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A NEW CHARACTERIZATION AND A RODRIGUES FORMULA FOR GENERALIZED HERMITE ORTHOGONAL POLYNOMIALS

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In this paper, we consider the raising operator $\mathcal{R}_\xi = \xi T_\mu + x\mathbb{I}$, $\xi \neq 0$, where T_μ and $\mathbb{I}$ are the Dunkl operator and the identity operator respectively. Our purpose is to determine all monic orthogonal polynomials sequences $\{P_n(x)\}_{n\geq 0}$ such that the sequence of polynomials $\{(\mathcal{R}_\xi P_n)(x)\}_{n\geq 0}$ is also orthogonal. We prove that the only sequence of polynomials satisfying this condition is, up to a dilation, the generalized Hermite polynomial sequence. Then, we explore our result to deduce a Rodrigues formula for the generalized Hermite polynomials sequence .

1. Introduction

Let $\mathcal{P}$ be the vector space of polynomials with complex coefficients and let $\mathcal{O}$ be an operator on $\mathcal{P}$. A monic orthogonal polynomial sequence (MOPS, for shorter) $\{P_n(x)\}_{n\geq 0}$ is called $\mathcal{O}$-classical polynomial sequence if $\{\mathcal{O}P_{n+1}(x)\}_{n\geq 0}$

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is orthogonal. The family of $\mathcal{O}$-classical polynomial sequences is wide enough to accommodate the most famous orthogonal polynomial sequences. For instance, when $\mathcal{O}$ is the derivative operator D, we find the continuous orthogonal polynomial sequences (Hermite, Laguerre, Bessel, Jacobi) [15]. When $\mathcal{O}$ is the difference operator Δ, i.e., $\Delta p(x) = p(x+1) - p(x)$, the discrete orthogonal polynomial sequences (Charlier, Meixner, Krawtchouk, Hahn) are the classical ones (see [12]). For the Dunkl operator T_μ defined as $T_\mu p(x) = p'(x) + \mu \frac{p(x)-p(-x)}{x}$, the Generalized Hermite and the Generalized Gegenbauer polynomial sequences are the unique symmetric T_μ-classical polynomial sequences [2]-[5]. A curious problem is to see if there exists an operator $\mathcal{O}$ such that the generalized Hermite polynomial sequence $\{\mathcal{H}_n^{(\mu)}(x)\}_{n\geq 0}$ is $\mathcal{O}$-classical. In the case of existence, an other problem arises: Can we give a Rodrigues formula for $\{\mathcal{H}_n^{(\mu)}(x)\}_{n\geq 0}$? The aim of this paper is to solve these problems.

Recall that generalized Hermite polynomials were introduced by G. Szegö in 1939 as a set of real polynomials orthogonal with respect to the weight function $|x|^{2\mu}e^{-x^2}$, $\mu > -\frac{1}{2}$ supported on the whole real line [20]. They can be written as

$$\begin{cases} \mathcal{H}_{2n}^{(\mu)}(x) = (-1)^n n! L_n^{(\mu-\frac{1}{2})}(x^2), n \geq 0 \\ \mathcal{H}_{2n+1}^{(\mu)}(x) = (-1)^n n! x L_n^{(\mu+\frac{1}{2})}(x^2), n \geq 0 \end{cases} \tag{1}$$

where $\{L_n^{(\alpha)}(x)\}_{n\geq 0}$ is the sequence of Laguerre polynomials given by the following generating function:

$$(1-t)^{-\alpha-1}\exp(-\frac{xt}{1-t}) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n \tag{2}$$

These polynomials satisfy the second order linear differential equation [20]

$$xD^2\mathcal{H}_{n+1}^{(\mu)}(x) + 2(\mu - x^2)D\mathcal{H}_{n+1}^{(\mu)}(x) + (2nx - \theta_n x^{-1})\mathcal{H}_{n+1}^{(\mu)}(x) = 0, n \geq 0,$$

where $\theta_n = \begin{cases} 0 \;\; if\, n\, even, \\ 2\mu \;\; if\, n\, odd, \end{cases}$

and the three term recurrence relation

$$\begin{cases} P_0(x) = 1, P_1(x) = x - \beta_0, \\ P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), n \geq 0 \end{cases} \tag{3}$$

with

$$\beta_n = 0;\ \gamma_{n+1} = \frac{n+1+\mu(1+(-1)^n)}{2},\ n \geq 0, \tag{4}$$

where the regularity condition is $\mu \neq -n-\frac{1}{2}$, $n \geq 0$.
In [8], T. S. Chihara established the following so-called structure relation for generalized Hermite polynomials.

