LE MATEMATICHE
Vol. LXIV (2009) – Fasc. I, pp. 25–45

LORENTZIAN BELTRAMI-EULER FORMULA AND GENERALIZED LORENTZIAN LAMARLE FORMULA IN $\mathscr{R}_1^n$

SOLEY ERSOY - MURAT TOSUN

In this paper, the sectional curvature of non-degenerate tangent sections of time-like ruled surface with the central ruled surface in n-dimensional Minkowski space, $\mathscr{R}_1^n$ is studied. The relationship between normal sectional curvature and the principal sectional curvatures of non-degenerate tangent sections of time-like ruled surfaces is obtained and called as Lorentzian Beltrami-Euler formula. Moreover, the relationship between the Gaussian curvature and the principal distribution parameter of the non-degenerate tangent sections of time-like ruled surfaces is obtained and called as generalized Lorentzian Lamarle formula.

1. Introduction

Analysis of curvature is an important study field in the realm of differential geometry because the theory of curvature has been used by various branches of sciences. Euler formula and Beltrami formula are well-known theorems from classical surface theory, [8]. Euler formula was applied to the tangent sections in [3] and called as Beltrami-Euler formula, which is a relationship between the normal curvature and the principal normal curvatures of the tangent sections of the generalized ruled surface with the central ruled surface in n-dimensional Euclid space, E^n. Moreover, Lamarle formula was given for curvatures of three

Entrato in redazione: 10 settembre 2008

AMS 2000 Subject Classification: 53B30, 53C21, 53C50.
Keywords: Sectional curvature, ruled surface, Beltrami-Euler formula, Lamarle formula.

dimensional surfaces in [2]. Generalized Lamarle formula was applied to the tangent sections, which is a relationship between the Gaussian curvature and the principal distribution parameter of the tangent sections of the generalized ruled surface with the central ruled surface in n-dimensional Euclid space E^n in [3].

2. Preliminaries

The Minkowski space $\mathscr{R}_1^n$ is the vector space $\mathscr{R}^n$ provided that the Lorentzian inner product $\langle\ ,\ \rangle$ is given by

$$\left\langle \vec{X}, \vec{X} \right\rangle = dx_1^2 + dx_2^2 + ... + dx_{n-1}^2 - dx_n^2$$

where $(x_1, x_2, ..., x_n)$ is a rectangular coordinate system of $\mathscr{R}_1^n$, [1]. Since $\langle\ ,\ \rangle$ is an indefinite metric, recall that a vector $\vec{v} \in \mathscr{R}_1^n$ can have one of three Lorentzian causal characters: it can be space-like if $\langle \vec{v}, \vec{v} \rangle > 0$ or $\vec{v} = 0$, time-like if $\langle \vec{v}, \vec{v} \rangle < 0$ and null (light-like) if $\langle \vec{v}, \vec{v} \rangle = 0$ and $\vec{v} \neq 0$, [1]. Similarly, an arbitrary curve $\alpha = \alpha(t) \subset \mathscr{R}_1^n$ can locally be space-like, time- like or null (light-like), if all of its velocity vectors $\dot{\alpha}(t)$ are respectively space-like, time-like or null (light-like). The norm of $\vec{v} \in \mathscr{R}_1^n$ is defined as

$$\|\vec{v}\| = \sqrt{|\langle \vec{v}, \vec{v} \rangle|}.$$

Let W be a subspace of $\mathscr{R}_1^n$ and denote $\langle\ ,\ \rangle|_W$ as reduced metric in subspace W of $\mathscr{R}_1^n$. A subspace W of $\mathscr{R}_1^n$ can be space-like, time-like or null (light-like) if $\langle\ ,\ \rangle|_W$ is positive definite, $\langle\ ,\ \rangle|_W$ is non-degenerate of index 1 or $\langle\ ,\ \rangle|_W$ is degenerate, respectively, [1]. Let the set of all time-like vectors in $\mathscr{R}_1^n$ be Γ. For $\vec{u} \in \Gamma$, we call

$$C(\vec{u}) = \{\, \vec{v} \in \Gamma |\ \langle \vec{v}, \vec{u} \rangle < 0\}$$

as time-conic of Minkowski space $\mathscr{R}_1^n$ including vector $\vec{u}$, [1].

Theorem 2.1. *Let $\vec{X}$ and $\vec{Y}$ be linearly independent space-like vectors in $\mathscr{R}_1^n$. Then the following are equivalent:*

1. *The vectors $\vec{X}$ and $\vec{Y}$ satisfy the equation $\left|\left\langle \vec{X}, \vec{Y} \right\rangle\right| \leq \left\|\vec{X}\right\| \left\|\vec{Y}\right\|$.*
2. *The vector subspace V spanned by $\vec{X}$ and $\vec{Y}$ is space-like.*
3. *The hyperplanes $\vec{P}$ and $\vec{Q}$ of H^n Lorentz orthogonal to $\vec{X}$ and $\vec{Y}$, respectively, intersect, [4].*

Let $\vec{X}$ and $\vec{Y}$ be space-like vectors in $\mathscr{R}_1^n$ that span a space-like plane. Then by Theorem 2.1, we have that

$$\left|\left\langle \vec{X},\vec{Y}\right\rangle\right| \leq \left\|\vec{X}\right\| \left\|\vec{Y}\right\|$$

with equality if and only if $\vec{X}$ and $\vec{Y}$ are linearly dependent. Hence, there is a unique $0 \leq \theta \leq \pi$ such that

$$\left\langle \vec{X},\vec{Y}\right\rangle = \left\|\vec{X}\right\| \left\|\vec{Y}\right\| \cos\theta. \tag{1}$$

The Lorentzian space-like angle between $\vec{X}$ and $\vec{Y}$ is defined to be θ, [4].

3. Generalized time-like surface with space-like generating space in $\mathscr{R}_1^n$

Let $\{e_1(t),...,e_k(t)\}$ be an orthonormal vector field, which is defined at each point $\alpha(t)$ of a time-like curve of $n-$dimensional Minkowski space $\mathscr{R}_1^n$. This system spanning at the point $\alpha(t) \in \mathscr{R}_1^n$ a k-dimensional subspace is denoted by $E_k(t)$ and is given by $E_k(t) = Sp\{e_1(t),...,e_k(t)\}$. If the space-like subspace $E_k(t)$ moves along time-like curve α, we obtain a $(k+1)$-dimensional surface in $\mathscr{R}_1^n$. This surface is called a $(k+1)$-dimensional time-like ruled surface of the n-dimensional Minkowski space $\mathscr{R}_1^n$ and is denoted by M. The subspace $E_k(t)$ and the curve α are called the generating space and the base curve, respectively. A parametrization of the ruled surface is the following:

$$\varphi(t,u_1,\ldots,u_k) = \alpha(t) + \sum_{\nu=1}^{k} u_\nu e_\nu(t) \tag{2}$$

Throughout the paper we assume that the system

$$\left\{\dot{\alpha}(t) + \sum_{\nu=1}^{k} u_\nu \dot{e}_\nu(t), e_1(t),...,e_k(t)\right\}$$

is linear independent, [6]. We call

$$Sp\{e_1(t),...,e_k(t), \dot{e}_1(t),...\dot{e}_k(t)\} \tag{3}$$

the asymptotic bundle of M with respect to $E_k(t)$ and denote it by $A(t)$. We have $\dim A(t) = k+m$, $0 \leq m \leq k$. There exists an orthonormal base of $A(t)$ that we denote as $\{e_1(t),...e_k(t), a_{k+1}(t),...a_{k+m}(t)\}$. It is clear that the asymptotic bundle is space-like subspace. The space

$$Sp\{e_1(t),...,e_k(t), \dot{e}_1(t),...\dot{e}_k(t), \dot{\alpha}(t)\} \tag{4}$$

includes the union of all the tangent spaces of $E_k(t)$ at a point p. This space is denoted by $T(t)$ and called the tangential bundle of M in $\mathscr{R}_1^n$. It can be easily seen that

