A NEW CHARACTERIZATION AND A RODRIGUES FORMULA FOR GENERALIZED HERMITE ORTHOGONAL POLYNOMIALS

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In this paper, we consider the raising operator $R_{\xi} = \xi T_{\mu} + x I$, $\xi \neq 0$, where $T_{\mu}$ and $I$ are the Dunkl operator and the identity operator respectively. Our purpose is to determine all monic orthogonal polynomials sequences $\{P_n(x)\}_{n \geq 0}$ such that the sequence of polynomials $\{(R_{\xi} P_n)(x)\}_{n \geq 0}$ is also orthogonal. We prove that the only sequence of polynomials satisfying this condition is, up to a dilation, the generalized Hermite polynomial sequence. Then, we explore our result to deduce a Rodrigues formula for the generalized Hermite polynomials sequence.

1. Introduction

Let $\mathcal{P}$ be the vector space of polynomials with complex coefficients and let $\mathcal{O}$ be an operator on $\mathcal{P}$. A monic orthogonal polynomial sequence (MOPS, for shorter) $\{P_n(x)\}_{n \geq 0}$ is called $\mathcal{O}$-classical polynomial sequence if $\{(\mathcal{O} P_{n+1})(x)\}_{n \geq 0}$

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is orthogonal. The family of $O$-classical polynomial sequences is wide enough to accommodate the most famous orthogonal polynomial sequences. For instance, when $O$ is the derivative operator $D$, we find the continuous orthogonal polynomial sequences (Hermite, Laguerre, Bessel, Jacobi) [15]. When $O$ is the difference operator $\Delta$, i.e., $\Delta p(x) = p(x + 1) - p(x)$, the discrete orthogonal polynomial sequences (Charlier, Meixner, Krawtchouk, Hahn) are the classical ones (see [12]). For the Dunkl operator $T_\mu$ defined as $T_\mu p(x) = p'(x) + \mu \frac{p(x) - p(-x)}{x}$, the Generalized Hermite and the Generalized Gegenbauer polynomial sequences are the unique symmetric $T_\mu$-classical polynomial sequences [2]-[5]. A curious problem is to see if there exists an operator $O$ such that the generalized Hermite polynomial sequence $\{H_n^{(\mu)}(x)\}_{n \geq 0}$ is $O$-classical. In the case of existence, another problem arises: Can we give a Rodrigues formula for $\{H_n^{(\mu)}(x)\}_{n \geq 0}$? The aim of this paper is to solve these problems.

Recall that generalized Hermite polynomials were introduced by G. Szegő in 1939 as a set of real polynomials orthogonal with respect to the weight function $|x|^{2\mu} e^{-x^2}$, $\mu > -\frac{1}{2}$ supported on the whole real line [20]. They can be written as

$$\left\{ \begin{array}{l}
H_{2n}^{(\mu)}(x) = (-1)^n n! L_n^{(\mu - \frac{1}{2})}(x^2), n \geq 0 \\
H_{2n+1}^{(\mu)}(x) = (-1)^n n! x L_n^{(\mu + \frac{1}{2})}(x^2), n \geq 0
\end{array} \right. \tag{1}$$

where $\{L_n^{(\alpha)}(x)\}_{n \geq 0}$ is the sequence of Laguerre polynomials given by the following generating function:

$$(1 - t)^{-\alpha - 1} \exp(-\frac{xt}{1-t}) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n \tag{2}$$

These polynomials satisfy the second order linear differential equation [20]

$$xD^2 H_{n+1}^{(\mu)}(x) + 2(\mu - x^2) D H_{n+1}^{(\mu)}(x) + (2n x - \theta_n x^{-1}) H_{n+1}^{(\mu)}(x) = 0, n \geq 0,$$

where $\theta_n = \left\{ \begin{array}{ll}
0 & \text{if } n \text{ even}, \\
2\mu & \text{if } n \text{ odd},
\end{array} \right.$

and the three term recurrence relation

$$\left\{ \begin{array}{l}
P_0(x) = 1, P_1(x) = x - \beta_0, \\
P_{n+2}(x) = (x - \beta_{n+1}) P_{n+1}(x) - \gamma_{n+1} P_n(x), n \geq 0
\end{array} \right. \tag{3}$$
with
\[ \beta_n = 0; \gamma_{n+1} = \frac{n + 1 + \mu(1 + (-1)^n)}{2}, n \geq 0, \] (4)

where the regularity condition is \( \mu \neq -n - \frac{1}{2}, n \geq 0. \)

In [8], T. S. Chihara established the following so-called structure relation for
generalized Hermite polynomials.
\[ xDH_{n+1}(\mu) = -\mu(1 + (-1)^n)H_{n+1}(\mu) + \left( n + 1 + \mu(1 + (-1)^n) \right) xH_n(\mu), n \geq 0. \] (5)

For more information in this subject, we refer the reader to [7, 14, 18].

