LINE COZERO-DIVISOR GRAPHS

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Let \( R \) be a commutative ring. The cozero-divisor graph of \( R \) denoted by \( \Gamma'(R) \) is a graph with the vertex set \( W^*(R) \), where \( W^*(R) \) is the set of all non-zero and non-unit elements of \( R \), and two distinct vertices \( x \) and \( y \) are adjacent if and only if \( x \notin R y \) and \( y \notin R x \). In this paper, we investigate when the cozero-divisor graph is a line graph. We completely present all commutative rings which their cozero-divisor graphs are line graphs. Also, we study when the cozero-divisor graph is the complement of a line graph.

1. Introduction

In 1988, Beck [12] introduced the concept of the zero-divisor graph. The zero-divisor graphs of commutative rings has been studied by several authors. We refer to the reader the papers [7, 8] and [9] for the properties of zero-divisor graphs. Also, the line zero divisor graphs was studied in [11]. For an arbitrary commutative ring \( R \), the cozero-divisor graph \( \Gamma'(R) \), as the dual notion of zero-divisor graphs, was introduced in [2]. Let \( W^*(R) \) be the set of all non-zero and non-unit elements of \( R \). The vertex set of \( \Gamma'(R) \) is \( W^*(R) \), and two distinct vertices \( x \) and \( y \) in \( W^*(R) \) are adjacent if and only if \( x \notin R y \) and \( y \notin R x \), where \( Rz \) is the ideal generated by the element \( z \) in \( R \). Many papers have been devoted to the study of cozero-divisor graphs, for instance see [1 – 6]. Motivated by
the previous works on the zero divisor graph and cozero-divisor graph, in this paper we study line cozero-divisor graphs. Throughout this paper, all graphs are simple with no loops and multiple edges and $R$ is a commutative ring with non-zero identity. We denote the set of all zero-divisor elements and the set of all unit elements of $R$ by $Z(R)$ and $U(R)$, respectively. If $R$ has a unique maximal ideal $m$, then $R$ is said to be a local ring and it is denoted by $(R,m)$. Also, $\mathbb{F}_q$ denotes a finite field with $q$ elements, for some positive integer $q$.

For basic definitions on graphs, one may refer to [14]. Let $G$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$. If $x$ is adjacent to $y$, then we write $x — y$ or $\{x,y\} \in E(G)$. A graph $G$ is complete if each pair of distinct vertices is joined by an edge. For a positive integer $n$, we use $K_n$ to denote the complete graph with $n$ vertices. Also, we say that $G$ is totally disconnected if no two vertices of $G$ are adjacent. Note that a graph whose vertex set is empty is an empty graph. The complement of $G$, denoted by $\overline{G}$ is a graph on the same vertices such that two distinct vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in $G$. If $|V(G)| \geq 2$, then a path from $x$ to $y$ is a series of adjacent vertices $x — x_1 — x_2 — \cdots — x_n — y$. A cycle is a path that begins and ends at the same vertex in which no edge is repeated and all vertices other than the starting and ending vertex are distinct. We use $P_n$ and $C_n$ to denote the path and the cycle with $n$ vertices, respectively. Suppose that $H$ is a non-empty subset of $V(G)$. The subgraph of $G$ whose vertex set is $H$ and whose edge set is the set of those edges of $G$ with both ends in $H$ is called the subgraph of $G$ induced by $H$. For every positive integer $r$, an $r$-partite graph is one whose vertex set can be partitioned into $r$ subsets, or parts, in such a way that no edge has both ends in the same part. An $r$-partite graph is complete $r$-partite if any two vertices in different parts are adjacent. We denote the complete $r$-partite graph, with part sizes $n_1, \ldots, n_r$ by $K_{n_1,\ldots,n_r}$. For every $n \geq 2$, the star graph with $n$ vertices is the complete bipartite graph with part sizes 1 and $n - 1$. The line graph $L(G)$ is a graph such that each vertex of $L(G)$ represents an edge of $G$, and two vertices of $L(G)$ are adjacent if and only if their corresponding edges are incident in $G$.

Here is a brief summary of the present paper. In this paper, we investigate when the cozero-divisor graph is a line graph. Also, we study when the cozero-divisor graph is the complement of a line graph. In Sec. 2, we characterize all finite rings whose cozero-divisor graphs are line graphs. In Sec. 3, we characterize all finite non-local rings whose cozero-divisor graphs are complements of line graphs. Also, we prove that if $(R,m)$ is a local ring with $m \neq 0$, $\Gamma'(R)$ is the complement of a line graph and $\{x,y\} \in E(\Gamma'(R))$, then $|Rx \cap Ry| \leq 2$. Finally, we determine a family of graphs can be occurred as the complement of line cozero-divisor graph of finite local rings.
2. When the Cozero-Divisor Graph is a Line Graph

In this section, we study when the graph $\Gamma'(R)$ is a line graph. We determine all finite commutative rings whose cozero-divisor graphs are line graphs. We will use one of the characterizations of line graphs which was proved in [13].

**Theorem 2.1.** Let $G$ be a graph. Then $G$ is the line graph of some graph if and only if none of the nine graphs in Fig. 1 is an induced subgraph of $G$.