$$xD\mathcal{H}_{n+1}^{(\mu)}(x) = -\mu(1+(-1)^n)\mathcal{H}_{n+1}^{(\mu)}(x) + \Big(n+1+\mu(1+(-1)^n)\Big)x\mathcal{H}_n^{(\mu)}(x),\ n \geq 0. \tag{5}$$

For more information in this subject, we refer the reader to [7, 14, 18].

This paper is organized as follows. In Section 2, we introduce some notations and preliminary results to be used in the sequel. In Section 3, we prove that $\{a^{-n}\mathcal{H}_n^{(\mu)}(ax)\}_{n\geq 0}$ is the unique $\mathcal{R}_\xi$-classical polynomial sequence, where $a \neq 0$, $a^2 = -\dfrac{1}{2\xi}$. Then we exploit these results to give a Rodrigues formula for the generalized Hermite polynomials.

2. Preliminaries and notations

Let $\mathcal{P}'$ be the algebraic linear dual of $\mathcal{P}$. The elements of the dual space are called linear functionals. We denote by $\langle u, f\rangle$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(u)_n = \langle u, x^n\rangle$, $n \geq 0$, the moments of u. For $f \in \mathcal{P}$, $a \in \mathbb{C}\setminus\{0\}$, we define the linear functionals fu and $h_a u$ as follows

$$\langle fu, p\rangle = \langle u, fp\rangle,\ \langle h_a u, p\rangle = \langle u, h_a p\rangle,\ p \in \mathcal{P},$$

where $(h_a p)(x) = p(ax)$.
Let $\{P_n(x)\}_{n\geq 0}$ be a sequence of monic polynomials (MPS, for shorter) with $\deg P_n = n$, $n \geq 0$. The dual sequence of $\{P_n(x)\}_{n\geq 0}$ is the sequence $\{u_n\}_{n\geq 0}$, $u_n \in \mathcal{P}'$, defined by $\langle u_n, P_m\rangle = \delta_{n,m}$, $n, m \geq 0$.
A linear functional u is said to be regular (quasi-definite) if there exists a MPS $\{P_n(x)\}_{n\geq 0}$ such that [8]

$$\langle u, P_m P_n\rangle = r_n\delta_{n,m},\ n, m \geq 0,\ r_n \neq 0, n \geq 0,$$

where $\delta_{n,m}$ is the Kronecker's symbol.
In this case,the sequence $\{P_n(x)\}_{n\geq 0}$ is said to be orthogonal with respect to u

and we have

$$u_n = (\langle u_0, P_n^2\rangle)^{-1} P_n u_0,\ n \geq 0. \tag{6}$$

Moreover, $u = \lambda u_0$, where $(u)_0 = \lambda \neq 0$ [17].
In the sequel, we will assume that all regular linear functionals u are normalized, i.e. $(u)_0 = 1$. Then, $u = u_0$
According to Favard's theorem, a sequence of monic orthogonal polynomials satisfies the following three-term recurrence relation [8]:

$$\begin{cases} P_0(x) = 1,\ P_1(x) = x - \beta_0, \\ P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x),\ n \geq 0, \end{cases} \tag{7}$$

with

$$\beta_n = \frac{\langle u_0, xP_n^2(x)\rangle}{\langle u_0, P_n^2(x)\rangle};\ \gamma_{n+1} = \frac{\langle u_0, P_{n+1}^2(x)\rangle}{\langle u_0, P_n^2(x)\rangle},\ n \geq 0.$$

A dilatation preserves the property of orthogonality. Indeed, the sequence $\{\widetilde{P}_n(x)\}_{n\geq 0}$ defined by $\widetilde{P}_n(x) = a^{-n}P_n(ax)$, $n \geq 0$, $a \in \mathbb{C}\setminus\{0\}$, satisfies the recurrence relation [16]

$$\begin{cases} \widetilde{P}_0(x) = 1,\ \widetilde{P}_1(x) = x - \widetilde{\beta}_0, \\ \widetilde{P}_{n+2}(x) = (x - \widetilde{\beta}_{n+1})\widetilde{P}_{n+1}(x) - \widetilde{\gamma}_{n+1}\widetilde{P}_n(x),\ n \geq 0, \end{cases} \tag{8}$$

$$\widetilde{\beta}_n = \frac{\beta_n}{a};\ \widetilde{\gamma}_{n+1} = \frac{\gamma_{n+1}}{a^2},\ n \geq 0. \tag{9}$$