This paper is organized as follows. In Section 2, we introduce some notations and preliminary results to be used in the sequel. In Section 3, we prove that \( \{a^{-n}H_n^{(\mu)}(ax)\}_{n \geq 0} \) is the unique \( R_\xi \)-classical polynomial sequence, where \( a \neq 0, a^2 = -\frac{1}{2\xi}. \) Then we exploit these results to give a Rodrigues formula for the generalized Hermite polynomials.

2. Preliminaries and notations

Let \( P' \) be the algebraic linear dual of \( P. \) The elements of the dual space are called linear functionals. We denote by \( \langle u, f \rangle \) the action of \( u \in P' \) on \( f \in P. \) In particular, we denote by \( \langle u, x^n \rangle, n \geq 0, \) the moments of \( u. \) For \( f \in P, \ a \in \mathbb{C} \setminus \{0\}, \) we define the linear functionals \( fu \) and \( h_a u \) as follows
\[ \langle fu, p \rangle = \langle u, fp \rangle, \ \langle h_a u, p \rangle = \langle u, h_a p \rangle, \ p \in P, \]
where \( (h_a p)(x) = p(ax). \)

Let \( \{P_n(x)\}_{n \geq 0} \) be a sequence of monic polynomials (MPS, for shorter) with \( \deg P_n = n, n \geq 0. \) The dual sequence of \( \{P_n(x)\}_{n \geq 0} \) is the sequence \( \{u_n\}_{n \geq 0}, \ u_n \in P', \) defined by \( \langle u_n, P_m \rangle = \delta_{n,m}, n, m \geq 0. \)

A linear functional \( u \) is said to be regular (quasi-definite) if there exists a MPS \( \{P_n(x)\}_{n \geq 0} \) such that [8]
\[ \langle u, P_m P_n \rangle = r_n \delta_{n,m}, n, m \geq 0, r_n \neq 0, n \geq 0, \]
where \( \delta_{n,m} \) is the Kronecker’s symbol.

In this case, the sequence \( \{P_n(x)\}_{n \geq 0} \) is said to be orthogonal with respect to \( u \)
and we have
\[
    u_n = (\langle u_0, P_n^2 \rangle)^{-1} P_n u_0, \ n \geq 0. \tag{6}
\]
Moreover, \( u = \lambda u_0 \), where \((u)_0 = \lambda \neq 0 \) [17].

In the sequel, we will assume that all regular linear functionals \( u \) are normalized, i.e. \((u)_0 = 1\). Then, \( u = u_0 \)

According to Favard’s theorem, a sequence of monic orthogonal polynomials satisfies the following three-term recurrence relation [8]:
\[
\begin{align*}
    P_0(x) &= 1, \quad P_1(x) = x - \beta_0, \\
    P_{n+2}(x) &= (x - \beta_{n+1}) P_{n+1}(x) - \gamma_{n+1} P_n(x), \ n \geq 0,
\end{align*}
\]
with
\[
\begin{align*}
    \beta_n &= \frac{\langle u_0, x P_n^2(x) \rangle}{\langle u_0, P_n^2(x) \rangle}; \quad \gamma_{n+1} = \frac{\langle u_0, P_{n+1}^2(x) \rangle}{\langle u_0, P_n^2(x) \rangle}, \ n \geq 0.
\end{align*}
\]