Throughout the paper $R$ is a finite commutative ring. By the structure theorem of Artinian rings [10, Theorem 8.7], there exists positive integer $n$ such that $R \cong R_1 \times R_2 \times \cdots \times R_n$ and $(R_i, m_i)$ is a local ring for all $1 \leq i \leq n$. We use this theorem in the rest of the paper. Also, let $e_i$ be the $1 \times n$ vector whose $i$th component is 1 and the other components are 0.

We first present the following lemma.

**Lemma 2.2.** Let $R \cong R_1 \times R_2 \times \cdots \times R_n$ and let $(R_i, m_i)$ be a local ring for all $1 \leq i \leq n$. If $n \geq 4$, then $\Gamma'(R)$ is not a line graph.

**Proof.** It is easy to see that $R(\sum_{i=1}^n e_i) \subseteqneq R(\sum_{i=3}^n e_i) \subseteqneq R(\sum_{i=2}^n e_i)$ and $e_1$ is adjacent to $\sum_{i=2}^n e_i$, $\sum_{i=3}^n e_i$ and $\sum_{i=4}^n e_i$. Hence the induced subgraph by the set $\{e_1, \sum_{i=2}^n e_i, \sum_{i=3}^n e_i, \sum_{i=4}^n e_i\}$ is isomorphic to $K_{1,3}$. Therefore by Theorem 2.1, $\Gamma'(R)$ is not a line graph. \hfill $\Box$

![Forbidden induced subgraphs of line graphs.](image)

**Fig. 1.** Forbidden induced subgraphs of line graphs.

**Lemma 2.3.** Let $R \cong R_1 \times R_2 \times R_3$ and let $(R_i, m_i)$ be a local ring for $i = 1, 2, 3$. Then $\Gamma'(R)$ is a line graph if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

**Proof.** Let $\Gamma'(R)$ be a line graph. If $|R_1| \geq 3$, then the induced subgraph by the set $\{e_2, e_3, e_1 + e_3, xe_1 + e_3\}$ is isomorphic to $K_{1,3}$, for every $x \in R_1 \setminus \{0, 1\}$.
which is impossible. Hence $|R_1| = 2$ and similarly, $|R_2| = |R_3| = 2$. Therefore $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. We draw the graph $\Gamma'(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ in Fig. 2. One can easily see that the graph $\Gamma'(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ is the line graph of the graph $K_{2,3}$ which is drawn in Fig. 2. The proof of converse is clear.

\[ \Gamma'(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \]

\[ K_{2,3} \]

**Fig. 2.** $\Gamma'(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ is the line graph of $K_{2,3}$.

**Lemma 2.4.** Let $R \cong R_1 \times R_2$ and let $(R_i, m_i)$ be a local ring for $i = 1, 2$. Then $\Gamma'(R)$ is a line graph if and only if $R$ is isomorphic to one of the rings $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$, and $\mathbb{Z}_3 \times \mathbb{Z}_3$.

**Proof.** One side is obvious. For the other side assume that $\Gamma'(R)$ is a line graph. We know that $|m_i| \leq |U(R_i)|$, for $i = 1, 2$. If $|m_1| \geq 2$, then we can put $a \in m_1^*$ and $u, v \in U(R_1)$. Then the induced subgraph on $\{ae_1, ue_1, ve_1, e_2\}$ is isomorphic to $K_{1,3}$, a contradiction. So, $R_1$ is a field. Similarly, $R_2$ is a field. Then $\Gamma'(R) = K_{|R_1| - 1, |R_2| - 1}$ and hence $R$ is isomorphic to one of the rings $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$.

The next theorem, follows immediately from the above lemmas.

**Theorem 2.5.** Let $R$ be a commutative non-local ring. Then $\Gamma'(R)$ is a line graph if and only if $R$ is isomorphic to one of the rings $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

For the last case of our discussion, we must assume that $n = 1$. So, $R$ is a local ring. Let $m$ be the only maximal ideal of $R$. We note that if $R$ is a field, then $W^*(R) = \emptyset$ which implies that $\Gamma'(R)$ is an empty graph and so it is the line graph of the graph $K_1$. So, we may assume that $R$ is a local ring which is not a field. This implies that $m \neq 0$. Also, it is clear that if $\Gamma'(R)$ is totally disconnected with $t$ vertices, for some positive integer $t$, then $\Gamma'(R)$ is the line graph of $\bigcup_{i=1}^t K_2$. In the rest of this section, we study the case that $R$ is a local ring with non-zero maximal ideal and $E(\Gamma'(R)) \neq \emptyset$. Our starting point is the following lemma.

**Lemma 2.6.** Let $(R, m)$ be a local ring with $m \neq 0$ and let $\Gamma'(R)$ be a line graph. If $\{x, y\} \in E(\Gamma'(R))$, then $|Rx \cap Ry| \leq 2$. 
Proof. By contradiction, suppose that $0 \neq a, b \in Rx \cap Ry$. If $a \in U(R)x$, then we have $y \in Ra \subseteq Rx$, which is impossible. Therefore $a \in my$. Similarly, $b \in my$ and so $R(y + a) = R(y + b) = Ry$. Now, the set $\{x, y, y + a, y + b\}$ determines an induced subgraph of the type $K_{1,3}$. Therefore by Theorem 2.1, $\Gamma'(R)$ is not a line graph, a contradiction. Hence $|Rx \cap Ry| \leq 2$. 

