

SOME RESULTS ASSOCIATED WITH A GENERALIZED BASIC HYPERGEOMETRIC FUNCTION

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In this paper, we define a q -extension of the new generalized hypergeometric function given by Saxena et al. in [13], and have investigated the properties of the above new function such as q -differentiation and q -integral representation. The results presented are of general character and the results given earlier by Saxena and Kalla in [14], Virchenko, Kalla and Al-Zamel in [15], Al-Musallam and Kalla in [2, 3], Kobayashi in [7, 8], Saxena et al. in [13], Kumbhat et al. in [11] follow as special cases.

1. Introduction

From the point of view of statistical distributions, the generalized form of the hypergeometric function ${}_2F_1(a, b; c; z)$ has been investigated by Dotsenko [4], Malovichko [12] and others. Recently, Kumbhat, Gupta and Surana [11] introduced a new generalized basic hypergeometric function in the following form:

$${}_2\Phi_1^\tau(a, b; c; \tau; q, z) = \frac{\Gamma_q(c)}{\Gamma_q(b)} \sum_{n=0}^{\infty} \frac{(a; q)_n \Gamma_q(b + \tau n) z^n}{\Gamma_q(c + \tau n) (q; q)_n} \quad (1)$$

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where a, b and c are real or complex numbers, $\tau \in \mathbb{R}$, $\tau > 0$, $c \neq 0, -1, -2, \dots$, $|z| < 1$, $|q| < 1$, $\Gamma_q(b + \tau n)$ and $\Gamma_q(c + \tau n)$ are finite for integer n .

Its q -integral representation is given by :

$${}_2\Phi_1^\tau(a, b; c; \tau; q, z) = \frac{\Gamma_q(c)}{\Gamma_q(b)\Gamma_q(c-b)} \int_0^1 t^{b-1} \frac{(q^a z t^\tau, tq; q)_\infty}{(z t^\tau, tq^{c-b}; q)_\infty} dq t \quad (2)$$

where $Re(c) > Re(b) > 0$, $\tau \in \mathbb{R}$, $\tau > 0$, $|z| < 1$, $|q| < 1$.

As $q \rightarrow 1$ in (1) and (2), we get the generalized hypergeometric function (cf. Virchenko et al. [15]).

The generalized basic hypergeometric series ${}_r\Phi_s(\cdot)$, (cf. Gasper and Rahman [5]) is :

$${}_r\Phi_s \left[\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, x \right] = \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_r; q)_n}{(q, b_1, \dots, b_s; q)_n} \left\{ (-1)^n q^{\frac{n(n-1)}{2}} \right\}^{1+s-r} x^n \quad (3)$$

where the convergence of the series (3) is true for $|q| < 1$, for all x if $r \leq s$, and for $|x| < 1$ if $r = s + 1$.

For real or complex a , $|q| < 1$, the q -shifted factorial is defined as

$$(a; q)_n = \begin{cases} 1 & ; \text{ if } n = 0 \\ (1 - q^a)(1 - q^{a+1}) \dots (1 - q^{a+n-1}) & ; \text{ if } n \in \mathbb{N} \end{cases} \quad (4)$$

In terms of the q -Gamma function, (4) can be expressed as :

$$(a; q)_n = \frac{\Gamma_q(a+n)(1-q)^n}{\Gamma_q(a)}, \quad n > 0 \quad (5)$$

where

$$\Gamma_q(\alpha) = \frac{(q; q)_\infty}{(q^\alpha; q)_\infty (1-q)^{\alpha-1}} \quad (6)$$

Indeed, it is easy to verify that

$$\lim_{q \rightarrow 1^-} \Gamma_q(a) = \Gamma(a) \quad (7)$$

and

$$\lim_{q \rightarrow 1^-} \frac{(q^a; q)_n}{(1-q)^n} = (a)_n \quad (8)$$

where

$$(a)_n = a(a+1)(a+2)\dots(a+n-1), \quad n \geq 1 \quad (9)$$

The q -binomial theorem is

$${}_1\Phi_0(a; -; q, x) = \frac{(ax; q)_\infty}{(x; q)_\infty}, \quad |x| < 1 \quad (10)$$

The fractional q -derivative of arbitrary order $\lambda > 0$, for a function $f(x) = x^\mu - 1$, is given by :

$$D_{q,x}^\lambda (x^{\mu-1}) = \frac{\Gamma_q(\mu)x^{\mu-\lambda-1}}{\Gamma_q(\mu-\lambda)} \quad (11)$$

where $\mu \neq 0, -1, -2, \dots$

For $\lambda = 1$, the equation (11) reduces to

$$D_{q,x}(x^{\mu-1}) = \frac{\Gamma_q(\mu)x^{\mu-2}}{\Gamma_q(\mu-1)} = \frac{(1-q^{\mu-1})x^{\mu-2}}{(1-q)} \quad (12)$$

The q -Beta function (cf. Gasper and Rahman [5]) is

$$\begin{aligned} B_q(x, y) &= \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)} \\ &= \int_0^1 t^{x-1} \frac{(tq; q)_\infty}{(tq^y; q)_\infty} d_q t, \end{aligned} \quad (13)$$

where $Re(x) > 0, Re(y) > 0$.