A dilatation preserves the property of orthogonality. Indeed, the sequence \( \{\tilde{P}_n(x)\}_{n \geq 0} \) defined by \( \tilde{P}_n(x) = a^{-n} P_n(ax) \), \( n \geq 0 \), \( a \in \mathbb{C} \setminus \{0\} \), satisfies the recurrence relation [16]
\[
\begin{align*}
    \tilde{P}_0(x) &= 1, \quad \tilde{P}_1(x) = x - \tilde{\beta}_0, \\
    \tilde{P}_{n+2}(x) &= (x - \tilde{\beta}_{n+1}) \tilde{P}_{n+1}(x) - \tilde{\gamma}_{n+1} \tilde{P}_n(x), \ n \geq 0,
\end{align*}
\]
with
\[
\begin{align*}
    \tilde{\beta}_n &= \frac{\beta_n}{a}; \quad \tilde{\gamma}_{n+1} = \frac{\gamma_{n+1}}{a^2}, \ n \geq 0.
\end{align*}
\]
Moreover, if \( \{P_n(x)\}_{n \geq 0} \) is orthogonal with respect to the regular linear functional \( u_0 \), then \( \{\tilde{P}_n(x)\}_{n \geq 0} \) is orthogonal with respect to the regular functional \( \tilde{u}_0 = h_{a^{-1}} u_0 \).

A polynomial set \( \{P_n\}_{n \geq 0} \) is called symmetric if \( P_n(-x) = (-1)^n P_n(x), \ n \geq 0 \), or equivalently, in (7), \( \beta_n = 0 \), \( n \geq 0 \).

A monic orthogonal polynomial sequence \( \{P_n(x)\}_{n \geq 0} \) is said to be \( T_\mu \)-classical \([2]\), (or Dunkl-classical) polynomial sequence, if \( \{(T_\mu P_n)(x)\}_{n \geq 1} \) is an orthogonal polynomial sequence. Here \( T_\mu \) is the Dunkl operator defined by [11]:
\[
    T_\mu = D + 2\mu H_{-1}, \ \mu > -\frac{1}{2},
\]
where \( H_{-1} \) is the Hahn’s operator defined by:
\[
    (H_{-1} f)(x) = \frac{f(x) - f(-x)}{2x}.
\]
For the Dunkl operator $T_\mu$ the following rules hold [6]

\begin{align}
T_\mu \circ h_a &= a h_a \circ T_\mu, \quad a \in \mathbb{C} \setminus \{0\}. \\
T_\mu f(x^2) &= 2xf'(x^2), \quad f \in \mathcal{P}. \\
T_\mu(xf(x^2)) &= (1 + 2\mu)f(x^2) + 2x^2f'(x^2), \quad f \in \mathcal{P}. \\
(T_\mu f)(x)g(x) &= f(x)(T_\mu g)(x) - 4\mu x(H_{-1}f)(x)(H_{-1}g)(x), \quad f, g \in \mathcal{P}.
\end{align}  

In particular, if $f$ (or $g$) is an even function, then

\begin{equation}
(T_\mu f)(x)g(x) = f(x)(T_\mu g)(x). \tag{14}
\end{equation}

In [14], A. Ghressi and L. Khérijji proved that the generalized Hermite polynomial sequence is, up to a dilation, the unique symmetric $T_\mu$-Appell MOPS. In particular, they showed that

\begin{equation}
T_\mu \mathcal{H}^{(\mu)}_{n+1}(x) = \mu_{n+1} \mathcal{H}^{(\mu)}_n(x), \quad n \geq 0, \tag{15}
\end{equation}

where

\begin{equation}
\mu_{n+1} = n + 1 + \mu(1 + (-1)^n), \quad n \geq 0.
\end{equation}

Application of (10), formula (15) becomes

\begin{equation}
T_\mu(\mathcal{H}^{(\mu)}_{n+1}(ax)) = a\mu_{n+1} \mathcal{H}^{(\mu)}_n(ax), \quad n \geq 0. \tag{16}
\end{equation}

Next, using the generating function of Laguerre polynomials, we will give an other proof of (16). From (2) and (11) we have

\begin{align}
\sum_{n \geq 0} T_\mu L_n^{(\mu-\frac{1}{2})}(a^2x^2)t^n &= T_\mu\left((1-t)^{-\mu-\frac{1}{2}} \exp\left(-\frac{a^2x^2t}{1-t}\right)\right) \text{ (}T_\mu \text{ acts on } x) \\
&= -2a^2xt(1-t)^{-\mu-\frac{1}{2}} \exp\left(-\frac{a^2x^2t}{1-t}\right) \\
&= -2a^2 \sum_{n \geq 0} xL_n^{(\mu+\frac{1}{2})}(a^2x^2)t^{n+1} \\
&= -2a \sum_{n \geq 1} axL_{n-1}^{(\mu+\frac{1}{2})}(a^2x^2)t^n. \tag{17}
\end{align}