Lemma 2.7. Let $(R, m)$ be a local ring with $m \neq 0$, $\Gamma'(R)$ be a line graph and let $\{x, y\} \in E(\Gamma'(R))$. If $Rx \cap Ry = \{0\}$, then the following hold:

(i) $Rx = \{0, x\}$ or $Rx = \{0, x, -x\}$.

(ii) $Ry = \{0, y\}$ or $Ry = \{0, y, -y\}$.

Proof. (i) We prove that $|Rx| \leq 3$. By contradiction, assume that $|Rx| \geq 4$. Let $a, b \in Rx \setminus \{0, x\}$. There are three following cases:

Case 1. $a, b \in U(R)x$. Then $Rx = Ra = Rb$ and the set $\{y, x, a, b\}$ determines an induced subgraph of the type $K_{1,3}$. This is a contradiction, by Theorem 2.1.

Case 2. $a, b \in mx$. Then $Rx = R(x + a) = R(x + b)$ and the set $\{y, x, x + a, x + b\}$ determines an induced subgraph of the type $K_{1,3}$, which is a contradiction, by Theorem 2.1.

Case 3. $a \in U(R)x$ and $b \in mx$. Then $Rx = Ra$ and $Rb \subseteq Rx$. Since $Ra = Rx$ and $\{x, y\} \in E(\Gamma'(R))$, $y$ is adjacent to $a$. If $y \in Rb$, then $y \in Rx$, which is impossible. On the other hand, if $b \in Ry$, then $b \in Rx \cap Ry = \{0\}$, a contradiction. Therefore $y$ is adjacent to $b$. Now, the set $\{y, x, a, b\}$ determines an induced subgraph of the type $K_{1,3}$, a contradiction.

By the above cases, we deduce that $|Rx| = 2, 3$. Clearly, if $|Rx| = 2$, then $Rx = \{0, x\}$. Also, it is not hard to see that if $|Rx| = 3$, then $Rx = \{0, x, -x\}$. This completes the proof.

(ii) It is similar to the proof of part (i). 

Now, we are in a position to prove one of the main results.

Lemma 2.8. Let $(R, m)$ be a local ring, $m \neq 0$, $E(\Gamma'(R)) \neq \emptyset$ and for every $\{x, y\} \in E(\Gamma'(R))$, let $Rx \cap Ry = \{0\}$. Then $\Gamma'(R)$ is a line graph if and only if it is a complete graph.

Proof. Suppose that $\Gamma'(R)$ is a line graph. Let $A = \{x \in V(\Gamma'(R)) | Rx = \{0, x\}\}$, $B = \{x \in V(\Gamma'(R)) | Rx = \{0, x, -x\}\}$ and let $C$ be the set of all isolated vertices of $\Gamma'(R)$. We note that the induced subgraph of $\Gamma'(R)$ by the set $A$ is a complete graph. Also, there exists $r \geq 0$ such that $|B| = 2r$. Because we have $x, -x \notin B$ or $x, -x \notin B$, for every $0 \neq x \in m$. Moreover, if $r > 0$, then the induced subgraph of $\Gamma'(R)$ by the set $B$ is complete $r$-partite graph and every part is equal to $\{x, -x\}$, for some $x \in B$. Furthermore, by Lemma 2.7, $V(\Gamma'(R)) = A \cup B \cup C$. We use these facts in the rest of the proof. Since $E(\Gamma'(R)) \neq \emptyset$, $A \cup B \neq \emptyset$. Consider two following cases:
Case 1. $A = \emptyset$. We note that $E(\Gamma'(R)) \neq \emptyset$. This yields that $|B| = 2r > 0$ and $B$ has two elements say $b_1$ and $b_2$ such that $b_1 \neq -b_2$ and $\{b_1, b_2\} \in E(\Gamma'(R))$. We claim that $C = \emptyset$. By contradiction, suppose that $c \in C$. If $c \in Rc_1$, then $c = b_1$ or $c = -b_1$. Hence $c$ is not an isolated vertex, which is a contradiction. Therefore $c \notin Rc_1$. Similarly, $c \notin Rc_2$. Since $c$ is an isolated vertex, we find that $b_1, b_2 \in Rc$. Assume that $b_1 = r_1c$ and $b_2 = r_2c$, for some $r_1, r_2 \in R$. If $r_1 \in U(R)$, then $Rc = Rc_1$. This implies that $c$ and $b_2$ are adjacent, which is impossible. Hence $r_1 \in m$ and similarly, $r_2 \in m$. Since $b_1$ and $b_2$ are adjacent, we deduce that $r_1$ and $r_2$ are adjacent. Therefore $r_1, r_2 \in B$. Moreover, we conclude that $r_1 \in \{b_1, -b_1\}$ and $r_2 \in \{b_2, -b_2\}$. It follows that $c = 0$, a contradiction. Therefore $C = \emptyset$ and the claim is proved. This implies that $\Gamma'(R)$ is a complete $r$-partite graph, because $|B| = 2r$. Also, as we mentioned before, every part of $\Gamma'(R)$ is equal to $\{b, -b\}$, for some $b \in B$. If $|B| \geq 8$, then there exists $b_1, b_2, b_3, b_4 \in B$ such that $b_i \neq -b_j$, for every $i \neq j$. Now, the induced subgraph by the set $\{b_1, b_2, b_3, -b_3, b_4\}$ is isomorphic to $G_3$ (see Fig. 3), a contradiction. Hence $|B| = 4, 6$ and so $\Gamma'(R) = K_{2, 2}$ or $\Gamma'(R) = K_{2, 2, 2}$. By [4, Lemma 2], we conclude that $\Gamma'(R) \neq K_{2, 2}$. Therefore $\Gamma'(R) = K_{2, 2, 2}$. It follows that $\Gamma'(R)$ is a complete 3-partite graph. By [6, Corollary 3], $\Gamma'(R)$ is a triangle, which is impossible.