$$k+m \leq \dim T(t) \leq k+m+1 \quad , \quad 0 \leq m \leq k.$$

In what follow we examine separately two cases. Let $\dim T(t) = k+m$ then $\{e_1(t), ...e_k(t), a_{k+1}(t), ...a_{k+m}(t)\}$ is an orthonormal bases of the asymptotic bundle $A(t)$ as well as of the tangential bundle $T(t)$. Let $\dim T(t) = k+m+1$. In this case, $\{e_1(t), ...e_k(t), a_{k+1}(t), ...a_{k+m}(t), a_{k+m+1}(t)\}$ is an orthonormal base of $T(t)$ and the tangential bundle $T(t)$ is a time-like subspace, [6]. If $\dim T(t) = k+m$, then $(k+1)$-dimensional time-like ruled surface M has a $(k-m)$-dimensional subspace and this subspace is called edge space of M and denoted as $K_{k-m}(t)$. Edge space $K_{k-m}(t) \subset E_k(t)$ is space-like subspace. If we take edge space $K_{k-m}(t)$ to be generating space and base curve α of M to be base curve, then there will be $(k-m+1)$-dimensional ruled surface contained by M. This surface is called edge ruled surface and the edge ruled surface is a time-like ruled surface, [6]. If $\dim T(t) = k+m+1$, then $(k+1)$-dimensional time-like ruled surface has a $(k-m)$-dimensional subspace called central space of M and denoted as $Z_{k-m}(t) \subset E_k(t)$. This space is a space-like subspace. Similarly, if we take base curve of M to be the base curve and $Z_{k-m}(t)$ to be the generating space, we get a $(k-m+1)$-dimensional ruled surface contained by M in $\mathscr{R}_1^n$ and this is called central ruled surface and denoted by Ω. The central ruled surface Ω is a time-like surface, too [6].

For the basis vectors of $E_k(t)$ we write the following derivative equations, [6];

$$\begin{aligned} \dot{e}_\sigma &= \sum_{\mu=1}^{k} \alpha_{\sigma\mu} e_\mu + \kappa_\sigma a_{k+\sigma} \quad , \quad 1 \leq \sigma \leq m \\ \dot{e}_{m+\rho} &= \sum_{\mu=1}^{k} \alpha_{(m+\rho)\mu} e_\mu \quad , \quad 1 \leq \rho \leq k-m \end{aligned} \tag{5}$$

where $\alpha_{\nu\mu} = -\alpha_{\mu\nu}$ and $\kappa_1 > \kappa_2 > \ldots > \kappa_m > 0$.

Let subspace $F_m(t) = Sp\{e_1(t), ...e_m(t)\}$ be totally orthogonal to generating space $Z_{k-m}(t)$ of Ω and orthogonal trajectories of central ruled surface Ω be r. If generating space $F_m(t)$ moves along base curve r it produces a $(m+1)$-dimensional ruled surface. This surface is known as principal ruled surface and denoted by Λ and $(m+1)$-dimensional principal ruled surface is a time-like ruled surface, [7].

In $\mathscr{R}_1^n$, 1-dimensional generating spaces $h_\sigma = Sp\{e_\sigma\}$, $1 \leq \sigma \leq m$, of base curve $\alpha(t)$ of time-like ruled surface M at the point $\zeta \in Z_{k-m}(t)$ are in $E_k(t)$. The generating space given by the parametric expression $\zeta + ue_\sigma(t)$ is called principal rays of $F_m(t)$, [7].

Let M be a generalized time-like ruled surface with the central ruled surface in

$\mathscr{R}_1^n$. σ^{th} principal rays $h_\sigma = Sp\{e_\sigma\}$, $1 \leq \sigma \leq m$, generates 2-dimensional principal surface M along the time-like base curve $\alpha(t)$ of central ruled surface Ω. This ruled surface is defined as the principal ray surface of M and parametrically given by

$$\varphi_\sigma(t,u) = \alpha(t) + ue_\sigma(t) \quad , \quad 1 \leq \sigma \leq m \quad , \quad (t,u) \in (I, \mathscr{R})$$

It is clear that the principal ray surface φ_σ, $1 \leq \sigma \leq m$, is time-like, [7]. If one choose base curve $\alpha(t)$ of central ruled surface Ω as the orthogonal trajectory of M, then $\alpha(t)$ becomes a striction line φ_σ. Therefore, every principal ray surface has its own striction line, [7].

Let M be a generalized time-like ruled surface with the central ruled surface Ω in $\mathscr{R}_1^n$. Every principal ray surface φ_σ, $1 \leq \sigma \leq m$, defined by the time-like base curve $\alpha(t)$ of central ruled surface Ω, has a striction line. If the base curve $\alpha(t)$ of central ruled surface Ω is an orthogonal trajectory of M, then base curve $\alpha(t)$ coincides with the striction line of principal ray surface φ_σ, [7].

If $(k+1)$-dimensional time-like ruled surface M is cylindrical (i.e., $m = 0$), then there is no principal ray ruled surface of M. A base curve α of $(k+1)$-dimensional ruled surface M is a base curve of edge or central surface $\Omega \subset M$, too iff its tangent vector has the form

$$\dot{\alpha}(t) = \sum_{\nu=1}^{k} \zeta_\nu e_\nu + \eta_{m+1} a_{k+m+1} \tag{6}$$

where $\eta_{m+1} \neq 0$, a_{k+m+1} is a unit vector well defined up to the sign with the property that $\{e_1, ..., e_k, a_{k+1}, ..., a_{k+m}, a_{k+m+1}\}$ is an orthonormal base of the tangential bundle of M. One shows: $\eta_{m+1} = 0$, in $t \in J$ iff generating $E_k(t)$ contains the edge space $K_{k-m}(t)$, [7].

If $\eta_{m+1} \neq 0$, we call $m-$magnitudes

$$P_\sigma = \frac{\eta_{m+1}}{\kappa_\sigma} \quad , \quad 1 \leq \sigma \leq m \tag{7}$$

the σ^{th} principal distribution parameter of M, [7]. Moreover in [7] the parameter of distribution of a generalized ruled surface M is given by

$$P = \sqrt[m]{|P_1 ... P_m|}. \tag{8}$$

The canonical base of the tangential bundle of generalized time-like ruled surface with space-like generating space is

$$\left\{ \sum_{\nu=1}^{k} \left(\zeta_\nu + \sum_{\mu=1}^{k} \alpha_{\nu\mu} u_\mu \right) e_\nu + \sum_{\sigma=1}^{m} u_\sigma \kappa_\sigma a_{k+\sigma} + \eta_{m+1} a_{k+m+1},\ e_1,\ e_2, \ldots, e_k \right\} \tag{9}$$

We can evaluate the first fundamental form and the metric coefficients of M. To use the conventional notation we can choose $u_0 = t$ and find metric coefficients of M as follows:

$$g_{00} = \langle \varphi_t, \varphi_t \rangle \quad , \quad g_{\nu 0} = \langle \varphi_{u_\nu}, \varphi_t \rangle \quad , \quad g_{\nu\mu} = \left\langle \varphi_{u_\nu}, \varphi_{u_\mu} \right\rangle \quad , \quad 1 \leq \nu, \mu \leq k.$$

Furthermore, considering $\dim T(t) = k+m+1$, i.e. a_{k+m+1} is time-like, we reach

$$\begin{aligned}
g_{00} &= \sum_{\nu=1}^{k} \left(\zeta_\nu + \sum_{\mu=1}^{k} \alpha_{\nu\mu} u_\mu \right)^2 + \sum_{\sigma=1}^{m} (u_\sigma \kappa_\sigma)^2 - (\eta_{m+1})^2, \\
g_{\nu 0} &= \zeta_\nu + \sum_{\mu=1}^{k} \alpha_{\nu\mu} u_\mu \quad , \quad 1 \leq \nu \leq k, \\
g_{\nu\mu} &= \delta_{\nu\mu} \qquad\qquad\qquad , \quad 1 \leq \nu, \mu \leq k.
\end{aligned}$$