Taking into account (1), we get

\begin{equation}
T_\mu(\mathcal{H}^{(\mu)}_{2n}(ax)) = a\mu_{2n} \mathcal{H}^{(\mu)}_{2n-1}(ax), \quad n \geq 1. \tag{18}
\end{equation}
Similarly, from (2) and (12) we have

\[
\sum_{n \geq 0} T_{\mu}(axL_n^{(\mu+\frac{1}{2})}(a^2x^2))t^n = a\left((1+2\mu)(1-t) - 2a^2x^2t\right)(1-t)^{-\mu-\frac{1}{2}} \exp\left(-\frac{a^2x^2t}{1-t}\right)
\]

\[
= a(2t \frac{\partial}{\partial t} + 1 + 2\mu) \left((1-t)^{-\mu-\frac{1}{2}} \exp\left(-\frac{a^2x^2t}{1-t}\right)\right)
\]

\[
= a(2t \frac{\partial}{\partial t} + 1 + 2\mu) \sum_{n \geq 0} L_n^{(\mu+\frac{1}{2})}(a^2x^2)t^n
\]

\[
= a \sum_{n \geq 0} (2n + 1 + 2\mu)L_n^{(\mu+\frac{1}{2})}(a^2x^2)t^n
\]

(19)

Taking into account (1), we get

\[
T_{\mu} \mathcal{H}^{(\mu)}_{2n+1}(ax) = a\mu_{2n+1} \mathcal{H}^{(\mu)}_{2n}(ax), \ n \geq 0,
\]

(20)

From (18) and (20) we obtain (16).

3. A new characterization and a Rodrigues formula for generalized Hermite polynomials

3.1. \( R_{\xi} \)-classical orthogonal polynomial

Let \( R_{\xi} \) be the linear operator defined on the vector space of polynomials as follows

\[
R_{\xi} : \mathcal{P} \rightarrow \mathcal{P}
\]

\[
f \mapsto R_{\xi}(f)(x) = \xi(T_{\mu}f)(x) + xf(x), \ \xi \neq 0,
\]

(21)

where \( T_{\mu} \) is the Dunkl operator.

**Lemma 3.1.** For \( a \in \mathbb{C} \setminus \{0\} \) and \( \xi = -\frac{1}{2a^2} \) we have

\[
R_{\xi}(\mathcal{H}^{(\mu)}_n(ax)) = a^{-1}\mathcal{H}^{(\mu)}_{n+1}(ax), \ n \geq 0,
\]

(22)

\[
R_{\xi}^n(\mathcal{H}^{(\mu)}_0(ax)) = a^{-n}\mathcal{H}^{(\mu)}_n(ax), \ n \geq 0,
\]

(23)

and

\[
R_{\xi}^m(\mathcal{H}^{(\mu)}_n(ax)) = a^{-m}\mathcal{H}^{(\mu)}_{n+m}(ax), \ n, m \geq 0,
\]

(24)

where \( R_{\xi}^n \) is defined by iteration as follows

\[
R_{\xi}^0 = I, \ R_{\xi}^{n+1} = R_{\xi} \circ R_{\xi}^n, \ n \geq 0.
\]
Applying the operator $R$, we obtain

\[ (1-t)^{-\mu - \frac{1}{2}} \exp\left(\frac{-a^2x^2t}{1-t}\right) = \sum_{n \geq 0} t_n^{(\mu - \frac{1}{2})}(a^2x^2)t^n \]  \hspace{1cm} (25)

Taking into account (1), we obtain

\[ \sum_{n \geq 0} R_\xi (L_n^{(\mu - \frac{1}{2})}(a^2x^2))t^n = (x + \frac{xt}{1-t})(1-t)^{-\mu - \frac{1}{2}} \exp\left(\frac{-a^2x^2t}{1-t}\right) \]
\[ = x(1-t)^{-\mu - \frac{1}{2}} \exp\left(\frac{-a^2x^2t}{1-t}\right) \]
\[ = a^{-1} \sum_{n \geq 0} axL_n^{(\mu + \frac{1}{2})}(a^2x^2)t^n \]  \hspace{1cm} (26)