Case 2. $A \neq \emptyset$. Let $a_1 \in A$. First, we prove that $C = \emptyset$. By contradiction, suppose that $C \neq \emptyset$. We know that $Ra_1 = \{0, a_1\}$. This yields that $a_1 \in Rc$, for every $c \in C$. Also, if $B \neq \emptyset$, then $b \in Rc$, for every $b \in B$ and every $c \in C$. Since $m$ is finite, we find that there exists $c_0 \in C$ such that $m = Rc_0$. On the other hand, by [2, Theorem 2.7], we conclude that $\Gamma'(R)$ is totally disconnected, a contradiction. Therefore $C = \emptyset$.

Now, we prove that $B = \emptyset$. By contradiction, assume that $|B| = 2r > 0$ and $B = \{b_1, \ldots, b_{2r}\}$. Since $a_1 + b_1$ is a vertex of $\Gamma'(R)$, $a_1 + b_1 \in V(\Gamma'(R)) = A \cup B$. If $a_1 + b_1 \in A$, then $R(a_1 + b_1) = \{0, a_1 + b_1\}$ and so $a_1 + b_1 = -(a_1 + b_1) = a_1 - b_1$. This yields that $b_1 = -b_1$, a contradiction. Therefore $a_1 + b_1 \in B$. With no loss of generality, we may assume that $a_1 + b_1 = b_2$. Then $a_1 = b_2 - b_1$. Since $2b_1 \neq 0, b_1$, we have $2b_1 = -b_1$. Hence $3b_1 = 0$. Similarly, $3b_2 = 0$. This implies that $3a_1 = 3(b_2 - b_1) = 0$. On the other hand, we have $2a_1 \in Ra_1 = \{0, a_1\}$ which shows that $2a_1 = 0$. Hence $a_1 = 0$, a contradiction. Thus $B = \emptyset$ and $V(\Gamma'(R)) = A$. Therefore $\Gamma'(R)$ is a complete graph.

From the above cases, we conclude that if $\Gamma'(R)$ is a line graph, then it is a complete graph. Clearly, if $\Gamma'(R) = K_t$, for some positive integer $t$, then it is the line graph of $K_{1,t}$. This completes the proof. \qed
A graph is said to be planar if it can be drawn in the plane so that its edges intersect only at their ends. A subdivision of a graph is a graph obtained by replacing edges of this graph with pairwise internally-disjoint paths. A remarkably simple characterization of planar graphs was given by Kuratowski in 1930, that says that a graph is planar if and only if it contains no subdivision of $K_5$ or $K_{3,3}$ [14].

Let $(R, m)$ be a local ring with $m \neq 0$, $|Rx \cap Ry| = 2$, for some $\{x, y\} \in E(\Gamma'(R))$ and let $\Gamma'(R)$ be a line graph. In the following theorem, first we prove that $\Gamma'(R)$ is planar. Then by using [1, Proposition 2.7], we characterize all local rings whose cozero-divisor graphs are line graphs.

**Lemma 2.9.** Let $(R, m)$ be a local ring with $m \neq 0$. If there exists $\{x, y\} \in E(\Gamma'(R))$ such that $|Rx \cap Ry| = 2$, then $\Gamma'(R)$ is a line graph if and only if $R$ is isomorphic to one of the following rings:

$$
\begin{align*}
\mathbb{Z}_2[x, y]/(x^2 - y^2, xy), & \quad \mathbb{Z}_2[x, y]/(x^2, y^3), \quad \mathbb{Z}_4[x, y]/(x^2 - 2xy, y^2 - 2, 2x), \\
\mathbb{Z}_4[x, y]/(x^2, xy - 2, y^2), & \quad \mathbb{Z}_4[x]/(x^2), \quad \mathbb{Z}_4[x]/(x^2 - 2x), \quad \mathbb{Z}_8[x]/(2x, x^2 - 4).
\end{align*}
$$