Thus, we find $[g_{ij}] =$

$$= \begin{bmatrix}
\sum_{\nu=1}^{k} \left(\zeta_\nu + \sum_{\mu=1}^{k} \alpha_{\nu\mu} u_\mu \right)^2 + \sum_{\sigma=1}^{m} (u_\sigma \kappa_\sigma)^2 - (\eta_{m+1})^2 & \zeta_1 + \sum_{\mu=1}^{k} \alpha_{1\mu} u_\mu & \zeta_2 + \sum_{\mu=1}^{k} \alpha_{2\mu} u_\mu & \cdots & \zeta_k + \sum_{\mu=1}^{k} \alpha_{k\mu} u_\mu \\
\zeta_1 + \sum_{\mu=1}^{k} \alpha_{1\mu} u_\mu & 1 & 0 & \cdots & 0 \\
\zeta_2 + \sum_{\mu=1}^{k} \alpha_{2\mu} u_\mu & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\zeta_k + \sum_{\mu=1}^{k} \alpha_{k\mu} u_\mu & 0 & 0 & \cdots & 1
\end{bmatrix}$$

From the last equation we get

$$g = \det [g_{ij}] = \sum_{\sigma=1}^{m} (u_\sigma \kappa_\sigma)^2 - \eta_{m+1}^2, \quad 0 \leq i, j \leq k. \tag{10}$$

As a consequence we find the following results:

$$\begin{aligned}
g_{00} &= g + \sum_{\nu=1}^{k} \left(\zeta_\nu + \sum_{\mu=1}^{k} \alpha_{\nu\mu} u_\mu \right)^2 \quad , \\
g_{\nu 0} &= \left(\zeta_\nu + \sum_{\mu=1}^{k} \alpha_{\nu\mu} u_\mu \right) \quad , \quad 1 \leq \nu \leq k, \\
g_{\nu\mu} &= \delta_{\nu\mu} \qquad\qquad\qquad , \quad 0 \leq \nu, \mu \leq k.
\end{aligned} \tag{11}$$

In addition to these, the coefficients of the inverse matrix $\left[g^{ij}\right]$ of the matrix $[g_{ij}]$, $0 \le i, j \le k$, are as follows

$$
\begin{aligned}
&g^{00} = g^{-1} \\
&g^{\nu 0} = -\left(\zeta_\nu + \sum_{\mu=1}^{k} \alpha_{\nu\mu} u_\mu\right) g^{-1}, \quad 1 \le \nu \le k \\
&g^{\nu\lambda} = \left(\delta_{\nu\lambda} g + \left(\zeta_\nu + \sum_{\mu=1}^{k} \alpha_{\nu\mu} u_\mu\right)\left(\zeta_\lambda + \sum_{\mu=1}^{k} \alpha_{\lambda\mu} u_\mu\right)\right) g^{-1}, \; 1 \le \nu, \lambda \le k.
\end{aligned} \tag{12}
$$

Considering the Koszul equation given by [1]

$$
\Gamma_{ij}^{k} = \frac{1}{2} \sum_{m} g^{km} \left[\frac{\partial g_{jm}}{\partial x_i} + \frac{\partial g_{im}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_m}\right] \tag{13}
$$

Christoffel symbols for $1 \le \nu, \mu, \lambda \le k$ are determined to be

$$
\begin{aligned}
&\Gamma_{00}^{0} = \tfrac{1}{2g}\left[\tfrac{\partial g}{\partial u_0} + \sum_{\nu=1}^{k}\left(\zeta_\nu + \sum_{\mu=1}^{k} \alpha_{\nu\mu} u_\mu\right)\tfrac{\partial g}{\partial u_\nu}\right], \\
&\Gamma_{00}^{\lambda} = \tfrac{1}{2g}\Bigg[-\left(\zeta_\lambda + \sum_{\mu=1}^{k} \alpha_{\lambda\mu} u_\mu\right)\left(\tfrac{\partial g}{\partial u_0} + \sum_{\nu=1}^{k}\left(\zeta_\nu + \sum_{\mu=1}^{k} \alpha_{\nu\mu} u_\mu\right)\tfrac{\partial g}{\partial u_\nu}\right) \\
&\qquad + 2g\left(\left(\dot{\zeta}_\lambda + \sum_{\mu=1}^{k} \dot{\alpha}_{\lambda\mu} u_\mu\right) + \sum_{\nu=1}^{k}\left(\zeta_\nu + \sum_{\mu=1}^{k} \alpha_{\nu\mu} u_\mu\right)\alpha_{\lambda\nu} - \tfrac{1}{2}\tfrac{\partial g}{\partial u_\lambda}\right)\Bigg], \\
&\Gamma_{\nu\mu}^{0} = \Gamma_{\mu\nu}^{0} = 0, \\
&\Gamma_{\nu\mu}^{\lambda} = \Gamma_{\mu\nu}^{\lambda} = 0, \\
&\Gamma_{\lambda 0}^{0} = \Gamma_{0\lambda}^{0} = \tfrac{1}{2g}\tfrac{\partial g}{\partial u_\lambda}, \\
&\Gamma_{\nu 0}^{\lambda} = \Gamma_{0\nu}^{\lambda} = \tfrac{1}{2g}\left[-\left(\zeta_\lambda + \sum_{\mu=1}^{k} \alpha_{\lambda\mu} u_\mu\right)\tfrac{\partial g}{\partial u_\nu} + 2g\left(\alpha_{\lambda\nu}\right)\right].
\end{aligned} \tag{14}
$$

Adopting that the base of tangential space in the neighborhood of coordinate system $\{u_0, u_1, \ldots, u_k\}$ is $\{\partial_0, \partial_1, \ldots, \partial_k\}$ $\left(\frac{\partial}{\partial u_i} = \partial_i, 0 \le i \le k\right)$, the Riemann curvature tensor of the generalized time-like surface M with space-like generating space is given by

$$
R_{\partial_i \partial_j}(\partial_l) = \sum_{r=0}^{k} R_{lij}^{r} \partial_r
$$

where the coefficient of the Riemann curvature tensor is

$$
R_{lij}^{r} = \frac{\partial}{\partial u_i}\Gamma_{jl}^{r} - \frac{\partial}{\partial u_j}\Gamma_{il}^{r} - \sum_{s=0}^{k} \Gamma_{il}^{s}\Gamma_{js}^{r} + \sum_{s=0}^{k} \Gamma_{jl}^{s}\Gamma_{is}^{r}.
$$

Hence the Riemann-Christoffel curvature tensor of becomes

$$R_{hlij} = \sum_{r=0}^{k} g_{rh} \left(\frac{\partial}{\partial u_i} \Gamma_{jl}^{r} - \frac{\partial}{\partial u_j} \Gamma_{il}^{r} - \sum_{s=0}^{k} \Gamma_{il}^{s} \Gamma_{js}^{r} + \sum_{s=0}^{k} \Gamma_{jl}^{s} \Gamma_{is}^{r} \right). \tag{15}$$

Furthermore, for the curvature tensor following relations hold

$$\begin{aligned} R_{hlij} &= R_{ijhl}, \\ R_{ijhl} &= -R_{jihl}. \end{aligned}$$

Taking equation (14) into consideration, the curvatures of R_{ij00}, $R_{ij\nu\mu}$, $R_{\nu 0\mu 0}$, $(0 \leq i, j \leq k,\ 1 \leq \nu, \mu \leq k)$ are found to be (in terms of the determinant of the first fundamental form of M, the first and second order partial differentials of g)

$$\begin{array}{rcl} R_{ij00} = 0 & , & 0 \leq i, j \leq k, \\ R_{ij\nu\mu} = 0 & , & 0 \leq i, j \leq k, 1 \leq \nu, \mu \leq k, \\ R_{\nu 0\mu 0} = -\frac{1}{2} \frac{\partial^2 g}{\partial u_\nu \partial u_\mu} + \frac{1}{4g} \frac{\partial g}{\partial u_\nu} \frac{\partial g}{\partial u_\mu} & , & 1 \leq \nu, \mu \leq k. \end{array} \tag{16}$$