Moreover, if $\{P_n(x)\}_{n\geq 0}$ is orthogonal with respect to the regular linear functional u_0, then $\{\widetilde{P}_n(x)\}_{n\geq 0}$ is orthogonal with respect to the regular functional $\widetilde{u}_0 = h_{a^{-1}}u_0$.
A polynomial set $\{P_n\}_{n\geq 0}$ is called symmetric if $P_n(-x) = (-1)^n P_n(x)$, $n \geq 0$, or equivalently, in (7), $\beta_n = 0$, $n \geq 0$.
A monic orthogonal polynomial sequence $\{P_n(x)\}_{n\geq 0}$ is said to be T_μ-classical [2], (or Dunkl-classical) polynomial sequence, if $\{(T_\mu P_n)(x)\}_{n\geq 1}$ is an orthogonal polynomial sequence. Here T_μ is the Dunkl operator defined by [11]:

$$T_\mu = D + 2\mu H_{-1},\ \mu > -\frac{1}{2},$$

where H_{-1} is the Hahn's operator defined by:

$$(H_{-1}f)(x) = \frac{f(x) - f(-x)}{2x}.$$

For the Dunkl operator T_μ the following rules hold [6]

$$T_\mu \circ h_a = a h_a \circ T_\mu,\ a \in \mathbb{C}\setminus\{0\}. \tag{10}$$

$$T_\mu f(x^2) = 2x f'(x^2),\ f \in \mathcal{P}. \tag{11}$$

$$T_\mu(x f(x^2)) = (1+2\mu) f(x^2) + 2x^2 f'(x^2),\ f \in \mathcal{P}. \tag{12}$$

$$(T_\mu fg)(x) = (T_\mu f)(x)g(x) + f(x)(T_\mu g)(x) - 4\mu x(H_{-1}f)(x)(H_{-1}g)(x),\ f,g \in \mathcal{P}. \tag{13}$$

In particular, if f (or g) is an even function, then

$$(T_\mu fg)(x) = (T_\mu f)(x)g(x) + f(x)(T_\mu g)(x). \tag{14}$$

In [14], A. Ghressi and L. Khériji proved that the generalized Hermite polynomial sequence is, up to a dilation, the unique symmetric T_μ-Appell MOPS. In particular, they showed that

$$T_\mu \mathcal{H}_{n+1}^{(\mu)}(x) = \mu_{n+1}\mathcal{H}_n^{(\mu)}(x),\ n \geq 0, \tag{15}$$

where

$$\mu_{n+1} = n+1+\mu(1+(-1)^n),\ n \geq 0.$$

Application of (10), formula (15) becomes

$$T_\mu(\mathcal{H}_{n+1}^{(\mu)}(ax)) = a\mu_{n+1}\mathcal{H}_n^{(\mu)}(ax),\ n \geq 0. \tag{16}$$

Next, using the generating function of Laguerre polynomials, we will give an other proof of (16). From (2) and (11) we have

$$\begin{aligned}
\sum_{n\geq 0} T_\mu L_n^{(\mu-\frac{1}{2})}(a^2x^2)t^n &= T_\mu\left((1-t)^{-\mu-\frac{1}{2}}\exp(-\tfrac{a^2x^2t}{1-t})\right)\ (T_\mu \text{acts on } x)\\
&= -2a^2xt(1-t)^{-\mu-\frac{3}{2}}\exp(-\frac{a^2x^2t}{1-t})\\
&= -2a^2\sum_{n\geq 0} xL_n^{(\mu+\frac{1}{2})}(a^2x^2)t^{n+1}\\
&= -2a\sum_{n\geq 1} axL_{n-1}^{(\mu+\frac{1}{2})}(a^2x^2)t^n
\end{aligned} \tag{17}$$