**Proof.** First assume that $(R, m)$ is a local ring, $\Gamma'(R)$ is a line graph, $\{x, y\} \in E(\Gamma'(R))$ and $Rx \cap Ry = \{0, a\}$. We note that every element of the set $Rx \setminus \{0, a\}$ is adjacent to every element of the set $Ry \setminus \{0, a\}$. Since $\Gamma'(R)$ is a line graph and $K_{1,3}$ is not an induced subgraph of $\Gamma'(R)$, we find that $Rx = \{0, a, x, x + a\}$ and $Ry = \{0, a, y, y + a\}$. Since $x \notin Ry$ and $y \notin Rx$, we conclude that $x + y \notin Rx \cup Ry$. If $x \in R(x + y)$, then $x = r(x + y)$, for some $r \in m$. Hence $(1 - r)x = ry$. This yields that $x = (1 - r)^{-1}ry \in Ry$, which is impossible. Therefore $x \notin R(x + y)$. Similarly, $y \notin R(x + y)$. Thus $x + y$ is adjacent to both $x$ and $y$. If $x + y$ is adjacent to $a$, then the set $\{x + y, x + a, a\}$ implies that $\Gamma'(R)$ has a $K_{1,3}$ as an induced subgraph, a contradiction. Therefore $a \in R(x + y)$. By the same argument as we saw before, $R(x + y) = \{0, a, x + y, x + y + a\}$. If $\Gamma'(R)$ has other vertex say $z$, then with no loss of generality, we may assume that there are the following cases:

**Case 1.** $z$ is adjacent to $x, y$ and $x + y$. Then the induced subgraph by the set $\{x, y, x + y, x + y + a, z\}$ is isomorphic to $G_3$ (see Fig. 4), a contradiction.

![Fig. 3](image1)

![Fig. 4](image2)
Case 2. \( z \) is adjacent to \( x \) and \( z \) is not adjacent to \( x + y \). Then \( x + y \in Rz \) and \( Rz = R(x + y + z) = R(a + z) \). The set \( \{x, z, x + y + z, a + z\} \) determines an induced subgraph of the type \( K_{1,3} \), which is contradiction.

Case 3. \( z \) is adjacent to \( x + y \) and \( z \) is not adjacent to \( x \). Then \( x \in Rz \) and \( Rz = R(x + z) = R(a + z) \). The set \( \{x + y, z, x + z, a + z\} \) implies that \( \Gamma'(R) \) has a \( K_{1,3} \) as an induced subgraph, which is contradiction.

Case 4. \( z \) is not adjacent to \( x, y \) and \( x + y \). Since \( x \) and \( z \) are not adjacent and \( z \in m \setminus (Rx \cup Ry \cup R(x + y)) \), \( x \in Rz \). This yields that \( x = x_1z \), for some \( x_1 \in m \). Similarly, \( y = y_1z \), for some \( y_1 \in m \). We note that \( x_1 \) and \( y_1 \) are adjacent and \( Rx_1 = R(x + x_1) = R(a + x_1) \). It follows that the induced subgraph by the set \( \{y_1, x_1, x + x_1, a + x_1\} \) is isomorphic to \( K_{1,3} \), a contradiction.

According to the above cases, we find that \( m = \{0, a, x, y, x + y, x + a, y + a, x + y + a\} \) and \( \Gamma'(R) = K_{2,2,2} \cup K_1 \). Since \( \Gamma'(R) \) is isomorphic to \( K_{2,2,2} \cup K_1 \), it is the line graph of \( K_4 \cup K_1 \). It is not hard to see that there exists a prime integer \( p \) and positive integers \( t, l, k \) such that \( \text{Char}(R) = p^t \), \( |m| = p^l \), \( |R| = p^k \) and \( \text{Char}(R/m) = p \). Since \( |m| = 2^3 \), we deduce that \( p = 2 \) and so \( \text{Char}(R/m) = 2 \). Also, we know that \( m \) is not principal and \( \Gamma'(R) \) is planar. In [1], the authors proved that the local rings of order \( 2^k \) for which their maximal ideal is not principal, their cozero-divisor graph is planar and \( \Gamma'(R) \) is isomorphic to \( K_{2,2,2} \cup K_1 \). The following rings:

\[
\begin{align*}
\mathbb{Z}_2[x,y]/(x^2 - y^2, xy), & \mathbb{Z}_2[x,y]/(x^2, y^2), \mathbb{Z}_4[x,y]/(x^2 - 2, xy, y^2 - 2, 2x), \\
\mathbb{Z}_4[x,y]/(x^2, xy - 2, y^2), & \mathbb{Z}_4[x]/(x^2), \mathbb{Z}_4[x]/(x^2 - 2x), \mathbb{Z}_8[x]/(2x, x^2 - 4).
\end{align*}
\]

In view of proof of [1, Proposition 2.7], we deduce that \( R \) is isomorphic to one of the above rings (see [1, Figure 1]). The proof of other side is clear.  

The following theorem can be obtained directly from Lemmas 2.8 and 2.9.

