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ON A SECOND ORDER DISCRETE PROBLEM

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By using variontional methods, we establish criteria for the existence of at least three solutions to a second order discrete problem with two parameters. Applications of the main theorem to several special cases of the problem are discussed and two examples are included to illustrate the results. This paper extends and complements some of our early work on related problems.

1. Introduction

In this paper, we study the existence of multiple solutions to the second order discrete problem

$$\begin{cases} -\Delta^2 u(k-1) = \lambda f(k,u(k)),\ k \in [1,N]_{\mathbb{Z}}, \\ u(0)=0,\ \Delta u(N) = \mu g(u(N+1)), \end{cases} \tag{1.1}$$

where $\lambda, \mu \in [0,\infty)$ are parameters, $N \in \mathbb{N}$, $[c,d]_{\mathbb{Z}} = \{z \in \mathbb{Z} \mid c \le z \le d\}$ for any $c,d \in \mathbb{Z}$ with $c \le d$, $f \in C([1,N]_{\mathbb{Z}} \times \mathbb{R}, \mathbb{R})$, $g \in C(\mathbb{R},\mathbb{R})$, and Δ is the forward difference operator defined by $\Delta u(k) = u(k+1) - u(k)$ and $\Delta^2 u(k) = \Delta(\Delta u(k))$. By a solution of problem (1.1), we mean a function $u : [0,N+1]_{\mathbb{Z}} \to \mathbb{R}$ such that u satisfies both the equation and the boundary conditions (BCs) in (1.1).

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In a recent paper [9], we studied the discrete problem

$$\begin{cases} -\Delta^2 u(k-1) = f(k,u(k)),\ k \in [1,N]_{\mathbb{Z}}, \\ u(0) = 0,\ u(N+1) = \nu u(N), \end{cases} \tag{1.2}$$

where $\nu \in [0,\infty)$. Notice that if $\mu \neq 1$, problem (1.1) with $\lambda = 1$ and $g(x) \equiv x$ on $\mathbb{R}$ can be rewritten in the form of (1.2) with $\nu = 1/(1-\mu)$. Several sufficient conditions were obtained in [9] for the existence of at least two and three solutions of problem (1.2) by employing variational approaches, combined with the classic mountain pass lemma and a result on the invariant set of descending flow. Specifically, in [9], solutions of problem (1.2) were sought in an N dimensional Banach space H_ν defined by

$$H_\nu = \{u : [0,N+1]_{\mathbb{Z}} \to \mathbb{R} \mid u(0) = 0,\ u(N+1) = \nu u(N)\} \tag{1.3}$$

and equipped with the norm $\|u\| = \left(\sum_{k=1}^N |u(k)|^2\right)^{1/2}$ for $u \in H_\nu$. In this paper, we study the existence of three solutions of problem (1.1) by applying a three critical points theorem due to Ricceri [16], which is recalled in Lemma 2.2 in Section 2. Our existence results for problem (1.1) are given in terms of the relationship between λ and μ. Under some assumptions, we show that for all λ in a suitable compact interval $[a,b]$, there exists $\delta > 0$ such that for every $\mu \in [0,\delta]$, problem (1.1) has at least three solutions. See Theorem 2.1 and Corollary 2.1 in Section 2 for details. To obtain such results, a space similar to H_ν cannot be used in our analysis since it depends on the parameter in the BC of the problem. In this paper, we instead introduce a new $N+1$ dimensional Banach space X (see (2.1) below) which is independent of the parameter μ in the BC. We then define suitable functionals on X to establish an equivalent variational structure for problem (1.1). We also derive a symmetric positive definite matrix A, defined by (2.7) below, to rewrite one of the functionals. The smallest and largest eigenvalues of the matrix A play a key role in the formulation of our results and their proofs.

As noted in Remark 2.1 below, the only condition imposed on the function g in the BC of problem (1.1) is that $g \in C(\mathbb{R},\mathbb{R})$. Thus, by taking g in different forms, our existence criteria can be applied to a variety of problems with different BCs. As an illustration of applications, new existence criteria are obtained for problem (1.2) in Corollaries 2.3 and 2.4 in Section 2.

