $Rx_1 \setminus \{0\},\dots,Rx_r \setminus \{0\}$. We claim that every two distinct elements of Rx_1 are non-adjacent. By contradiction, assume that $0 \neq a,b \in Rx_1$ and $\{a,b\} \in E(\Gamma'(R))$. If $a,b \in \mathfrak{m}x_1$, then the induced subgraph by the set $\{a,b,x_1,a+x_1,b+x_1\}$ is isomorphic to $\overline{G_3}$, a contradiction. If $a \in \mathfrak{m}x_1$ and $b \in U(R)x_1$, then $a \in Rb$, which is a contradiction. Also, $Ra = Rb = Rx_1$, where $a,b \in U(R)x_1$, which is a contradiction. Therefore the claim is proved. By the same argument, we have that every two distinct elements of Rx_i are non-adjacent, for $i = 1,\dots,r$. By the maximality of Rx_i and Rx_j, we find that x_i and x_j are adjacent, for every i,j, $1 \leq i < j \leq r$. Since $\{x_i,x_j\} \in E(\Gamma'(R))$, by our assumption we have $Rx_i \cap Rx_j = \{0\}$, for every i,j, $1 \leq i < j \leq r$. This yields that every elements of $Rx_i \setminus \{0\}$ and $Rx_j \setminus \{0\}$ are adjacent, where $1 \leq i < j \leq r$. Therefore $\Gamma'(R)$ is a complete r-partite graph with parts $Rx_1 \setminus \{0\},\dots,Rx_r \setminus \{0\}$. Let $|Rx_i \setminus \{0\}| = n_i$, for $i = 1,\dots,r$. Then $\Gamma'(R) = K_{n_1,\dots,n_r} = \overline{L(\bigcup_{i=1}^{r} K_{1,n_i})}$.
Conversely, suppose that $\Gamma'(R)$ is a complete r-partite graph with parts $V_1,\dots,V_r$, for some positive integer r and $\{x,y\} \in E(\Gamma'(R))$. We prove that $Rx \cap Ry = \{0\}$. By contradiction, suppose that $0 \neq a \in Rx \cap Ry$. Since $\{x,y\} \in E(\Gamma'(R))$, $x \in V_i$ and $y \in V_j$, for some $i \neq j$. On the other hand, a is adjacent neither x nor y, because $a \in Rx \cap Ry$. This implies that $a \in V_i \cap V_j$, a contradiction. Therefore $Rx \cap Ry = \{0\}$ and the proof is complete. □

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