**Theorem 2.10.** Let \( R \) be a commutative local ring. Then \( \Gamma'(R) \) is a line graph if and only if \( \Gamma'(R) \) is totally disconnected, \( \Gamma'(R) \) is complete graph or \( R \) is isomorphic to one of the rings \( \mathbb{F}_q, \mathbb{Z}_2[x,y]/(x^2 - y^2, xy), \mathbb{Z}_2[x,y]/(x^2, y^2), \mathbb{Z}_4[x,y]/(x^2 - 2, xy, y^2 - 2, 2x), \mathbb{Z}_4[x,y]/(x^2, xy - 2, y^2), \mathbb{Z}_4[x]/(x^2), \mathbb{Z}_4[x]/(x^2 - 2x), \mathbb{Z}_8[x]/(2x, x^2 - 4) \).

Finally, in the following theorem, we characterize all commutative rings such that their cozero-divisor graphs are line graphs.

**Theorem 2.11.** Let \( R \) be a commutative ring. Then \( \Gamma'(R) \) is a line graph if and only if \( \Gamma'(R) \) is totally disconnected, \( \Gamma'(R) \) is complete graph or \( R \) is isomorphic to one of the following rings:

\[
\begin{align*}
\mathbb{F}_q, & \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_3, \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_2[x,y]/(x^2 - y^2, xy), \\
\mathbb{Z}_2[x,y]/(x^2, y^2), & \mathbb{Z}_4[x,y]/(x^2 - 2, xy, y^2 - 2, 2x), \mathbb{Z}_4[x,y]/(x^2, xy - 2, y^2), \\
\mathbb{Z}_4[x]/(x^2), & \mathbb{Z}_4[x]/(x^2 - 2x), \mathbb{Z}_8[x]/(2x, x^2 - 4).
\end{align*}
\]
3. When the Cozero-Divisor Graph is the Complement of a Line Graph

In this section, we investigate when the graph $\Gamma'(R)$ is the complement of a line graph. We use the following version of Theorem 2.1.

**Theorem 3.1.** A graph $G$ is the complement of a line graph if and only if none of the nine graphs $G_i$ of Fig. 5 is an induced subgraph of $G$.

\begin{center}
\begin{tabular}{c c c c c}
$G_1$ & $G_2$ & $G_3$ & $G_4$ & $G_5$
\end{tabular}
\end{center}

\begin{center}
\begin{tabular}{c c c c}
$G_6$ & $G_7$ & $G_8$ & $G_9$
\end{tabular}
\end{center}

Fig. 5. Forbidden induced subgraphs of complement of line graphs.

**Lemma 3.2.** Let $R \cong R_1 \times R_2 \times \cdots \times R_n$ and let $(R_i, m_i)$ be a local ring for all $1 \leq i \leq n$. If $\Gamma'(R)$ is the complement of a line graph, then $n \leq 3$.

**Proof.** By contradiction, suppose that $n \geq 4$. Then the graph $\Gamma'(R)$ has an induced subgraph which is isomorphic to $\overline{G_1}$ (see Fig. 6). This is a contradiction. Hence $n \leq 3$.

\begin{center}
\begin{tabular}{c c c}
\includegraphics[width=0.2\textwidth]{fig6.png} & \includegraphics[width=0.2\textwidth]{fig7.png} & \includegraphics[width=0.2\textwidth]{fig8.png}
\end{tabular}
\end{center}

Fig. 6 \hspace{1cm} Fig. 7 \hspace{1cm} Fig. 8 $\Gamma'(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$

**Lemma 3.3.** Let $R \cong R_1 \times R_2 \times R_3$ and let $(R_i, m_i)$ be a local ring for $i = 1, 2, 3$. Then $\Gamma'(R)$ is the complement of a line graph if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

**Proof.** Let $\Gamma'(R)$ be the complement of a line graph. We prove that $|U(R_1)| = 1$. By contradiction, suppose that $1 \neq u \in U(R_1)$. Then the induced subgraph by the set $\{e_1, e_2, e_3, u e_1, e_1 + e_2, e_1 + e_3\}$ is isomorphic to $\overline{G_4}$ (see Fig. 7), a contradiction. Therefore $|U(R_1)| = 1$. This yields that $R_1 \cong \mathbb{Z}_2$. Similarly, $R_2 \cong R_3 \cong \mathbb{Z}_2$ and so $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. The graph $\Gamma'(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ was drawn in Fig. 2. It is not hard to see that $\Gamma'(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) = C_6$, and so $\Gamma'(R)$ is the
complement of the line graph of the graph \( C_6 \) (see Fig. 8). This completes the proof.

\[
\begin{align*}
&x_2e_1 \\
&x_1e_1 & \bullet (1 + x_2)e_1 \\
& e_1 & \bullet (1 + x_1)e_1 \\
\end{align*}
\]

Fig. 9

\[
\begin{align*}
&y_1e_2 \\
&y_2e_2 & (1 + y_1)e_2 \\
&e_1 & \bullet e_2 \\
& e_1 + y_1e_2 & \bullet e_1 \\
\end{align*}
\]

Fig. 10

**Lemma 3.4.** Let \( R \cong R_1 \times R_2 \) and let \( (R_i, m_i) \) be a local ring for \( i = 1, 2 \). Then \( \Gamma'(R) \) is the complement of a line graph if and only if \( R \) is isomorphic to one of the rings \( \mathbb{F}_{q_1} \times \mathbb{F}_{q_2}, \mathbb{Z}_2 \times \mathbb{Z}_4 \) and \( \mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2) \).