Finally, in this section, we point out that, as is well known, nonlinear difference equations have extensive applications in many fields, some of which are documented in [7, 8]. In the literature, second order discrete problems with various BCs have been actively investigated by many researchers using a variety of methods such as fixed point theory, lower and upper solution methods, and critical point theory. We refer the reader to [1, 3, 5, 9–12, 17, 18] for a small sample of the work on the subject.

2. Main results

Let X be a set defined by

$$X = \{u : [0, N+1]_{\mathbb{Z}} \to \mathbb{R} \mid u(0) = 0\}. \tag{2.1}$$

Then, X is a vector space with $au + bv = \{au(k) + bv(k)\}$ for any $u, v \in X$ and $a, b \in \mathbb{R}$. It is easy to see that X is an $N+1$ dimensional Banach space equipped with the norm $\|u\| = \left(\sum_{k=1}^{N+1} u^2(k)\right)^{1/2}$, $u \in X$.

Let the functionals $\Phi, \Psi, J : X \to \mathbb{R}$ be defined by

$$\begin{cases} \Phi(u) = \dfrac{1}{2} \displaystyle\sum_{k=1}^{N+1} |\Delta u(k-1)|^2, \\ \Psi(u) = \displaystyle\int_0^{u(N+1)} g(s)ds, \\ J(u) = \displaystyle\sum_{k=1}^{N} F(k, u(k)), \end{cases} \tag{2.2}$$

where $u \in X$ and

$$F(k, x) = \int_0^x f(k, s)ds \quad \text{for all } x \in \mathbb{R}. \tag{2.3}$$

Then, Φ, Ψ, J are well defined and continuously differentiable whose derivatives are the linear functionals $\Phi'(u)$, $\Psi'(u)$, and $J'(u)$ given by

$$\Phi'(u)(v) = \sum_{k=1}^{N+1} \Delta u(k-1)\Delta v(k-1), \tag{2.4}$$

$$\Psi'(u)(v) = g(u(N+1))v(N+1), \tag{2.5}$$

and

$$J'(u)(v) = \sum_{k=1}^{N} f(k, u(k))v(k) \tag{2.6}$$

for all $u, v \in X$.

Lemma 2.1. *The function $u \in X$ is a critical point of the functional $\Phi - \lambda J - \mu\Psi$ if and only if u is a solution of problem* (1.1).

Proof. Let $u \in X$. Then, from (2.4)-(2.6) and the summation by parts formula, we see that u is a critical point of $\Phi - \lambda J - \mu\Psi$ if and only if

$$\begin{aligned}
0 &= \Phi'(u)(v) - \lambda J'(u)(v) - \mu\Psi'(u)(v) \\
&= \sum_{k=1}^{N+1} \Delta u(k-1)\Delta v(k-1) - \sum_{k=1}^{N} \lambda f(k,u(k))v(k) - \mu g(u(N+1))v(N+1) \\
&= \Delta u(N)v(N+1) - \Delta u(0)v(0) - \sum_{k=1}^{N} \Delta^2 u(k-1)v(k) \\
&\quad - \sum_{k=1}^{N} \lambda f(k,u(k))v(k) - \mu g(u(N+1))v(N+1) \\
&= \big(\Delta u(N) - \mu g(u(N+1))\big)v(N+1) - \sum_{k=1}^{N} \big(\Delta^2 u(k-1) + \lambda f(k,u(k))\big)v(k)
\end{aligned}$$

for any $v \in X$. By the arbitrariness of $v \in X$, the above equality is equivalent to

$$\Delta u(N) = \mu g(u(N+1)) \text{ and } -\Delta^2 u(k-1) = \lambda f(k,u(k)) \quad \text{for all } k \in [1,N]_{\mathbb{Z}},$$

which implies that u is a solution of problem (1.1) by noting that $u(0) = 0$ (since $u \in X$). This completes the proof of the lemma. □