Taking into account (1), we get

$$T_\mu(\mathcal{H}_{2n}^{(\mu)}(ax)) = a\mu_{2n}\mathcal{H}_{2n-1}^{(\mu)}(ax),\ n \geq 1. \tag{18}$$

Similarly, from (2) and (12) we have

$$\begin{aligned}
\sum_{n\geq 0} T_\mu(axL_n^{(\mu+\frac{1}{2})}(a^2x^2))t^n &= a\Big((1+2\mu)(1-t)-2a^2x^2t\Big)(1-t)^{-\mu-\frac{5}{2}}\exp(-\frac{a^2x^2t}{1-t})\\
&= a(2t\frac{\partial}{\partial t}+1+2\mu)\Big((1-t)^{-\mu-\frac{1}{2}}\exp(\frac{-a^2x^2t}{1-t})\Big)\\
&= a(2t\frac{\partial}{\partial t}+1+2\mu)\sum_{n\geq 0} L_n^{(\mu-\frac{1}{2})}(a^2x^2)t^n\\
&= a\sum_{n\geq 0}(2n+1+2\mu)L_n^{(\mu-\frac{1}{2})}(a^2x^2)t^n
\end{aligned} \tag{19}$$

Taking into account (1), we get

$$T_\mu \mathcal{H}_{2n+1}^{(\mu)}(ax) = a\mu_{2n+1}\mathcal{H}_{2n}^{(\mu)}(ax),\, n\geq 0, \tag{20}$$

From (18) and (20) we obtain (16).

3. A new characterization and a Rodrigues formula for generalized Hermite polynomials

3.1. $\mathcal{R}_\xi$-classical orthogonal polynomial

Let $\mathcal{R}_\xi$ be the linear operator defined on the vector space of polynomials as follows

$$\begin{aligned}
\mathcal{R}_\xi : \mathcal{P} &\longrightarrow \mathcal{P}\\
f &\longmapsto \mathcal{R}_\xi(f)(x) = \xi(T_\mu f)(x) + xf(x),\, \xi\neq 0,
\end{aligned} \tag{21}$$

where T_μ is the Dunkl operator.

Lemma 3.1. *For $a\in\mathbb{C}\setminus\{0\}$ and $\xi = -\frac{1}{2a^2}$ we have*

$$\mathcal{R}_\xi(\mathcal{H}_n^{(\mu)}(ax)) = a^{-1}\mathcal{H}_{n+1}^{(\mu)}(ax),\, n\geq 0, \tag{22}$$

$$\mathcal{R}_\xi^n(\mathcal{H}_0^{(\mu)}(ax)) = a^{-n}\mathcal{H}_n^{(\mu)}(ax),\, n\geq 0, \tag{23}$$

and

$$\mathcal{R}_\xi^m(\mathcal{H}_n^{(\mu)}(ax)) = a^{-m}\mathcal{H}_{n+m}^{(\mu)}(ax),\, n,m\geq 0, \tag{24}$$

where $\mathcal{R}_\xi^n$ is defined by iteration as follows

$$\mathcal{R}_\xi^0 = \mathbb{I},\ \mathcal{R}_\xi^{n+1} = \mathcal{R}_\xi\circ\mathcal{R}_\xi^n,\, n\geq 0.$$

Proof. In (2), we replace x by a^2x^2 and we take $\alpha = \mu - \frac{1}{2}$, we get

$$(1-t)^{-\mu-\frac{1}{2}} \exp(\frac{-a^2x^2t}{1-t}) = \sum_{n\geq 0} L_n^{(\mu-\frac{1}{2})}(a^2x^2)t^n \tag{25}$$

Applying the operator $\mathcal{R}_\xi$, where $\xi = -\dfrac{1}{2a^2}$, and using (11), we obtain

$$\begin{aligned}\sum_{n\geq 0} \mathcal{R}_\xi(L_n^{(\mu-\frac{1}{2})}(a^2x^2))t^n &= (x+\frac{xt}{1-t})(1-t)^{-\mu-\frac{1}{2}} \exp(-\frac{a^2x^2t}{1-t}) \\ &= x(1-t)^{-\mu-\frac{3}{2}} \exp(-\frac{a^2x^2t}{1-t}) \\ &= a^{-1}\sum_{n\geq 0} axL_n^{(\mu+\frac{1}{2})}(a^2x^2)t^n\end{aligned} \tag{26}$$