**Proof.** Let \( \Gamma'(R) \) be the complement of a line graph. First, we claim that \( \Gamma'(R_1) \) is totally disconnected or \( R_1 \) is a field. If \( \{x_1, x_2\} \in E(\Gamma'(R_1)) \), then the induced subgraph by the set \( \{e_1, x_1e_1, x_2e_1, (1 + x_1)e_1, (1 + x_2)e_1\} \) is isomorphic to \( \overline{G_3} \) (see Fig. 9), which is a contradiction. Therefore \( \Gamma'(R_1) \) has not any edge. This implies that \( \Gamma'(R_1) \) is totally disconnected or \( R_1 \) is a field and the claim is proved. Similarly, \( \Gamma'(R_2) \) is totally disconnected or \( R_2 \) is a field. We divide the proof into three following cases:

**Case 1.** \( R_1 \) and \( R_2 \) are fields. Let \( R_1 = \mathbb{F}_{q_1} \) and \( R_2 = \mathbb{F}_{q_2} \), for some positive integers \( q_1 \) and \( q_2 \). Let \( A = \{xe_1 | 0 \neq x \in \mathbb{F}_{q_1}\} \) and let \( B = \{ye_2 | 0 \neq y \in \mathbb{F}_{q_2}\} \). Clearly, \( V(\Gamma'(R)) = A \cup B \) and \( \Gamma'(R) \) is a complete bipartite graph with parts \( A \) and \( B \). It follows that \( \Gamma'(R) : K_{q_1-1,q_2-1} \) and it is the complement of the line graph of the union of two stars \( K_{1,q_1-1} \) and \( K_{1,q_2-1} \).

**Case 2.** \( R_1 \) is a field and \( \Gamma'(R_2) \) is totally disconnected. We prove that \( |m_2| = 2 \). Assume, on the contrary, \( 0 \neq y_1, y_2 \in m_2 \). With no loss of generality, we may assume that \( y_2 \in R_{y_1} \). Then the induced subgraph by the set \( \{e_1, e_2, y_1e_2, y_2e_2, e_1 + y_1e_2, (1 + y_1)e_2\} \) is isomorphic to \( \overline{G_5} \) (see Fig. 10), which is a contradiction. Therefore \( |m_2| = 2 \). Let \( m_2 = \{0,y_1\} \). We note that \( m_2 = Z(R_2) \) and by [7, Remark 1], we find that \( |R_2| \leq |m_2|^2 \) and so \( R_2 \cong \mathbb{Z}_4 \) or \( R_2 \cong \mathbb{Z}_2[x]/(x^2) \). If \( x \in R_1 \setminus \{0,1\} \), then the induced subgraph by the set \( \{e_1, xe_1, e_2, y_1e_2, e_1 + y_1e_2, xe_1 + y_1e_2\} \) is isomorphic to \( \overline{G_5} \) (see Fig. 11), which is a contradiction. Therefore \( R_1 \cong \mathbb{Z}_2 \) and so \( R \) is isomorphic to one of the rings \( \mathbb{Z}_2 \times \mathbb{Z}_4 \) and \( \mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2) \). Clearly, \( \Gamma'(\mathbb{Z}_2 \times \mathbb{Z}_4) \cong \Gamma'(\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)) \). The graph \( \Gamma'(\mathbb{Z}_2 \times \mathbb{Z}_4) \) was drawn in Fig. 13. It is not hard to see that it is the complement of the line graph of the graph \( H \) (see Fig. 13).

**Case 3.** \( \Gamma'(R_1) \) and \( \Gamma'(R_2) \) are totally disconnected. Since \( R_1 \) and \( R_2 \) are not fields, \( |m_1|, |m_2| \geq 2 \). Let \( 0 \neq x_1 \in m_1 \) and \( 0 \neq y_1 \in m_2 \). The induced subgraph by the set \( \{e_2, x_1e_1 + y_1e_2, y_1e_2, x_1e_1 + (1 + y_1)e_2, x_1e_1 + e_2\} \) is isomorphic to \( \overline{G_3} \) (see Fig. 12), which is a contradiction.

\[\text{\(\square\)}\]
From the above cases, we find that if $\Gamma'(R)$ is the complement of a line graph, then $R$ is isomorphic to one of the rings $\mathbb{F}_{q_1} \times \mathbb{F}_{q_2}, \mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$. The proof of converse is clear. \hfill \Box

\begin{align*}
e_2
\end{align*}

Fig. 11 \quad Fig. 12

\begin{align*}
e_2
\end{align*}

Fig. 13. $\Gamma'(\mathbb{Z}_2 \times \mathbb{Z}_4) \cong \Gamma'(\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)) = \overline{L(H)}$.

Now, we have the following conclusion which completely characterizes all finite commutative non-local rings $R$ whose cozero-divisor graphs are the complement of line graphs.

**Theorem 3.5.** Let $R$ be a commutative non-local ring. Then $\Gamma'(R)$ is the complement of a line graph if and only if $R$ is isomorphic to one of the rings $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{F}_{q_1} \times \mathbb{F}_{q_2}, \mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$.