4. Sectional curvatures of the time-like generalized ruled surface with space-like generating space in $\mathscr{R}_1^n$

Two-dimensional subspace Π of $(k+1)$-dimensional time-like ruled surface at the point $\xi \in T_M(\xi)$ is called tangent section of M at point ξ. If $\vec{v}$ and $\vec{w}$ form a basis of the tangent section Π, then $Q(\vec{v}, \vec{w}) = \langle \vec{v}, \vec{v} \rangle \langle \vec{w}, \vec{w} \rangle - \langle \vec{v}, \vec{w} \rangle^2$ is a non-zero quantity if and only if Π is non-degenerate. This quantity represents the square of the Lorentzian area of the parallelogram determined by $\vec{v}$ and $\vec{w}$. Using the square of the Lorentzian area of the parallelogram determined by the basis vectors $\{\vec{v}, \vec{w}\}$, one has the following classification for the tangent sections of the time-like ruled surfaces:

$$\begin{array}{lcl} Q(\vec{v}, \vec{w}) = \langle \vec{v}, \vec{v} \rangle \langle \vec{w}, \vec{w} \rangle - \langle \vec{v}, \vec{w} \rangle^2 < 0 & , & (\text{time} - \text{like} \quad \text{plane}), \\ Q(\vec{v}, \vec{w}) = \langle \vec{v}, \vec{v} \rangle \langle \vec{w}, \vec{w} \rangle - \langle \vec{v}, \vec{w} \rangle^2 = 0 & , & (\text{degenerate} \quad \text{plane}), \\ Q(\vec{v}, \vec{w}) = \langle \vec{v}, \vec{v} \rangle \langle \vec{w}, \vec{w} \rangle - \langle \vec{v}, \vec{w} \rangle^2 > 0 & , & (\text{space} - \text{like} \quad \text{plane}). \end{array}$$

For the non-degenerate tangent section Π given by the basis $\{\vec{v}, \vec{w}\}$ of M at point ξ

$$K_\xi(\vec{v}, \vec{w}) = \frac{\langle R_{\vec{v}\vec{w}} \vec{v}, \vec{w} \rangle}{\langle \vec{v}, \vec{v} \rangle \langle \vec{w}, \vec{w} \rangle - \langle \vec{v}, \vec{w} \rangle^2} = \frac{\sum R_{ijkm} \beta_i \gamma_j \beta_k \gamma_m}{\langle \vec{v}, \vec{v} \rangle \langle \vec{w}, \vec{w} \rangle - \langle \vec{v}, \vec{w} \rangle^2} \tag{17}$$

is called sectional curvature of M at the point ξ. Here, the coordinates of the basis vectors $\vec{v}$ and $\vec{w}$ are $(\beta_0, \beta_1, \ldots, \beta_k)$ and $(\gamma_0, \gamma_1, \ldots, \gamma_k)$, respectively, [1]. A normal tangent vector

$$n = \sum_{\sigma=1}^{m} u_\sigma \kappa_\sigma(t)\, a_{k+\sigma}(t) + \eta_{m+1} a_{k+m+1}(t) \quad , \quad (\eta_{m+1} \neq 0) \tag{18}$$

is time-like or space-like vector. This means that the tangent sectional (e_ν, n), $1 \leq \nu \leq k$, at the point $\forall \xi \in M$ is time-like or space-like. This tangent section is called ν^{th} principal tangent section of M. Thus, whether ν^{th} principal tangent section is time-like or space-like, we can give following theorems and corollaries.

Theorem 4.1. *Let M be generalized time-like ruled surface with central ruled surface and n be non-degenerate normal tangent vector in $\mathscr{R}_1^n$. Curvature of (e_ν, n), $1 \leq \nu \leq k$, non-degenerate ν^{th} principal section of M, at the point $\forall \xi \in M$ is*

$$K_\xi(e_\nu, n) = -\frac{1}{2g}\frac{\partial^2 g}{\partial u_\nu^2} + \frac{1}{4g^2}\left(\frac{\partial g}{\partial u_\nu}\right)^2 \quad , \quad 1 \leq \nu \leq k. \tag{19}$$

Proof. Let the coordinates of e_ν and n generating the base of ν^{th} principal section of generalized time-like ruled surface M with central ruled surface and space-like generating space be $(\beta_0, \beta_1, \ldots, \beta_\nu, \ldots, \beta_k)$ and $(\gamma_0, \gamma_1, \ldots, \gamma_\nu, \ldots, \gamma_k)$, respectively. According to the canonical base of the tangential bundle of M given by equation (9), correspondence of these coordinates are $(0, \ldots, 1, \ldots, 0)$ and $(1, \ldots, 0, \ldots, 0)$, respectively. Therefore the curvature of the ν^{th} principal section (e_ν, n), $1 \leq \nu \leq k$, is

$$K_\xi(e_\nu, n) = \frac{R_{\nu 0 \nu 0}}{\langle e_\nu, e_\nu \rangle \langle n, n \rangle - \langle e_\nu, n \rangle^2}.$$

Substituting equations (16) and (18) into last equation we find

$$K_\xi(e_\nu, n) = \frac{-\frac{1}{2}\frac{\partial^2 g}{\partial u_\nu^2} + \frac{1}{4g}\left(\frac{\partial g}{\partial u_\nu}\right)^2}{\sum_{\sigma=1}^{m} (u_\sigma \kappa_\sigma)^2 - \eta_{m+1}^2} \quad , \quad 1 \leq \nu \leq k.$$

The last equation and the equation (10) complete the proof. $\square$

First and second partial derivatives of g, given by equation (10), with respect to σ and $(m+\rho)$ are as follows

$$\frac{\partial g}{\partial u_\sigma} = 2u_\sigma \kappa_\sigma^2 \quad , \quad \frac{\partial^2 g}{\partial u_\sigma^2} = 2\kappa_\sigma^2 \quad , \quad \left(\frac{\partial g}{\partial u_\sigma}\right)^2 = 4\left(u_\sigma \kappa_\sigma^2\right)^2 \kappa_\sigma^2 \quad , 1 \leq \sigma \leq m$$

and

$$\frac{\partial g}{\partial u_{m+\rho}} = 0 \quad , \quad \frac{\partial^2 g}{\partial u_{m+\rho}^2} = 0 \quad , \quad \left(\frac{\partial g}{\partial u_{m+\rho}}\right)^2 = 0 \quad , \quad 1 \leq \rho \leq k-m.$$

Substituting these equations into equation (10) σ^{th} principal sectional curvature becomes $K_\xi\left(e_\sigma,n\right)=$

$$=-\frac{2\left(\kappa_\sigma\right)^2}{2\left(\sum_{\iota=1}^{m}\left(u_\iota\kappa_\iota\right)^2-\eta_{m+1}^2\right)}+\frac{4\left(u_\sigma\kappa_\sigma\right)^2\left(\kappa_\sigma\right)^2}{4\left(\sum_{\iota=1}^{m}\left(u_\iota\kappa_\iota\right)^2-\eta_{m+1}^2\right)^2}\quad,\ 1\leq\sigma\leq m.$$

After simplifying the last equation we reach

$$K_\xi\left(e_\sigma,n\right)=-\frac{\left(\kappa_\sigma\right)^2\left[\sum_{\iota=1}^{m}\left(u_\iota\kappa_\iota\right)^2-\eta_{m+1}^2-\left(u_\sigma\kappa_\sigma\right)^2\right]}{\left(\sum_{\iota=1}^{m}\left(u_\iota\kappa_\iota\right)^2-\eta_{m+1}^2\right)^2}.$$

It can be easily seen that $(m+\rho)^{\text{th}}$ principal sectional curvature is

$$K_\xi\left(e_{m+\rho},n\right)=0\quad,\quad 1\leq\rho\leq k-m.$$

Corollary 4.2. *Let M be generalized time-like ruled surface with central ruled surface and n be non-degenerate normal tangent vector in $\mathscr{R}_1^n$. σ^{th}, $1\leq\sigma\leq m$, principal sectional curvature and $(m+\rho)^{\text{th}}$, $1\leq\rho\leq k-m$, principal sectional curvature of M at the point $\forall\xi\in M$ are*

$$\begin{array}{ll} K_\xi\left(e_\sigma,n\right)=-\frac{\left(\kappa_\sigma\right)^2\left[\sum_{\iota=1}^{m}\left(u_\iota\kappa_\iota\right)^2-\eta_{m+1}^2-\left(u_\sigma\kappa_\sigma\right)^2\right]}{\left(\sum_{\iota=1}^{m}\left(u_\iota\kappa_\iota\right)^2-\eta_{m+1}^2\right)^2} & ,\quad 1\leq\sigma\leq m, \\ K_\xi\left(e_{m+\rho},n\right)=0 & ,\quad 1\leq\rho\leq k-m, \end{array}\tag{20}$$

respectively.