The paper will be organized as follows. In section 2, we define a q -extension of the new generalized hypergeometric function and derive its q -integral representations. Certain analytic properties of this special function are described in section 3.

2. Definition and q -integral representations of ${}_3\Phi_2^\tau(\cdot)$.

A new generalized basic hypergeometric function is defined in terms of series representation in the form:

$${}_3\Phi_2^\tau(\lambda, a, b; c, d; \tau; q, z) = \frac{\Gamma_q(c)\Gamma_q(d)}{\Gamma_q(a)\Gamma_q(b)} \sum_{n=0}^{\infty} \frac{(\lambda; q)_n \Gamma_q(a + \tau n) \Gamma_q(b + \tau n) z^n}{\Gamma_q(c + \tau n) \Gamma_q(d + \tau n) (q; q)_n} \quad (14)$$

where $|z| < 1$, $\tau \in \mathbb{R}$, $\tau > 0$, $|q| < 1$, and

$${}_3\Phi_2^\tau(\lambda, a, b; c, d; \tau; q, z) \quad (15)$$

$$= \frac{\Gamma_q(d)}{\Gamma_q(a)\Gamma_q(d-a)} \int_0^1 t^{a-1} \frac{(tq; q)_\infty}{(tq^{d-a}; q)_\infty} {}_2\Phi_1^\tau(\lambda, b; c; \tau; q, zt) d_q t$$

where $Re(d) > Re(a) > 0$, $Re(c) > Re(b) > 0$, $\tau \in \mathbb{R}$, $\tau > 0$, $|z| < 1$, $|q| < 1$.

It is interesting to note that for $b = d$, (14) reduces to the generalized basic hypergeometric function studied by Kumbhat et al. [11].

If we let $q \rightarrow 1^-$ and make use of the limit formula (8), (14) reduces to the generalized hypergeometric function studied by Saxena et al. [13].

Proof of (15) :

To prove (15) we have :

$$\begin{aligned} {}_3\Phi_2^\tau(\lambda, a; b; c, d; \tau; q, z) &= \frac{\Gamma_q(c)\Gamma_q(d)}{\Gamma_q(a)\Gamma_q(b)} \sum_{n=0}^{\infty} \frac{(\lambda; q)_n \Gamma_q(a + \tau n) \Gamma_q(b + \tau n) z^n}{\Gamma_q(c + \tau n) \Gamma_q(d + \tau n) (q; q)_n} \\ &= \frac{\Gamma_q(d)}{\Gamma_q(a)\Gamma_q(d-a)} \sum_{n=0}^{\infty} \frac{(\lambda; q)_n \Gamma_q(c) \Gamma_q(b + \tau n) z^n}{(q; q)_n \Gamma_q(b) \Gamma_q(c + \tau n)} B_q(a + \tau n, d - n) \end{aligned}$$

Applying the property of integral of q -Beta function (13), we obtain :

$$\frac{\Gamma_q(d)}{\Gamma_q(a)\Gamma_q(d-a)} \int_0^1 t^{a+\tau n-1} \frac{(tq; q)_\infty}{(tq^{d-a}; q)_\infty} \cdot \frac{\Gamma_q(c) \Gamma_q(b + \tau n) (\lambda; q)_n z^n}{\Gamma_q(b) \Gamma_q(c + \tau n) (q; q)_n} d_q t$$

Interchanging the order of integration and summation; we get

$$\frac{\Gamma_q(d)}{\Gamma_q(a)\Gamma_q(d-a)} \int_0^1 t^{a-1} \frac{(tq; q)_\infty}{(tq^{d-a}; q)_\infty} \sum_{n=0}^{\infty} \frac{(\lambda; q)_n \Gamma_q(c) \Gamma_q(b + \tau n) z^n}{\Gamma_q(b) \Gamma_q(c + \tau n) (q; q)_n} \cdot (zt^\tau)^n d_q t$$

After some simplification, we get :

$$\frac{\Gamma_q(d)}{\Gamma_q(a)\Gamma_q(d-a)} \int_0^1 t^{a-1} \frac{(tq; q)_\infty}{(tq^{d-a}; q)_\infty} {}_2\Phi_1^\tau(\lambda, b; c; \tau; q, zt^\tau) d_q t$$

which completes the proof of (15).