Similarly, from (2), where $x \to a^2x^2$ and $\alpha = \mu + \frac{1}{2}$, we have

\[ (1-t)^{-\mu - \frac{3}{2}} x \exp\left(\frac{-a^2x^2t}{1-t}\right) = \sum_{n \geq 0} xL_n^{(\mu + \frac{1}{2})}(a^2x^2)t^n \]  \hspace{1cm} (28)

Applying the operator $R_\xi$, where $\xi = -\frac{1}{2a^2}$, and using (12), we obtain

\[ \sum_{n \geq 0} R_\xi (xL_n^{(\mu + \frac{1}{2})}(a^2x^2))t^n = \left( x^2 + \xi (1+2\mu)(1-t) \right)(1-t)^{-\mu - \frac{5}{2}} \exp\left(\frac{-a^2x^2t}{1-t}\right) \]  \hspace{1cm} (29)

But,

\[ \left( x^2 + \xi (1+2\mu)(1-t) \right)(1-t)^{-\mu - \frac{5}{2}} \exp\left(\frac{-a^2x^2t}{1-t}\right) = -a^{-2} \frac{\partial}{\partial t} \left( (1-t)^{-\mu - \frac{1}{2}} \exp\left(\frac{-a^2x^2t}{1-t}\right) \right) \]

Then,

\[ a^2 \sum_{n \geq 0} R_\xi \left( xL_n^{(\mu + \frac{1}{2})}(a^2x^2) \right)t^n = -\frac{\partial}{\partial t} \left( (1-t)^{-\mu - \frac{1}{2}} \exp\left(\frac{-a^2x^2t}{1-t}\right) \right) \]
\[ = -\frac{\partial}{\partial t} \sum_{n \geq 0} L_n^{(\mu - \frac{1}{2})}(a^2x^2)t^n \]
\[ = \sum_{n \geq 1} -nL_n^{(\mu - \frac{1}{2})}(a^2x^2)t^{n-1} \]
\[ = \sum_{n \geq 0} -(n+1)L_{n+1}^{(\mu - \frac{1}{2})}(a^2x^2)t^n \]
So, by virtue of (1), we get

\[ R_\xi (H_{2n+1}^{(\mu)} (ax)) = a^{-1} H_{2n+2}^{(\mu)} (ax) \]  

(30)

From (27) and (30), formula (22) follows.

Relations (23) and (24) are straightforward consequences of (22).

Next, we will determine all \( R_\xi \)-classical polynomial sequences.

**Definition 3.1.** A MOPS \( \{P_n(x)\}_{n \geq 0} \) is called \( R_\xi \)-classical polynomial sequence if the sequence \( \{S_n(x)\}_{n \geq 0} \) defined by

\[ S_0(x) = 1, \quad S_{n+1}(x) = (R_\xi P_n)(x), \quad n \geq 0, \]

is orthogonal.

**Remarks 3.1.**

1. The orthogonality of the sequence \( \{S_n(x)\}_{n \geq 0} \) is not necessarily with respect to the same linear functional as \( \{P_n(x)\}_{n \geq 0} \).

2. According to Lemma 3.1, \( \{H_n^{(\mu)} (ax)\}_{n \geq 0} \) is an \( R_\xi \)-classical MOPS, where

\[ \xi = -\frac{1}{2a^2}. \]

**Theorem 3.2.** For any non-zero complex number \( \xi \) and any MOPS \( \{P_n(x)\}_{n \geq 0} \), the following statements are equivalent.

(a) \( \{P_n(x)\}_{n \geq 0} \) is \( R_\xi \)-classical.