Corollary 4.3. *$(m+\rho)^{\text{th}}$, $1\leq\rho\leq k-m$, principal sectional curvature of generalized time-like ruled surface with central ruled surface is zero at the point $\forall\xi\in M$ in $\mathscr{R}_1^n$.*

Theorem 4.4. *Let M be generalized time-like ruled surface with central ruled surface and n be non-degenerate normal tangent vector in $\mathscr{R}_1^n$. σ^{th}, $1\leq\sigma\leq m$, principal sectional curvature and $(m+\rho)^{\text{th}}$, $1\leq\rho\leq k-m$, principal sectional curvature of M at the point $\forall\zeta\in M$ are*

$$\begin{array}{ll} K_\zeta\left(e_\sigma,n\right)=\frac{1}{P_\sigma^2} & ,\quad 1\leq\sigma\leq m, \\ K_\zeta\left(e_{m+\rho},n\right)=0 & ,\quad 1\leq\rho\leq k-m, \end{array}\tag{21}$$

respectively, where $P_\sigma=\frac{\eta_{m+1}}{\kappa_\sigma}$, $1\leq\sigma\leq m$, is the σ^{th} principal distribution parameter of M.

Proof. Let M be generalized time-like ruled surface with space-like generating space and central ruled surface in $\mathscr{R}_1^n$. Considering equation (20) and the condition $u_\sigma = 0$, $1 \leq \sigma \leq m$, at the central point $\zeta \in \Omega$, we find the σ^{th} sectional curvature of M to be

$$K_\zeta\left(e_\sigma, n\right) = \frac{\left(-\kappa_\sigma^2\right)\left(-\eta_{m+1}^2\right)}{\left(-\eta_{m+1}^2\right)^2} \quad , \quad 1 \leq \sigma \leq m.$$

Taking the last equation with equation (7) into consideration, we see that the following relation

$$K_\zeta\left(e_\sigma, n\right) = \frac{1}{P_\sigma^2} \quad , \quad 1 \leq \sigma \leq m$$

holds between the σ^{th} principal sectional curvature of M and the σ^{th} principal distribution parameter of M. Furthermore, from Corollary 4.2, it is obvious that the $(m+\rho)^{\text{th}}$ principal sectional curvature of M is

$$K_\zeta\left(e_{m+\rho}, n\right) = 0 \quad , \quad 1 \leq \rho \leq k-m.$$

□

In $\mathscr{R}_1^n$ if 1-dimensional generator $h_\sigma = Sp\{e_\sigma\}$, $1 \leq \sigma \leq m$, (σ^{th} principal ray) moves along the orthogonal trajectory of M, then 2-dimensional ray surface is obtained. This surface is called σ^{th} principal ray surface and denoted by φ_σ, $1 \leq \sigma \leq m$. A parametrization of φ_σ is

$$\varphi_\sigma(t,u) = \alpha(t) + u e_\sigma(t) \quad , \quad 1 \leq \sigma \leq m.$$

Now the theorem about non-degenerate sectional curvature of φ_σ can be given in the following.

Theorem 4.5. *Let M be generalized time-like ruled surface with central ruled surface and φ_σ, $1 \leq \sigma \leq m$, be 2-dimensional time-like σ^{th} principal ray surface in $\mathscr{R}_1^n$. For $\zeta \in \Omega \subset M$, $u \in \mathscr{R}$, the sectional curvature of φ_σ at the point $\zeta + u e_\sigma$ on generator $h_\sigma = Sp\{e_\sigma\}$ is*

$$K_{\zeta+ue_\sigma}\left(e_\sigma, n\right) = \frac{P_\sigma^2}{\left(u^2 - P_\sigma^2\right)^2} \quad , \quad 1 \leq \sigma \leq m \tag{22}$$

where P_σ, $1 \leq \sigma \leq m$, is the σ^{th} principal distribution parameter of M.

Proof. The determinant of the first fundamental form of φ_σ, $1 \leq \sigma \leq m$, generated by the principal ray $h_\sigma = Sp\{e_\sigma\}$ along the orthogonal trajectory of M in n-dimensional Minkowski space $\mathscr{R}_1^n$ is expressed to be

$$g = \left(u\kappa_\sigma\right)^2 - \eta_{m+1}^2 \quad , \quad 1 \leq \sigma \leq m.$$

Therefore, the first and the second order partial derivatives of g are

$$\frac{\partial g}{\partial u}=2u\kappa_\sigma^2 \quad , \quad \frac{\partial^2 g}{\partial u^2}=2\kappa_\sigma^2 \quad , \quad \left(\frac{\partial g}{\partial u}\right)^2=4\left(u\kappa_\sigma^2\right)^2\kappa_\sigma^2.$$

To find the curvature of (e_σ,n), $1\leq\sigma\leq m$, section at the point $\zeta+ue_\sigma$ on generator $h_\sigma=Sp\{e_\sigma\}$, i.e. $\zeta\in\Omega\subset M$, $u\in\mathscr{R}$, we substitute the last equation into equation (17) and get

$$\begin{aligned}
K_{\zeta+ue_\sigma}(e_\sigma,n) &= \frac{R_{\sigma 0\sigma 0}}{\langle e_\sigma,e_\sigma\rangle\langle n,n\rangle-\langle e_\sigma,n\rangle^2}\\
&= -\frac{1}{2g}\frac{\partial^2 g}{\partial u^2}+\frac{1}{4g^2}\left(\frac{\partial g}{\partial u}\right)^2\\
&= -\frac{2(\kappa_\sigma)^2}{2\left((u\kappa_\sigma)^2-\eta_{m+1}^2\right)}+\frac{4(u\kappa_\sigma)^2(\kappa_\sigma)^2}{4\left((u\kappa_\sigma)^2-\eta_{m+1}^2\right)^2}\\
&= -\frac{(\kappa_\sigma)^2\left[(u\kappa_\sigma)^2-\eta_{m+1}^2-(u\kappa_\sigma)^2\right]}{\left((u\kappa_\sigma)^2-\eta_{m+1}^2\right)^2}\\
&= \frac{(\kappa_\sigma\eta_{m+1})^2}{(u\kappa_\sigma)^4-2(u\kappa_\sigma\eta_{m+1})^2+(\eta_{m+1})^4}.
\end{aligned}$$

If we simplify the last equation by dividing numerator and denominator κ_σ^4 we reach

$$K_{\zeta+ue_\sigma}(e_\sigma,n)=\frac{\left(\frac{\eta_{m+1}}{\kappa_\sigma}\right)^2}{u^4-2u^2\left(\frac{\eta_{m+1}}{\kappa_\sigma}\right)^2+\left(\frac{\eta_{m+1}}{\kappa_\sigma}\right)^4}$$

At this point, if we consider equation (17) then we find

$$K_{\zeta+ue_\sigma}(e_\sigma,n)=\frac{P_\sigma^2}{u^4-2u^2P_\sigma^2+P_\sigma^4} \quad , \quad 1\leq\sigma\leq m.$$

Therefore, the sectional curvature of two-dimensional time-like principal ruled surface φ_σ at the point $\zeta+ue_\sigma$ on generator $h_\sigma=Sp\{e_\sigma\}$ is found to be

$$K_{\zeta+ue_\sigma}(e_\sigma,n)=\frac{P_\sigma^2}{(u^2-P_\sigma^2)^2} \quad , \quad 1\leq\sigma\leq m.$$

□

Let M_σ, $1\leq\sigma\leq m$, be σ^{th} principal ray surface produced by $h_\sigma=Sp\{e_\sigma\}\subset E_k(t)$ along the orthogonal trajectory of M and P_σ, $1\leq\sigma\leq m$ be the σ^{th} principal distribution parameter of M in $\mathscr{R}_1^n$. The sectional curvature given in equation (22) is the generalized form of the Lamarle formula in $\mathscr{R}_1^3$, which is the relationship between the Gaussian curvature and principal parameter of 2-dimensional ruled surface. Thus, equation (22) is named as **generalized**

Lorentzian Lamarle formula by us.
Now we find the curvature of non-degenerate section (e,n), e being a unit vector within the space-like generating space $E_k(t)$ of generalized time-like ruled surface with central ruled surface M and n being non-degenerate normal tangent vector of M orthogonal to $E_k(t)$. Here while the normal tangent vector n is space-like or time-like, the following equations are hold

$$\langle e,e\rangle\langle n,n\rangle-\langle e,n\rangle^2=1>0$$

and

$$\langle e,e\rangle\langle n,n\rangle-\langle e,n\rangle^2=-1<0$$

respectively. This means that the tangent section (e,n) is either space-like or time-like plane. The curvature of space-like and time-like section are same for both cases. Therefore, instead of space-like or time-like section (e,n), we may give a theorem regarding to the non-degenerate section (e,n).