3. q -Differentiation Formulas of ${}_3\Phi_2^\tau(\cdot)$

$$D_{q,z} [{}_3\Phi_2^\tau(\lambda, a, b; c, d; \tau; q, z)] = \frac{(\lambda; q) \Gamma_q(d) \Gamma_q(c) \Gamma_q(a + \tau) \Gamma_q(b + \tau)}{\Gamma_q(a) \Gamma_q(b) \Gamma_q(c + \tau) \Gamma_q(d + \tau)}. \quad (16)$$

$$\cdot {}_3\Phi_2^\tau(\lambda q, a + \tau, b + \tau; c + \tau, d + \tau; \tau; q, z)$$

$$D_{q,z} \left[z^\lambda {}_3\Phi_2^\tau(\lambda, a, b; c, d; \tau; q, z) \right] = (\lambda; q) z^{\lambda-1} {}_3\Phi_2^\tau(\lambda q, a, b; c, d; \tau; q, z) \quad (17)$$

$$D_{q,z}^n \left[z^{\lambda+n-1} {}_3\Phi_2^\tau(\lambda, a, b; c, d; \tau; q, z) \right] = (\lambda; q) z^{\lambda-1} {}_3\Phi_2^\tau(\lambda q^n, a, b; c, d; \tau; q, z) \quad (18)$$

$$D_{q,z}^n [{}_3\Phi_2^\tau(\lambda, a, b; c, d; \tau; q, z)] = \frac{\Gamma_q(c) \Gamma_q(d) \Gamma_q(a + \tau n) \Gamma_q(b + \tau n) (\lambda; q)_n}{\Gamma_q(a) \Gamma_q(b) \Gamma_q(c + \tau n) \Gamma_q(d + \tau n)}. \quad (19)$$

$$\cdot {}_3\Phi_2^\tau(\lambda q^n, a + \tau n, b + \tau n; c + \tau n, d + \tau n; \tau; q, z)$$

Proof of (16) :

To prove (16) we see that the left hand side of (16) can be written as :

$$\begin{aligned} & \frac{\Gamma_q(c) \Gamma_q(d)}{\Gamma_q(a) \Gamma_q(b)} \sum_{n=0}^{\infty} \frac{(\lambda; q)_n \Gamma_q(a + \tau n) \Gamma_q(b + \tau n)}{\Gamma_q(c + \tau n) \Gamma_q(d + \tau n) (q; q)_n} \cdot D_{q,z}(z^n) \\ &= \frac{\Gamma_q(c) \Gamma_q(d)}{\Gamma_q(a) \Gamma_q(b)} \sum_{n=1}^{\infty} \frac{(\lambda; q)_n \Gamma_q(a + \tau n) \Gamma_q(b + \tau n)}{\Gamma_q(c + \tau n) \Gamma_q(d + \tau n) (q; q)_n} \cdot \frac{(1 - q^n)}{(1 - q)} z^{n-1} \end{aligned}$$

Replacing n by $n + 1$; we get :

$$= \frac{\Gamma_q(c) \Gamma_q(d)}{\Gamma_q(a) \Gamma_q(b)} \sum_{n=0}^{\infty} \frac{(\lambda; q)_{n+1} \Gamma_q(a + \tau n + \tau) \Gamma_q(b + \tau n + \tau)}{\Gamma_q(c + \tau n + \tau) \Gamma_q(d + \tau n + \tau) (q; q)_{n+1}} \cdot \frac{(1 - q^{n+1})}{(1 - q)} z^n$$

After some simplification; we get :

$$= \frac{(\lambda; q) \Gamma_q(c) \Gamma_q(d)}{\Gamma_q(a) \Gamma_q(b)} \sum_{n=0}^{\infty} \frac{(\lambda; q)_n \Gamma_q(a + \tau n + \tau) \Gamma_q(b + \tau n + \tau)}{\Gamma_q(c + \tau n + \tau) \Gamma_q(d + \tau n + \tau)} \cdot \frac{z^n}{(q; q)_n}$$

Using (14), the right hand side of (16) is obtained.

The proof of (17) to (19) are straight forward and hence are omitted.