(b) \( P_n(x) = a^{-n} H_n^{(\mu)} (ax), \quad n \geq 0 \), where \( a^2 = -(2\xi)^{-1} \)

**Proof.** (a) \( \Rightarrow \) (b) Let \( \{P_n(x)\}_{n \geq 0} \) be a MOPS fulfilling (7) and \( \{S_n(x)\}_{n \geq 0} \) be the MPS defined by

\[ S_0(x) = 1, \quad S_{n+1}(x) = \xi (T_\mu P_n)(x) + xP_n(x), \quad n \geq 0. \]  

(31)

According to Favard’s theorem, \( \{S_n(x)\}_{n \geq 0} \) satisfies

\[
\begin{cases}
S_0(x) = 1, S_1(x) = x - \alpha_0, \\
S_{n+2}(x) = (x - \alpha_{n+1})S_{n+1}(x) - \lambda_{n+1}S_n(x), \lambda_{n+1} \neq 0, \quad n \geq 0.
\end{cases}
\]  

(32)

Applying the operator \( T_\mu \) to (7), where \( n \to n - 1 \) and using (13), we get

\[
(T_\mu P_{n+1})(x) = (1 + 2\mu)P_n(x) + (x - \beta_n)(T_\mu P_n)(x) - 4\mu x(H_{-1}P_n)(x) - \gamma_n(T_\mu P_{n-1}), \quad n \geq 1.
\]
Equivalently,

\[(T_\mu P_{n+1})(x) = P_n(x) + 2\mu P_n(-x) + (x - \beta_n)(T_\mu P_n)(x) - \gamma_n(T_\mu P_{n-1}), \quad n \geq 1.\]  

(33)

Multiplying (33) and (7) by \(\xi\) and \(x\), respectively, and summing the result, we obtain

\[\xi (T_\mu P_{n+1})(x) + xP_{n+1}(x) = (x - \beta_n)\left(\xi (T_\mu P_n)(x) + xP_n\right) - \gamma_n(xP_{n-1} + \xi (T_\mu P_{n-1})(x)) + \xi (P_n(x) + 2\mu P_n(-x)), \quad n \geq 1.\]

(34)

Taking into account (31), we get

\[S_{n+2}(x) = (x - \beta_n)S_{n+1}(x) - \gamma_nS_n(x) + \xi (P_n(x) + 2\mu P_n(-x)), \quad n \geq 1.\]

(35)

Identification of (32) with the last equation gives

\[(x - \alpha_{n+1})S_{n+1}(x) - \lambda_{n+1}S_n(x) = (x - \beta_n)S_{n+1}(x) - \gamma_nS_n(x) + \xi (P_n(x) + 2\mu P_n(-x)), \quad n \geq 1.\]

(36)

Or, equivalently,

\[(\beta_n - \alpha_{n+1})S_{n+1}(x) + (\gamma_n - \lambda_{n+1})S_n(x) = \xi (P_n(x) + 2\mu P_n(-x)), \quad n \geq 1.\]

(37)

Comparing the degrees in the last equation, we obtain

\[\beta_n = \alpha_{n+1}, \quad n \geq 1,\]

(38)

and

\[\gamma_n - \lambda_{n+1} = \xi (1 + 2\mu (-1)^n), \quad n \geq 1.\]

(39)

Notice that from Definition 3.1 we get \(S_1(x) = x\), i.e. \(\alpha_0 = 0\). Since \(\xi \neq 0\), then (37) becomes

\[\xi (1 + 2\mu (-1)^n)S_n(x) = P_n(x) + 2\mu P_n(-x), \quad n \geq 1.\]

(40)

On the other hand, for \(n = 1\), the last equality gives \((1 + 2\mu)\beta_0 = 0\). Taking into account the fact that \(\mu \neq -1/2\), we get \(\beta_0 = 0\).

Using (31) and (40), we will prove by recurrence on \(n\), that

\[S_n(-x) = (-1)^nS_n(x), \quad n \geq 0.\]

(41)
and
\[ P_n(-x) = (-1)^n P_n(x), \quad n \geq 0. \] (42)

The assumption is true for \( n = 0 \). In fact,
\[ S_0(-x) = 1 = (-1)^0 S_0(x), \quad P_0(-x) = 1 = (-1)^0 P_0(x). \]

Suppose that the assumption is true until \( n \) and let us prove it for \( n + 1 \). Using the definition of the operator \( T_\mu \), (31) can be written as
\[ S_{n+1}(x) = \xi \left( P_n'(x) + \mu \frac{P_n(x) - P_n(-x)}{x} \right) + xP_n(x), \quad n \geq 0. \] (43)

Taking into account (42), we get
\[ S_{n+1}(x) = \xi \left( P_n'(x) + \mu (1 - (-1)^n \frac{P_n(x)}{x}) \right) + xP_n(x), \quad n \geq 0. \] (44)