Theorem 4.6. *Let M be generalized time-like ruled surface with space-like generating space and central ruled surface and e be unit vector within $E_k(t)$ in n-dimensional Minkowski space $\mathscr{R}_1^n$. Taking n to be non-degenerate normal tangent vector of M orthogonal to $E_k(t)$, we write the relation*

$$K_\zeta(e,n)=\sum_{\sigma=1}^{m}\cos^2\theta_\sigma K_\zeta(e_\sigma,n) \tag{23}$$

between the curvature of non-degenerate section (e,n) and curvatures of non-degenerate principal sections (e_σ,n) at the point $\zeta\in\Omega\subset M$, where

$$e=\sum_{\nu=1}^{k}\cos\theta_\nu e_\nu \quad , \quad \sum_{\nu=1}^{k}\cos^2\theta_\nu=1$$

in which the angles between unit vector e and the base $e_1,e_2,\ldots,e_k$ are θ_1, θ_2, ..., θ_k, respectively.

Proof. Since the vector $e=\sum\limits_{\nu=1}^{k}\beta_\nu e_\nu$ is a unit vector within the space-like generating space $E_k(t)$ of generalized time-like ruled surface M with central ruled surface Ω, the relation

$$\beta_1^2+\beta_2^2+\ldots+\beta_k^2=1$$

is hold, i.e.

$$\beta_\nu^2\leq 1 \quad , \quad 1\leq\nu\leq k$$

and

$$\langle e,e\rangle\langle e_\nu,e_\nu\rangle-\langle e,e_\nu\rangle^2=1-\beta_\nu^2>0.$$

Therefore, (e_ν, n), $1 \leq \nu \leq k$, plane is space-like. From equation (1) we may write

$$\beta_\nu = \langle e, e_\nu \rangle = \cos\theta_\nu \quad , \quad 1 \leq \nu \leq k$$

where the θ_ν, $1 \leq \nu \leq k$, are the angles between space-like unit vector e and space-like base vectors e_ν, $1 \leq \nu \leq k$, in the space-like plane (e, e_ν), $1 \leq \nu \leq k$. From the last relation one may get

$$e = \sum_{\nu=1}^{k} \cos\theta_\nu e_\nu \quad , \quad \sum_{\nu=1}^{k} \cos^2\theta_\nu = 1. \tag{24}$$

Suppose that the coordinates of the space-like vector e and non-degenerate normal tangent vector n, which are the base of non-degenerate section (e, n), are $(\beta_0, \beta_1, \ldots, \beta_k)$ and $(\gamma_0, \gamma_1, \ldots, \gamma_k)$, respectively. From equations (18) and (24), we find

$$\beta_0 = \langle e, e_0 \rangle = 0 \quad , \quad \beta_\nu = \langle e, e_\nu \rangle = \cos\theta_\nu \quad , \quad 1 \leq \nu \leq k$$

and

$$\gamma_0 = \langle n, e_0 \rangle = 1 \quad , \quad \gamma_\nu = \langle n, e_\nu \rangle = 0 \quad , \quad 1 \leq \nu \leq k.$$

Substituting these equations into equations in to equation (17) for the central point $\zeta \in \Omega$, we obtain

$$K_\zeta(e, n) = \frac{\sum_{\sigma=1}^{m} \cos^2\theta_\sigma R_{\sigma 0 \sigma 0}}{\langle e, e \rangle \langle n, n \rangle - \langle e, n \rangle^2} \quad , \quad 1 \leq \sigma \leq m.$$

Therefore, the curvature of non-degenerate section (e, n) becomes

$$K_\zeta(e, n) = g^{-1}\left(\cos^2\theta_1 R_{1010} + \cos^2\theta_2 R_{2020} + \ldots + \cos^2\theta_m R_{m0m0}\right)$$

If we consider the curvature formula of non-degenerate principal section (e_σ, n), $1 \leq \sigma \leq m$, given by equation (19) at the point $\forall \zeta \in \Omega$, we find that the following relation is hold between curvature of non-degenerate section (e, n) and principal sectional curvature of generalized time-like ruled surface M with space-like generating space

$$K_\zeta(e, n) = \sum_{\sigma=1}^{m} \cos^2\theta_\sigma K_\zeta(e_\sigma, n).$$

□

This relation is called **Lorentzian Beltrami-Euler formula** for the generalized time-like ruled surface M with space-like generating space and central

ruled surface at the point $\zeta \in \Omega$.
Additionally, sectional curvature of (e,n) at any non-central point $\xi \in M$ is

$$K_{\xi}(e,n) = \sum_{\sigma=1}^{m} \cos^2\theta_\sigma \left[-\frac{1}{2g}\frac{\partial^2 g}{\partial u_\sigma^2} + \frac{1}{4g^2}\left(\frac{\partial g}{\partial u_\sigma}\right)^2 \right].$$

If we substitute the determinant of the first fundamental form g of M and the first and second order partial derivatives of g into the last equation sectional curvature of (e,n) is found to be

$$K_{\xi}(e,n) = -\frac{\sum\limits_{\sigma=1}^{m}(\cos\theta_\sigma \kappa_\sigma)^2}{\sum\limits_{\iota=1}^{m}(u_\iota \kappa_\iota)^2 - \eta_{m+1}^2} + \frac{\sum\limits_{\sigma,\iota=1}^{m}(\kappa_\sigma\kappa_\iota)^2 \cos\theta_\sigma \cos\theta_\iota u_\sigma u_\iota}{\left(\sum\limits_{\iota=1}^{m}(u_\iota\kappa_\iota)^2 - \eta_{m+1}^2\right)^2}.$$

It is clear from the last equation that at non-central points Lorentzian Beltrami-Euler Formula does not exist.

Theorem 4.7. *Let M be 2-dimensional time-like ruled surface with the space-like generators $h_\zeta(e) \subset E_k(t)$ and P be the distribution parameter of M at the central point $\zeta \in \Omega$ in n-dimensional Minkowski space $\mathscr{R}_1^n$. The non-degenerate (space-like or time-like) sectional curvature of (e,n) of M at the point $\zeta \in \Omega$ is*

$$K_\zeta(e,n) = \frac{1}{P^2}. \tag{25}$$

Proof. 2-dimensional ruled surface M with space-like generator $h_\zeta(e) \subset E_k(t)$ in is given parametrically by

$$\varphi(t,u) = \alpha(t) + ue(t).$$

Since $u_\sigma = 0$, $1 \le \sigma \le m$, at the central point $\zeta \in \Omega$, evaluating equation (23) gives the sectional curvature of (e,n) to be

$$K_\zeta(e,n) = \sum_{\sigma=1}^{m} \cos^2\theta_\sigma \left[\frac{\left(-\kappa_\sigma^2\right)\left(-\eta_{m+1}^2\right)}{\left(-\eta_{m+1}^2\right)^2} \right].$$

Simplifying this equation gives us

$$K_\zeta(e,n) = \frac{\sum\limits_{\sigma=1}^{m}\cos^2\theta_\sigma\kappa_\sigma^2}{\eta_{m+1}^2}. \tag{26}$$