Taking into account (1), we obtain

$$\mathcal{R}_\xi \mathcal{H}_{2n}^{(\mu)}(ax) = a^{-1}\mathcal{H}_{2n+1}^{(\mu)}(ax) \tag{27}$$

Similarly, from (2), where $x \to a^2x^2$ and $\alpha = \mu + \dfrac{1}{2}$, we have

$$(1-t)^{-\mu-\frac{3}{2}} x\exp(\frac{-a^2x^2t}{1-t}) = \sum_{n\geq 0} xL_n^{(\mu+\frac{1}{2})}(a^2x^2)t^n \tag{28}$$

Applying the operator $\mathcal{R}_\xi$, where $\xi = -\dfrac{1}{2a^2}$, and using (12), we obtain

$$\sum_{n\geq 0} \mathcal{R}_\xi(xL_n^{(\mu+\frac{1}{2})}(a^2x^2))t^n = \Big(x^2+\xi(1+2\mu)(1-t)\Big)(1-t)^{-\mu-\frac{5}{2}} \exp(\frac{-a^2x^2t}{1-t}) \tag{29}$$

But,

$$\Big(x^2+\xi(1+2\mu)(1-t)\Big)(1-t)^{-\mu-\frac{5}{2}} \exp(\frac{-a^2x^2t}{1-t}) = -a^{-2}\frac{\partial}{\partial t}\Big((1-t)^{-\mu-\frac{1}{2}} \exp(\frac{-a^2x^2t}{1-t})\Big)$$

Then,

$$\begin{aligned}a^2\sum_{n\geq 0} \mathcal{R}_\xi\Big(xL_n^{(\mu+\frac{1}{2})}(a^2x^2)\Big)t^n &= -\frac{\partial}{\partial t}\Big((1-t)^{-\mu-\frac{1}{2}} \exp(\frac{-a^2x^2t}{1-t})\Big) \\ &= -\frac{\partial}{\partial t}\sum_{n\geq 0} L_n^{(\mu-\frac{1}{2})}(a^2x^2)t^n \\ &= \sum_{n\geq 1} -nL_n^{(\mu-\frac{1}{2})}(a^2x^2)t^{n-1} \\ &= \sum_{n\geq 0} -(n+1)L_{n+1}^{(\mu-\frac{1}{2})}(a^2x^2)t^n\end{aligned}$$

So, by virtue of (1), we get

$$\mathcal{R}_\xi(\mathcal{H}_{2n+1}^{(\mu)}(ax)) = a^{-1}\mathcal{H}_{2n+2}^{(\mu)}(ax) \tag{30}$$

From (27) and (30), formula (22) follows.
Relations (23) and (24) are straightforward consequences of (22). □

Next, we will determine all $\mathcal{R}_\xi$-classical polynomial sequences.

Definition 3.1. *A MOPS $\{P_n(x)\}_{n\geq 0}$ is called $\mathcal{R}_\xi$-classical polynomial sequence if the sequence $\{S_n(x)\}_{n\geq 0}$ defined by*

$$S_0(x) = 1,\ S_{n+1}(x) = (\mathcal{R}_\xi P_n)(x), n \geq 0,$$

is orthogonal.

Remarks 3.1.

1. *The orthogonality of the sequence $\{S_n(x)\}_{n\geq 0}$ is not necessarily with respect to the same linear functional as $\{P_n(x)\}_{n\geq 0}$.*
2. *According to Lemma 3.1, $\{\mathcal{H}_n^{(\mu)}(ax)\}_{n\geq 0}$ is an $\mathcal{R}_\xi$-classical MOPS, where $\xi = -\dfrac{1}{2a^2}$.*

Theorem 3.2. *For any non-zero complex number ξ and any MOPS $\{P_n(x)\}_{n\geq 0}$, the following statements are equivalent.*
(a) $\{P_n(x)\}_{n\geq 0}$ is $\mathcal{R}_\xi$-classical.
(b) $P_n(x) = a^{-n}\mathcal{H}_n^{(\mu)}(ax), n \geq 0$, where $a^2 = -(2\xi)^{-1}$