Using (42) and (44), we get
\[ S_{n+1}(-x) = (-1)^{n+1} S_{n+1}(x). \] (45)

The change of indices \( n \to n + 1 \) in (40) yields
\[ (1 + 2\mu (-1)^{n+1}) S_{n+1}(x) = P_{n+1}(x) + 2\mu P_{n+1}(-x), \quad n \geq 0. \] (46)

Replacing \( x \) by \(-x\) in (46), we obtain
\[ (1 + 2\mu (-1)^{n+1}) S_{n+1}(-x) = P_{n+1}(-x) + 2\mu P_{n+1}(x), \quad n \geq 0. \] (47)

From (45), (46) and (47), we get
\[ P_{n+1}(-x) + 2\mu P_{n+1}(x) = (-1)^{n+1} \left( P_{n+1}(x) + 2\mu P_{n+1}(-x) \right). \]

Or, equivalently,
\[ ((-1)^n + 2\mu) P_{n+1}(x) = (2\mu (-1)^{n+1} - 1) P_{n+1}(-x). \]

So,
\[ P_{n+1}(-x) = (-1)^{n+1} P_{n+1}(x). \]
This completes the recurrence proof.

Notice that (41) and (42) mean that \( \{S_n(x)\}_{n \geq 0} \) and \( \{P_n(x)\}_{n \geq 0} \) are two symmetric sequences. Furthermore, according to (40) and (42), we have

\[ S_n(x) = P_n(x), \quad n \geq 0. \]  

(48)

Therefore,

\[ \beta_n = \alpha_n = 0, \quad n \geq 0, \] 

(49)

and

\[ \gamma_n = \lambda_n, \quad n \geq 1. \] 

(50)

Thus, (39) becomes

\[ \gamma_n - \gamma_{n+1} = \xi (1 + 2\mu (-1)^n), \quad n \geq 1. \] 

(51)

Writing

\[ 1 + 2\mu (-1)^n = \mu_{n+1} - \mu_n, \quad n \geq 0, \mu_0 = 0, \]

Equality (51) becomes

\[ \gamma_{n+1} - \gamma_n = -\xi (\mu_{n+1} - \mu_n), \quad n \geq 1. \]

Hence,

\[ \gamma_n - \gamma_1 = -\xi (\mu_n - \mu_1), \quad n \geq 1. \] 

(52)

But, from (31), where \( n = 1 \), we have

\[ \gamma_1 = -\xi \mu_1. \]

Therefore, (52) becomes

\[ \gamma_n = -\xi \mu_n, \quad n \geq 1. \]

Application of (9) with

\[ a^2 = -(2\xi)^{-1}, \] 

(53)

we obtain

\[ \tilde{\beta}_n = 0; \quad \tilde{\gamma}_{n+1} = \frac{\mu_{n+1}}{2}, \quad n \geq 0. \]

So, according to (4),

\[ P_n(x) = a^{-n} \mathcal{H}_n^{(\mu)}(ax), \quad a^2 = -(2\xi)^{-1}, \quad n \geq 0, \]

where \( \{\mathcal{H}_n^{(\mu)}\}_{n \geq 0} \) is the generalized Hermite sequence.

(b) \( \Rightarrow \) (a) The proof is an immediate consequence of Remarks 3.1 \( \square \)
3.2. Rodrigues formula of generalized Hermite polynomials

In this subsection, we will write \( H_{n}^{(\mu)}(x), n \geq 0 \) as follows
\[
H_{n}^{(\mu)}(x) = \frac{1}{\zeta_{n}\omega(x)} T_{\mu}^{n} \left( \omega(x) \pi^{n}(x) \right), n \geq 0,
\]
where \( \zeta_{n} \) is a normalization factor, \( \pi(x) \) is a polynomial of degree less than or equal to 2, and \( \omega(x) \) is an integrable function supported on a subset of the real line. The last equality is known as Rodrigues formula of generalized Hermite polynomials. We need the two following results:

**Proposition 3.3.** For \( a \in \mathbb{C} \setminus \{0\} \) and \( \xi = -\frac{1}{2a^2} \), the operator \( R_{\xi} \) satisfies the following relation
\[
(R_{\xi}^{n}f)(x) = \frac{(-1)^{n}}{2^{n}a^{2n}} e^{a^{2}x^{2}} T_{\mu}^{n} \left( f(x)e^{-a^{2}x^{2}} \right), f \in \mathcal{P}, n \geq 0. \tag{54}
\]