In addition to that, a space-like unit vector in e is

$$e=\sum_{\nu=1}^{k}\cos\theta_{\nu}e_{\nu}$$

where $\theta_1,\theta_2,\ldots,\theta_k$ denote the angles between the space-like unit vector e and the base vectors $e_1,e_2,\ldots,e_k$. Thus the tangential vector of the space-like vector e has the form

$$\dot{e}=\sum_{\nu=1}^{k}\cos\theta_{\nu}\dot{e}_{\nu}.$$

Furthermore, the asymptotic bundle $A(t)=Sp\{e_1,\ldots,e_k,\dot{e}_1,\ldots,\dot{e}_k\}$ has an orthogonal base as $\left\{e_1,\ldots,e_k,\overset{\circ}{e}_1,\ldots,\overset{\circ}{e}_m\right\}$, we write

$$\begin{aligned}\overset{\circ}{e}&=\dot{e}-\sum_{\mu=1}^{k}\left\langle\dot{e},e_{\mu}\right\rangle e_{\mu}\\&=\sum_{\nu=1}^{k}\cos\theta_{\nu}\dot{e}_{\nu}-\sum_{\mu=1}^{k}\left\langle\sum_{\nu=1}^{k}\cos\ {}_{\nu}\dot{e}_{\nu},e_{\mu}\right\rangle e_{\mu}\\&=\sum_{\nu=1}^{k}\cos\theta_{\nu}\left(\dot{e}_{\nu}-\sum_{\mu=1}^{k}\left\langle\dot{e}_{\nu},e_{\mu}\right\rangle e_{\mu}\right).\end{aligned}$$

Since

$$\overset{\circ}{e}=\sum_{\sigma=1}^{m}\cos\theta_{\sigma}\overset{\circ}{e}_{\sigma}$$

and we find

$$\left\|\overset{\circ}{e}\right\|^2=\sum_{\sigma=1}^{m}\cos^2\theta_{\sigma}\left\|\overset{\circ}{e}_{\sigma}\right\|^2$$

In addition to these, since $\left\|\overset{\circ}{e}\right\|=\kappa$ and $\left\|\overset{\circ}{e}_{\sigma}\right\|=\kappa_{\sigma}$, $1\leq\sigma\leq m$, we obtain

$$\kappa^2=\sum_{\sigma=1}^{m}\cos^2\theta_{\sigma}\kappa_{\sigma}^2.$$

If the last equation is substituted in to the equation (26) we reach

$$K_{\zeta}(e,n)=\left(\frac{\kappa}{\eta_{m+1}}\right)^2.$$

Since the distribution parameter of M has the form $P=\frac{\eta_{m+1}}{\kappa}$, finally we reach to

$$K_{\zeta}(e,n)=\frac{1}{P^2}.$$

□

This equation is called **Lorentzian Lamarle formula** for the curvature of non-degenerate sections (e,n) of two-dimensional time-like ruled surface with space-like generator at the central point $\zeta \in \Omega$.

The sectional curvature of two dimensional time-like surface generated by the movement of space-like generator $h_\zeta(e) \subset E_k(t)$ along the time-like orthogonal trajectory of M in n-dimensional Minkowski space $\mathscr{R}_1^n$ degenerates to Gauss curvature two-dimensional time-like surfaces with space-like generators in 3-dimensional Minkowski space $\mathscr{R}_1^3$.

Example 4.8. Considering $X = (x_1, x_2, x_3, x_4, x_5)$ and $Y = (y_1, y_2, y_3, y_4, y_5)$, let us take 5-dimensional Minkowski space $\mathscr{R}_1^5$ given by Lorentz metric

$$\langle X, Y\rangle = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 - x_5y_5.$$

Suppose that the curve $\alpha : I \to \mathscr{R}_1^5$ is given by

$$\alpha(t) = \frac{1}{\varepsilon}\left(2\varepsilon^2 t, \kappa\sin\varepsilon t - \kappa\cos\varepsilon t, \tau\sin\varepsilon t - \tau\cos\varepsilon t, -\varepsilon\cos\varepsilon t - \varepsilon\sinh\varepsilon t, 0\right)$$

and the subspace $E_2(t) = Sp\{e_1(t), e_2(t)\}$ defined at every point of curve α is given by

$$\begin{aligned} e_1(t) &= \tfrac{1}{\sqrt{3}\varepsilon}(\varepsilon,\ \kappa\cos\varepsilon t - \tau,\ \tau\cos\varepsilon t + \kappa,\ \varepsilon\sin\varepsilon t, 0) \\ e_2(t) &= \tfrac{1}{\sqrt{3}\varepsilon}(\varepsilon,\ \kappa\sin\varepsilon t + \tau,\ \tau\sin\varepsilon t - \kappa,\ -\varepsilon\cos\varepsilon t, 0) \end{aligned}$$

where κ, τ and $\varepsilon = \sqrt{\kappa^2 + \tau^2}$ are arbitrary constants. Since $\langle\dot{\alpha}, \dot{\alpha}\rangle = -3\varepsilon^2 < 0$, α is time-like curve and since $\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = 1$, $E_2(t)$ is a space-like subspace. In this case the transformation

$$\varphi(t, u_1, u_2) = \alpha(t) + \sum_{v=1}^{2} u_v e_v(t)$$

defines 3-dimensional time-like ruled surface with time-like base curve and space-like generating space in $\mathscr{R}_1^5$. Let $\{e_1(t), e_2(t)\}$ be the principal frame of generating space $E_2(t)$ and $\{e_1(t), e_2(t), \dot{e}_1(t), \dot{e}_2(t), \dot{\alpha}(t)\}$ be the base of tangential bundle of 3-dimensional time-like ruled surface M. From the Gram-Schmidt method, we find

$$\begin{aligned} a_3(t) &= \frac{1}{\sqrt{6}\varepsilon}(\varepsilon,\ -2\kappa\sin\varepsilon t + \tau,\ -2\tau\sin\varepsilon t - \kappa,\ 2\varepsilon\cos\varepsilon t, 0), \\ a_4(t) &= \frac{1}{\sqrt{6}\varepsilon}(-\varepsilon,\ 2\kappa\cos\varepsilon t + \tau,\ 2\tau\cos\varepsilon t - \kappa,\ 2\varepsilon\sin\varepsilon t, 0), \\ a_5(t) &= (0,\ 0,\ 0,\ 0,\ 1). \end{aligned}$$

From these equations we establish the orthonormal base

$$\{e_1(t), e_2(t), a_3(t), a_4(t), a_5(t)\}$$

of tangential bundle of M. Therefore, derivative equations for principal frame $\{e_1(t), e_2(t)\}$ of generating space of M are found to be

$$\begin{aligned}\dot{e}_1(t) &= -\tfrac{\varepsilon}{3}e_2(t) + \tfrac{\sqrt{2}}{3}\varepsilon a_3(t),\\ \dot{e}_2(t) &= \tfrac{\varepsilon}{3}e_1(t) + \tfrac{\sqrt{2}}{3}\varepsilon a_4(t).\end{aligned}$$

In addition to that the velocity vector of the base curve of M is evaluated

$$\dot{\alpha} = \sqrt{3}\varepsilon e_1 + \sqrt{3}\varepsilon e_2 + 3\varepsilon a_5$$

In this case the metric coefficients of 3-dimensional time-like ruled surface with space-like generating space are

$$\begin{aligned}g_{00} &= \frac{2\sqrt{3}}{3}\varepsilon^2 u_1 - \frac{2\sqrt{3}}{3}\varepsilon^2 u_2 + \frac{\varepsilon^2}{3}u_1^2 + \frac{\varepsilon^2}{3}u_2^2 - 3\varepsilon^2,\\ g_{10} &= g_{01} = \sqrt{3}\varepsilon - \frac{\varepsilon}{3}u_2,\\ g_{20} &= g_{02} = \sqrt{3}\varepsilon + \frac{\varepsilon}{3}u_1,\\ g_{12} &= g_{21} = 0,\\ g_{11} &= g_{22} = 1.\end{aligned}$$

and first fundamental form is

$$[g_{ij}]_{3\times 3} = \begin{bmatrix} \frac{2\sqrt{3}}{3}\varepsilon^2 u_1 - \frac{2\sqrt{3}}{3}\varepsilon^2 u_2 + \frac{\varepsilon^2}{3}u_1^2 + \frac{\varepsilon^2}{3}u_2^2 - 3\varepsilon^2 & \sqrt{3}\varepsilon - \frac{\varepsilon}{3}u_2 & \sqrt{3}\varepsilon + \frac{\varepsilon}{3}u_1 \\ \sqrt{3}\varepsilon - \frac{\varepsilon}{3}u_2 & 1 & 0 \\ \sqrt{3}\varepsilon + \frac{\varepsilon}{3}u_1 & 0 & 1 \end{bmatrix}$$