Proof. $(a) \Rightarrow (b)$ Let $\{P_n(x)\}_{n\geq 0}$ be a MOPS fulfilling (7) and $\{S_n(x)\}_{n\geq 0}$ be the MPS defined by

$$S_0(x) = 1,\ S_{n+1}(x) = \xi(T_\mu P_n)(x) + xP_n(x), n \geq 0. \tag{31}$$

According to Favard's theorem, $\{S_n(x)\}_{n\geq 0}$ satisfies

$$\begin{cases} S_0(x) = 1, S_1(x) = x - \alpha_0, \\ S_{n+2}(x) = (x - \alpha_{n+1})S_{n+1}(x) - \lambda_{n+1}S_n(x),\ \lambda_{n+1} \neq 0,\ n \geq 0. \end{cases} \tag{32}$$

Applying the operator T_μ to (7), where $n \to n-1$ and using (13),we get

$$(T_\mu P_{n+1})(x) = (1+2\mu)P_n(x) + (x-\beta_n)(T_\mu P_n)(x) - 4\mu x(H_{-1}P_n)(x) - \gamma_n(T_\mu P_{n-1}), n \geq 1.$$

Equivalently,

$$(T_\mu P_{n+1})(x) = P_n(x) + 2\mu P_n(-x) + (x-\beta_n)(T_\mu P_n)(x) - \gamma_n(T_\mu P_{n-1}),\ n \geq 1. \tag{33}$$

Multiplying (33) and (7) by ξ and x, respectively, and summing the result, we obtain

$$\begin{aligned}\xi(T_\mu P_{n+1})(x) + xP_{n+1}(x) &= (x-\beta_n)\Big(\xi(T_\mu P_n)(x) + xP_n\Big) \\ &\quad -\gamma_n(xP_{n-1} + \xi(T_\mu P_{n-1})(x)) + \xi(P_n(x) + 2\mu P_n(-x)),\ n \geq 1.\end{aligned} \tag{34}$$

Taking into account (31), we get

$$S_{n+2}(x) = (x-\beta_n)S_{n+1}(x) - \gamma_n S_n(x) + \xi(P_n(x) + 2\mu P_n(-x)),\ n \geq 1. \tag{35}$$

Identification of (32) with the last equation gives

$$(x-\alpha_{n+1})S_{n+1}(x) - \lambda_{n+1}S_n(x) = (x-\beta_n)S_{n+1}(x) - \gamma_n S_n(x) + \xi(P_n(x) + 2\mu P_n(-x)),\ n \geq 1. \tag{36}$$

Or, equivalently,

$$(\beta_n - \alpha_{n+1})S_{n+1}(x) + (\gamma_n - \lambda_{n+1})S_n(x) = \xi(P_n(x) + 2\mu P_n(-x)),\ n \geq 1. \tag{37}$$

Comparing the degrees in the last equation, we obtain

$$\beta_n = \alpha_{n+1},\ n \geq 1, \tag{38}$$

and

$$\gamma_n - \lambda_{n+1} = \xi(1 + 2\mu(-1)^n),\ n \geq 1. \tag{39}$$

Notice that from Definition 3.1 we get $S_1(x) = x$, i. e. $\alpha_0 = 0$. Since $\xi \neq 0$, then (37) becomes

$$(1 + 2\mu(-1)^n)S_n(x) = P_n(x) + 2\mu P_n(-x),\ n \geq 1. \tag{40}$$

On the other hand, for $n = 1$, the last equality gives $(1+2\mu)\beta_0 = 0$. Taking into account the fact that $\mu \neq -1/2$, we get $\beta_0 = 0$.
Using (31) and (40), we will prove by recurrence on n, that

$$S_n(-x) = (-1)^n S_n(x),\ n \geq 0. \tag{41}$$

and

$$P_n(-x) = (-1)^n P_n(x), n \geq 0. \tag{42}$$

The assumption is true for $n = 0$. In fact,

$$S_0(-x) = 1 = (-1)^0 S_0(x),\ P_0(-x) = 1 = (-1)^0 P_0(x).$$

Suppose that the assumption is true until n and let us prove it for $n+1$.
Using the definition of the operator T_μ, (31) can be written as

$$S_{n+1}(x) = \xi(P_n'(x) + \mu\frac{P_n(x) - P_n(-x)}{x}) + xP_n(x), n \geq 0. \tag{43}$$