**Proof.** The proof is based on the following result: If \( A \) is an operator on functions and \( g(x) \neq 0 \) for all \( x \) then \( g^{-1}Ag \), the conjugate of \( A \), satisfies
\[
(g^{-1}Ag)^{n} = g^{-1}A^{n}g, n \geq 0. \tag{55}
\]
Putting
\[
g(x) = \exp \left( -\frac{1}{2} a^{2}x^{2} \right) \tag{56}
\]
On account of (53), we can easily see that
\[
g^{-2}T_{\mu}g^{2} = (-2a^{2})R_{\xi} \tag{57}
\]
So, application of (55) with \( A = T_{\mu} \) and \( g \rightarrow g^{2} \), the desired result follows. \( \square \)

**Corollary 3.4.** For \( a \in \mathbb{C} \setminus \{0\} \) and \( \xi = -\frac{1}{2a^2} \), the generalized Hermite polynomial sequence \( \{H_{n}^{(\mu)}(x)\}_{n \geq 0} \) satisfies
\[
R_{\xi}^{n}(H_{m}^{(\mu)}(ax)) = \frac{(-1)^{n}}{2^{n}a^{2n}} e^{a^{2}x^{2}} T_{\mu}^{n} \left( H_{m}^{(\mu)}(ax)e^{-a^{2}x^{2}} \right), n, m \geq 0. \tag{58}
\]

**Theorem 3.5.** For \( a \in \mathbb{C} \setminus \{0\} \) and \( \xi = -\frac{1}{2a^2} \), the generalized Hermite polynomials sequence satisfies the following Rodrigues formula:
\[
H_{n+m}^{(\mu)}(ax) = \frac{(-1)^{n}}{2^{n}a^{n}} e^{a^{2}x^{2}} T_{\mu}^{n} \left( H_{m}^{(\mu)}(ax)e^{-a^{2}x^{2}} \right), n, m \geq 0. \tag{59}
\]
In particular,
\[
H_{n}^{(\mu)}(ax) = \frac{(-1)^{n}}{2^{n}a^{n}} e^{a^{2}x^{2}} T_{\mu}^{n} (e^{-a^{2}x^{2}}), n \geq 0. \tag{60}
\]
Proof. From (58) and (24), we obtain (59).

From (59), where \( m = 0 \), we get (60).

\[ \square \]

**Remarks 3.2.**

1. When \( \mu = 0 \), we recover the classical Rodrigues formula for the Hermite polynomials sequence [19].

2. Notice that the results of this paper generalize those in [1].

To conclude this paper, we will give some relations between \( T_\mu \) and \( R_\xi \) and their applications in physics. Fore more information the reader can be referred to [13]. Put

\[ A_\alpha^+ = a^2xI + T_\mu, \ A_\alpha^- = a^2xI - T_\mu \]

Then, we have

\[ g^{-1}A_\alpha^+g = T_\mu, \ g^{-1}A_\alpha^-g = 2a^2R_\xi \]

These formulas can be used for an interesting application to the quantum harmonic oscillator. First, note that

\[ \frac{1}{2}(A_\alpha^+A_\alpha^- + A_\alpha^-A_\alpha^+) = -T_\mu^2 + a^4x^2I \]

and \( (-T_\mu^2 + a^4x^2I)\Psi(x) = E\Psi(x) \) is a modified form of the Schrödinger equation for the harmonic oscillator on the line, namely \( (-\frac{d}{dx})^2 + a^4x^2I)\Psi(x) = E\Psi(x) \), with the potential \( a^4x^2 \) (and \( a > 0 \)). The eigenfunctions are of the form \( p(x)g(x) \). Moreover, using the definition of the operator \( R_\xi \) and (14), we can see that

\[ (-T_\mu^2 + a^4x^2I)p(x)g(x) = a^2g(x)(T_\mu R_\xi + R_\xi T_\mu)p(x) \] (61)

Consequently, using (15) and (16), we get

\[ (-T_\mu^2 + a^4x^2I)\mathcal{H}_n^{(\mu)}(ax)g(x) = a^2(2n + 1 + 2\mu)\mathcal{H}_n^{(\mu)}(ax)g(x). \]

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REFERENCES


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