From these equations we see that the determinant of the first fundamental form of M is

$$g = \det[g_{ij}] = \frac{2}{9}\varepsilon^2 u_1^2 + \frac{2}{9}\varepsilon^2 u_2^2 - 9\varepsilon^2$$

the normal tangent vector of 3-dimensional time-like ruled surface M which is orthogonal to the generating space $E_2(t)$ at the point $\forall \xi \in M$ is defined to be

$$n = \frac{\sqrt{2}}{3}\varepsilon u_1 a_3 + \frac{\sqrt{2}}{3}\varepsilon u_2 a_4 + 3\varepsilon a_5$$

so that, since

$$\langle n, n\rangle = \frac{2}{3}\varepsilon^2\left(u_1^2 + u_2^2\right) - 9\varepsilon^2$$

normal tangent vector n to be non-degenerate (space-like or time-like) the relation $u_1^2+u_2^2 \neq \frac{81}{2}$ should be satisfied. Thus, from Theorem 4.1, the sectional curvature of non-degenerate first principal section (e_1,n) and non-degenerate second principal section (e_1,n) of M regarding to the principal frame of $E_2(t)$ are given by

$$K_\xi(e_1,n) = -\frac{1}{2g}\frac{\partial^2 g}{\partial u_1^2} + \frac{1}{4g^2}\left(\frac{\partial g}{\partial u_1}\right)^2$$

and

$$K_\xi(e_2,n) = -\frac{1}{2g}\frac{\partial^2 g}{\partial u_2^2} + \frac{1}{4g^2}\left(\frac{\partial g}{\partial u_2}\right)^2$$

respectively. Since

$$\frac{\partial g}{\partial u_1} = \frac{4}{9}\varepsilon^2 u_1 \quad , \quad \frac{\partial^2 g}{\partial u_1^2} = \frac{4}{9}\varepsilon^2 \quad , \quad \left(\frac{\partial g}{\partial u_1}\right)^2 = \frac{16}{81}\varepsilon^4 u_1^2$$

and

$$\frac{\partial g}{\partial u_2} = \frac{4}{9}\varepsilon^2 u_2 \quad , \quad \frac{\partial^2 g}{\partial u_2^2} = \frac{4}{9}\varepsilon^2 \quad , \quad \left(\frac{\partial g}{\partial u_2}\right)^2 = \frac{16}{81}\varepsilon^4 u_2^2$$

the sectional curvature of non-degenerate 1^{st} principal section (e_1,n) and non-degenerate 2^{nd} principal section (e_2,n) are found to be

$$K_\xi(e_1,n) = \frac{-4u_2^2+162}{\left(2u_1^2+2u_2^2-81\right)^2} \quad , \quad u_1^2+u_2^2 \neq \frac{81}{2}$$

and

$$K_\xi(e_2,n) = \frac{-4u_1^2+162}{\left(2u_1^2+2u_2^2-81\right)^2} \quad , \quad u_1^2+u_2^2 \neq \frac{81}{2}$$

respectively. Considering $u_1 = u_2 = 0$ at the central point $\forall\zeta \in \Omega$, the principal curvatures $K_\zeta(e_1,n)$ and $K_\zeta(e_2,n)$ are found to be

$$K_\zeta(e_1,n) = \frac{2}{81} \quad \text{and} \quad K_\zeta(e_2,n) = \frac{2}{81}.$$

Furthermore, since 1^{st} and 2^{nd} principal distribution parameters of M at point $\forall\zeta \in \Omega$ are

$$P_1 = \frac{3\varepsilon}{\frac{\sqrt{2}\varepsilon}{3}} = \frac{9}{\sqrt{2}} \quad \text{and} \quad P_2 = \frac{3\varepsilon}{\frac{\sqrt{2}\varepsilon}{3}} = \frac{9}{\sqrt{2}}$$

respectively. From Theorem 4.4, the result is obtained as

$$K_\zeta(e_1,n) = \frac{1}{P_1^2} = \frac{2}{81} \quad \text{and} \quad K_\zeta(e_2,n) = \frac{1}{P_2^2} = \frac{2}{81}.$$

Now we consider 2-dimensional principal ray surfaces in $\mathscr{R}_1^5$. Let φ_1 be time-like principal ray surface with space-like generator given by

$$\varphi_1(t,u) = \alpha(t) + ue_1(t).$$

From Theorem 4.5, curvature of non-degenerate section (e_1,n) of φ_1 at the point $\zeta + ue_1$ is found to be

$$K_{\zeta+ue_1}(e_1,n) = \frac{162}{(2u^2-81)^2} \quad , \quad u \neq \frac{9}{\sqrt{2}}.$$

Let φ_2 be time-like principal ray surface with space-like generator given by the parametrical equation of

$$\varphi_2(t,u) = \alpha(t) + ue_2(t).$$

Similarly from Theorem 4.5, curvature of non-degenerate section (e_2,n) of φ_2 at the point $\zeta + ue_2$ is evaluated to be

$$K_{\zeta+ue_2}(e_2,n) = \frac{162}{(2u^2-81)^2} \quad , \quad u \neq \frac{9}{\sqrt{2}}.$$

Now we consider space-like unit vector e within $E_2(t)$. Taking angles θ_1 and θ_2 between the unit vector e and the base vectors e_1 and e_2, respectively, we write the space-like unit vector as

$$e = \cos\theta_1 e_1 + \cos\theta_2 e_2 \quad , \quad \cos^2\theta_1 + \cos^2\theta_2 = 1.$$

From Theorem 4.6, since there exists a relation between sectional curvature and principal sectional curvature of M at the central point as

$$K_\zeta(e,n) = \cos^2\theta_1 K_\zeta(e_1,n) + \cos^2\theta_2 K_\zeta(e_2,n)$$

we find for the curvature of non-degenerate section (e,n) the following result

$$K_\zeta(e,n) = \frac{2}{81}\cos^2\theta_1 + \frac{2}{81}\cos^2\theta_2 = \frac{2}{81}$$

this result is consistent with the Theorem 4.5, because the distribution parameter of M is

$$P = \sqrt{|P_1.P_2|} = \sqrt{\left|\frac{9}{\sqrt{2}}.\frac{9}{\sqrt{2}}\right|} = \frac{9}{\sqrt{2}}$$

Therefore, it can be seen that the sectional curvature of (e,n) is

$$K_\zeta(e,n) = \frac{1}{P^2} = \frac{2}{81}.$$

REFERENCES

[1] B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, New York, 1983.

[2] E. Kruppa, *Analytische und Konstruktive Differentialgeometrie*, Wien Springer-Verlag, 1957, 26–28 and 74.

[3] H. Frank - O. Giering, *Zur Schnittkrmmung verallgemeinerter Regelflachen*, Archiv Der Mathematik, Fasc.1, 32, (1979), 86–90.

[4] J.G. Ratcliffe, *Foundations of Hyperbolic Manifolds*, Graduate Texts in Mathematics 149, Springer, New York, 1994, 63–76.

[5] J.K. Beem - P. E. Ehrlich - K. L. Easley, *Lorentzian Geometry*, Marcel Dekker, New York, 1981.

[6] M. Tosun - N. Kuruoglu, *On $(k+1)$-dimensional time-like ruled surface in the Minkowski space R_1^n*, J. Inst. Math. Comput. Sci. Math. Ser. 11 n. 1 (1998), 1–9.

[7] M. Tosun - N. Kuruoglu, *On properties of generalized time like ruled surface in the Minkowski space R_1^n*, Antarct. J. Math. 2 n. 2 (2005), 173–180.

[8] W. Blaschke, *Vorlesungen ber Differentialgeometrie*, 14 Ayfl, Berlin, 1945.

SOLEY ERSOY
Department of Mathematics
Faculty of Arts and Sciences
Sakarya University
54187 Sakarya TURKEY
e-mail: `sersoy@sakarya.edu.tr`

MURAT TOSUN
Department of Mathematics
Faculty of Arts and Sciences
Sakarya University
54187 Sakarya TURKEY
e-mail: `tosun@sakarya.edu.tr`