Taking into account (42), we get

$$S_{n+1}(x) = \xi(P_n'(x) + \mu(1 - (-1)^n)\frac{P_n(x)}{x}) + xP_n(x), n \geq 0. \tag{44}$$

Using (42) and (44), we get

$$S_{n+1}(-x) = (-1)^{n+1} S_{n+1}(x). \tag{45}$$

The change of indices $n \to n+1$ in (40) yields

$$(1 + 2\mu(-1)^{n+1})S_{n+1}(x) = P_{n+1}(x) + 2\mu P_{n+1}(-x), n \geq 0. \tag{46}$$

Replacing x by $-x$ in (46), we obtain

$$(1 + 2\mu(-1)^{n+1})S_{n+1}(-x) = P_{n+1}(-x) + 2\mu P_{n+1}(x), n \geq 0. \tag{47}$$

From (45), (46) and (47), we get

$$P_{n+1}(-x) + 2\mu P_{n+1}(x) = (-1)^{n+1}\big(P_{n+1}(x) + 2\mu P_{n+1}(-x)\big).$$

Or, equivalently,

$$((-1)^n + 2\mu)P_{n+1}(x) = (2\mu(-1)^{n+1} - 1)P_{n+1}(-x).$$

So,

$$P_{n+1}(-x) = (-1)^{n+1} P_{n+1}(x).$$

This completes the recurrence proof.
Notice that (41) and (42) mean that $\{S_n(x)\}_{n\geq 0}$ and $\{P_n(x)\}_{n\geq 0}$ are two symmetric sequences. Furthermore, according to (40) and (42), we have

$$S_n(x) = P_n(x),\, n \geq 0. \tag{48}$$

Therefore,

$$\beta_n = \alpha_n = 0,\, n \geq 0, \tag{49}$$

and

$$\gamma_n = \lambda_n,\, n \geq 1. \tag{50}$$

Thus, (39) becomes

$$\gamma_n - \gamma_{n+1} = \xi(1 + 2\mu(-1)^n),\, n \geq 1. \tag{51}$$

Writing

$$1 + 2\mu(-1)^n = \mu_{n+1} - \mu_n,\, n \geq 0, \mu_0 = 0,$$

Equality (51) becomes

$$\gamma_{n+1} - \gamma_n = -\xi(\mu_{n+1} - \mu_n),\, n \geq 1.$$

Hence,

$$\gamma_n - \gamma_1 = -\xi(\mu_n - \mu_1),\, n \geq 1. \tag{52}$$

But, from (31), where $n = 1$, we have

$$\gamma_1 = -\xi\mu_1.$$

Therefore, (52) becomes

$$\gamma_n = -\xi\mu_n,\, n \geq 1.$$

Application of (9) with

$$a^2 = -(2\xi)^{-1}, \tag{53}$$

we obtain

$$\widetilde{\beta}_n = 0;\, \widetilde{\gamma}_{n+1} = \frac{\mu_{n+1}}{2},\, n \geq 0.$$

So, according to (4),

$$P_n(x) = a^{-n}\mathcal{H}_n^{(\mu)}(ax),\, a^2 = -(2\xi)^{-1},\, n \geq 0,$$

where $\{\mathcal{H}_n^{(\mu)}\}_{n\geq 0}$ is the generalized Hermite sequence.
$(b) \Rightarrow (a)$ The proof is an immediate consequence of Remarks 3.1 □

3.2. Rodrigues formula of generalized Hermite polynomials

In this subsection, we will write $\mathcal{H}_n^{(\mu)}(x),\ n \geq 0$ as follows

$$\mathcal{H}_n^{(\mu)}(x) = \frac{1}{\zeta_n \omega(x)} T_\mu^n\Big(\omega(x)\pi^n(x)\Big), n \geq 0,$$

where ζ_n is a normalization factor, $\pi(x)$ is a polynomial of degree less than or equal to 2, and $\omega(x)$ is an integrable function supported on a subset of the real line. The last equality is known as Rodrigues formula of generalized Hermite polynomials. We need the two following results:

Proposition 3.3. *For $a \in \mathbb{C} \setminus \{0\}$ and $\xi = -\frac{1}{2a^2}$, the operator $\mathcal{R}_\xi$ satisfies the following relation*

$$(\mathcal{R}_\xi^n f)(x) = \frac{(-1)^n}{2^n a^{2n}} e^{a^2x^2} T_\mu^n\Big(f(x)e^{-a^2x^2}\Big),\ f \in \mathcal{P},\ n \geq 0. \tag{54}$$