The only remaining case is that $R$ is a local ring. As we mentioned in the previous section, if $R$ is a field, then $\Gamma'(R)$ is an empty graph. It follows that $\Gamma'(R)$ is the complement of the line graph of the graph $K_1$. So, we may assume that $R$ is a local ring with $m \neq 0$. In the following results, we characterize a family of graphs can be occurred as the complement of line cozero-divisor graph of local rings.

\begin{align*}
e_2
\end{align*}

Fig. 14 \quad Fig. 15
Lemma 3.6. \( \text{Let } (R, m) \text{ be a local ring and } m \neq 0. \text{ If } \Gamma'(R) \text{ is the complement of a line graph and } \{x, y\} \in E(\Gamma'(R)), \text{ then } |Rx \cap Ry| \leq 2. \)

Proof. By contradiction, assume that \(0 \neq a, b \in Rx \cap Ry\). There are two following cases:

Case 1. \(a\) and \(b\) are adjacent. Then the induced subgraph by the set \(\{a, b, x, y, x + a\}\) is isomorphic to \(\overline{G_2}\) (see Fig. 14), a contradiction.

Case 2. \(a\) and \(b\) are not adjacent. Then the induced subgraph by the set \(\{a, b, x, y, x + a, y + a\}\) is isomorphic to \(\overline{G_6}\) (see Fig. 15), a contradiction. \(\square\)

We close this paper by the following theorem.

Theorem 3.7. \( \text{Let } (R, m) \text{ be a local ring with } m \neq 0 \text{ and let } \Gamma'(R) \text{ be the complement of a line graph. Then } Rx \cap Ry = \{0\}, \text{ for every } \{x, y\} \in E(\Gamma'(R)) \text{ if and only if } \Gamma'(R) \text{ is a complete } r\text{-partite graph, for some positive integer } r. \)

Proof. Assume that \(\Gamma'(R)\) is the complement of a line graph and \(Rx \cap Ry = \{0\}\), for every \(\{x, y\} \in E(\Gamma'(R))\). Since \(R\) is finite, \(A = \{Rx | 0 \neq x \in m\}\) with the inclusion relation has maximal element. Let \(\{Rx_1, \ldots, Rx_r\}\) be the set of all maximal elements of \(A\), for some positive integer \(r\). We show that \(\Gamma'(R)\) is a complete \(r\)-partite graph with parts \(Rx_1 \setminus \{0\}, \ldots, Rx_r \setminus \{0\}\). We claim that every two distinct elements of \(Rx_i\) are non-adjacent. By contradiction, assume that \(0 \neq a, b \in Rx_1\) and \(\{a, b\} \in E(\Gamma'(R))\). If \(a, b \in mRx_1\), then the induced subgraph by the set \(\{a, b, x_1, a + x_1, b + x_1\}\) is isomorphic to \(\overline{G_3}\), a contradiction. If \(a \in mRx_1\) and \(b \in U(R)x_1\), then \(a \in Rb\), which is a contradiction. Also, \(Ra = Rb = Rx_1\), where \(a, b \in U(R)x_1\), which is a contradiction. Therefore the claim is proved. By the same argument, we have that every two distinct elements of \(Rx_i\) are non-adjacent, for \(i = 1, \ldots, r\). By the maximality of \(Rx_i\) and \(Rx_j\), we find that \(x_i\) and \(x_j\) are adjacent, for every \(i, j, 1 \leq i < j \leq r\). Since \(\{x_i, x_j\} \in E(\Gamma'(R))\), by our assumption we have \(Rx_i \cap Rx_j = \{0\}\), for every \(i, j, 1 \leq i < j \leq r\). This yields that every elements of \(Rx_i \setminus \{0\}\) and \(Rx_j \setminus \{0\}\) are adjacent, where \(1 \leq i < j \leq r\). Therefore \(\Gamma'(R)\) is a complete \(r\)-partite graph with parts \(Rx_1 \setminus \{0\}, \ldots, Rx_r \setminus \{0\}\). Let \(|Rx_i \setminus \{0\}| = n_i\), for \(i = 1, \ldots, r\). Then \(\Gamma'(R) = K_{n_1, \ldots, n_r} = \overline{L(\cup_{i=1}^r K_{1, n_i})}\). Conversely, suppose that \(\Gamma'(R)\) is a complete \(r\)-partite graph with parts \(V_1, \ldots, V_r\), for some positive integer \(r\) and \(\{x, y\} \in E(\Gamma'(R))\). We prove that \(Rx \cap Ry = \{0\}\). By contradiction, suppose that \(0 \neq a \in Rx \cap Ry\). Since \(\{x, y\} \in E(\Gamma'(R))\), \(x \in V_i\) and \(y \in V_j\), for some \(i \neq j\). On the other hand, \(a\) is adjacent neither \(x\) nor \(y\), because \(a \in Rx \cap Ry\). This implies that \(a \in V_i \cap V_j\), a contradiction. Therefore \(Rx \cap Ry = \{0\}\) and the proof is complete. \(\square\)
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