4. Application. The q -distribution

Using the result given by Saxena et al. [13], the q -Laplace type integral

$$I = \int_0^{\infty} e_q \{-\gamma (1-q) z\} z^{\beta-1} \xi (z) d (z; q)$$

Where $\xi (z) = {}_3\Phi_2^{\tau} (\lambda, a, b; c, d; \tau; q, kz^{\tau})$

On using the definition (14), it yields

$$\int_0^{\infty} e_q \{-\gamma (1-q) z\} z^{\beta-1} \frac{\Gamma_q (c) \Gamma_q (d)}{\Gamma_q (a) \Gamma_q (b)} \\ \cdot \sum_{n=0}^{\infty} \frac{(\lambda; q)_n \Gamma_q (a + \tau n) \Gamma_q (b + \tau n) (kz^{\tau})^n}{\Gamma_q (c + \tau n) \Gamma_q (d + \tau n) (q; q)_n} d (z; q)$$

On interchanging the order of integration and summation and then using the equation namely [cf. Abdi [1]],

$$z^{\beta-1} \underset{Q}{\supset} (q; q)_{\beta-1} q^{-\frac{\beta(\beta-1)}{2}} s^{-\beta} \text{Re} (\beta) > 0 \\ \underset{Q}{\supset} \frac{(q; q)_{\beta-1} (-sq^{\beta} : q)_{\infty} \left(-\frac{q^{1-\beta}}{s} : q\right)_{\infty}}{(-s : q)_{\infty} \left(-\frac{q}{s} : q\right)_{\infty}}$$

It reduces to

$$q^{\frac{(\beta-\beta^2)}{2}} \frac{\Gamma_q (\beta)}{\gamma^{\beta}} \cdot {}_4\Phi_2^{\tau} \left(\lambda, a, b, \beta; c, d; \tau; q, \frac{kq^{-\frac{\tau(2\beta+\tau n-1)}{2}}}{\gamma^{\tau}} \right)$$

Thus the function

$$f (z; \beta, \gamma, \lambda, a, b, c, d, \tau, k, q)$$

$$= \frac{e_q \{-\gamma(1-q)z\} z^{\beta-1} {}_3\Phi_2^{\tau} (\lambda, a, b; c, d; \tau; q, kz^{\tau})}{q^{\frac{(\beta-\beta^2)}{2}} \gamma^{-\beta} \Gamma_q (\beta) {}_4\Phi_2^{\tau} \left(\lambda, a, b, \beta; c, d; \tau; q, \frac{kq^{-\frac{\tau(2\beta+\tau n-1)}{2}}}{\gamma^{\tau}} \right)}$$

Where

$$\text{Re} (\beta) > 0, \text{Re} (\gamma) > 0, \text{Re} (\gamma) > \text{Re} (k), \tau > 0, 0 < |q| < 1, 0 < z < \infty,$$

$$c, d \neq 0, -1, -2, \dots, \quad = 0 \text{ elsewhere}$$

provides a basic analogue of probability density function.

The generalized basic hypergeometric function ${}_3\Phi_2^{\tau}(\lambda, a, b; c, d; \tau; q, kz^{\tau})$ can be reduced to basic hypergeometric function ${}_2\Phi_2(a, b, c, d; q, z)$ by taking $\tau = 1$ and replacing k by $\frac{k}{\lambda}$ then taking the limit $\lambda \rightarrow \infty$. The basic analogue of probability density function changes to

$$f(z; \beta, \gamma, a, b, c, d, 1, k, q) \tag{20}$$

$$= \frac{e_q\{-\gamma(1-q)z\} z^{\beta-1} {}_2\Phi_2(a, b; c, d; q, kz)}{\Gamma_q(\beta) q^{\frac{(\beta-\beta^2)}{2}} \gamma^{-\beta} {}_3\Phi_2\left(a, b, \beta; c, d; q, \frac{kq^{-2\beta}}{\gamma}\right)}$$

If we set $\gamma = \mu$ in (20), we obtain q -extension of the function of the type George and Mathai [6].

$$f(z; \beta, \gamma, a, c, 1, k, q) \tag{21}$$

$$= \frac{e_q\{-\gamma(1-q)z\} {}_1\Phi_1(a; c; q, kz)}{q^{\frac{(\beta-\beta^2)}{2}} \gamma^{-\beta} \Gamma_q(\beta) {}_2\Phi_1\left(a, \beta; c; q, \frac{kq^{-2\beta}}{\gamma}\right)}$$

If we take $k \rightarrow 0$ in (20), we obtain two parameters q -gamma density type function

$$f(z; \beta, \gamma, q) \tag{22}$$

$$= \frac{e_q\{-\gamma(1-q)z\} z^{\beta-1}}{\Gamma_q(\beta) q^{\frac{(\beta-\beta^2)}{2}} \gamma^{-\beta}}$$

Indeed, if we let $q \rightarrow 1^-$ and make use of the relation (8), in (20), we get known results due to Kumbhat and Shanu [10]. On the other hand, if we put $b = d$ and $q \rightarrow 1^-$ in (20), we obtain a known result due to George and Mathai [6].

Finally if we take $k \rightarrow 0$ or either $a = 0$ or $b = 0$ and $q \rightarrow 1^-$, we obtain a known result due to Kumbhat and Shanu [10].

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