Proof. The proof is based on the following result: If A is an operator on functions and $g(x) \neq 0$ for all x then $g^{-1}Ag$, the conjugate of A, satisfies

$$(g^{-1}Ag)^n = g^{-1}A^n g,\ n \geq 0 \tag{55}$$

Putting

$$g(x) = \exp(-\frac{1}{2}a^2x^2) \tag{56}$$

On account of (53), we can easily see that

$$g^{-2}T_\mu g^2 = (-2a^2)\mathcal{R}_\xi \tag{57}$$

So, application of (55) with $A = T_\mu$ and $g \to g^2$, the desired result follows. □

Corollary 3.4. *For $a \in \mathbb{C} \setminus \{0\}$ and $\xi = -\frac{1}{2a^2}$, the generalized Hermite polynomial sequence $\{\mathcal{H}_n^{(\mu)}(x)\}_{n\geq 0}$ satisfies*

$$\mathcal{R}_\xi^n(\mathcal{H}_m^{(\mu)}(ax)) = \frac{(-1)^n}{2^n a^{2n}} e^{a^2x^2} T_\mu^n\Big(\mathcal{H}_m^{(\mu)}(ax)e^{-a^2x^2}\Big),\ n, m \geq 0. \tag{58}$$

Theorem 3.5. *For $a \in \mathbb{C} \setminus \{0\}$ and $\xi = -\frac{1}{2a^2}$, the generalized Hermite polynomials sequence satisfies the following Rodrigues formula:*

$$\mathcal{H}_{n+m}^{(\mu)}(ax) = \frac{(-1)^n}{2^n a^n} e^{a^2x^2} T_\mu^n\Big(\mathcal{H}_m^{(\mu)}(ax)e^{-a^2x^2}\Big),\ n, m \geq 0. \tag{59}$$

In particular,

$$\mathcal{H}_n^{(\mu)}(ax) = \frac{(-1)^n}{2^n a^n} e^{a^2x^2} T_\mu^n(e^{-a^2x^2}),\ n \geq 0. \tag{60}$$

Proof. From (58) and (24), we obtain (59).
From (59), where $m = 0$. we get (60). □

Remarks 3.2. *1. When $\mu = 0$, we recover the classical Rodrigues formula for the Hermite polynomials sequence [19].*

2. Notice that the results of this paper generalize those in [1].

To conclude this paper, we will give some relations between T_μ and $\mathcal{R}_\xi$ and their applications in physics. Fore more information the reader can be referred to [13]. Put

$$A_a^+ = a^2 x\mathbb{I} + T\mu,\ A_a^- = a^2 x\mathbb{I} - T_\mu$$

Then, we have

$$g^{-1}A_a^+ g = T\mu,\ g^{-1}A_a^- g = 2a^2\mathcal{R}_\xi$$

These formulas can be used for an interesting application to the quantum harmonic oscillator. First, note that

$$\frac{1}{2}(A_a^+A_a^- + A_a^-A_a^+) = -T_\mu^2 + a^4x^2\mathbb{I}$$

and $\Big(-T_\mu^2 + a^4x^2\mathbb{I}\Big)\Psi(x) = E\Psi(x)$ is a modified form of the Schrödinger equation for the harmonic oscillator on the line, namely $\Big(-(\frac{d}{dx})^2 + a^4x^2\mathbb{I}\Big)\Psi(x) = E\Psi(x)$, with the potential a^4x^2 (and $a > 0$). The eigenfunctions are of the form $p(x)g(x)$. Moreover, using the definition of the operator R_ξ and (14), we can see that

$$\Big(-T_\mu^2 + a^4x^2\mathbb{I}\Big)p(x)g(x) = a^2g(x)(T_\mu R_\xi + R_\xi T_\mu)p(x) \tag{61}$$

Consequently, using (15) and (16), we get

$$\Big(-T_\mu^2 + a^4x^2\mathbb{I}\Big)\mathcal{H}_n^{(\mu)}(ax)g(x) = a^2(2n+1+2\mu)\mathcal{H}_n^{(\mu)}(ax)g(x).$$

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