Now, we present an equivalent form of the functional Φ. Let

$$u = (u(0), u(1), \cdots, u(N), u(N+1)) \in X.$$

Since $u(0) = 0$ and X is isomorphic to $\mathbb{R}^{N+1}$, in the sequel, we always identify u with the vector $u = (u(1), \cdots, u(N+1)) \in \mathbb{R}^{N+1}$. Let

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}_{(N+1)\times(N+1)}. \tag{2.7}$$

Then, A is a symmetric positive definite matrix. Moreover, by direct calculations, we have

$$\Phi(u) = \frac{1}{2} u A u^T \qquad \text{for all } u \in X.$$

Let λ_i be the eigenvalues of A ordered as follows

$$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_{N+1} < \infty.$$

Then, we can verify that

$$\frac{1}{2}\lambda_1\|u\|^2 \le \Phi(u) \le \frac{1}{2}\lambda_{N+1}\|u\|^2 \quad \text{for all } u \in X. \tag{2.8}$$

Next, we state the main result of the paper. Below, X and F are defined by (2.1) and (2.3), respectively.

Theorem 2.1. Assume that the following conditions hold

(H1) $\rho := \max\left\{ \limsup_{x\to 0} \frac{\max_{k\in[1,N]_{\mathbb{Z}}} F(k,x)}{x^2},\ \limsup_{|x|\to\infty} \frac{\max_{k\in[1,N]_{\mathbb{Z}}} F(k,x)}{x^2} \right\} < \infty;$

(H2) $\sup_{u\in\Phi^{-1}(0,\infty)} \frac{\sum_{k=1}^N F(k,u(k))}{\|u\|^2} > \frac{\rho\lambda_{N+1}}{\lambda_1}.$

Then for each compact interval $[a,b]$ satisfying

$$[a,b] \subset \left(\frac{\lambda_{N+1}}{2\sup_{u\in\Phi^{-1}(0,\infty)} \frac{\sum_{k=1}^N F(k,u(k))}{\|u\|^2}},\ \frac{\lambda_1}{2\rho} \right),$$

there exists $r > 0$ with the following property: for every $\lambda \in [a,b]$ and $g \in C(\mathbb{R},\mathbb{R})$, there exists $\delta > 0$ such that for any $\mu \in [0,\delta]$, problem (1.1) has at least three solutions in X whose norms are less than r.

The following corollaries are direct consequences of Theorem 2.1.

Corollary 2.1. Assume that $\rho = 0$ and there exists $w \in X$ such that

$$\sum_{k=1}^N F(k,w(k)) > 0.$$

Then for each compact interval $[a,b]$ satisfying

$$[a,b] \subset \left(\frac{\lambda_{N+1}\|w\|^2}{2\sum_{k=1}^N F(k,w(k))},\ \infty \right),$$

there exists $r > 0$ with the following property: for every $\lambda \in [a,b]$ and $g \in C(\mathbb{R},\mathbb{R})$, there exists $\delta > 0$ such that for any $\mu \in [0,\delta]$, problem (1.1) has at least three solutions in X whose norms are less than r.

Corollary 2.2. Assume that (H1) holds and there exists $w \in X$ such that

$$0 < \frac{\lambda_{N+1}\|w\|^2}{\sum_{k=1}^N F(k,w(k))} < 2 < \frac{\lambda_1}{\rho}. \tag{2.9}$$

Then there exists $r > 0$ such that for every $g \in C(\mathbb{R},\mathbb{R})$, there exists $\delta > 0$ such that for any $\mu \in [0,\delta]$, the problem

$$\begin{cases} -\Delta^2 u(k-1) = f(k,u(k)),\ k \in [1,N]_{\mathbb{Z}}, \\ u(0) = 0,\ \Delta u(N) = \mu g(u(N+1)) \end{cases}$$

has at least three solutions in X whose norms are less than r.

Remark 2.1. In Theorem 2.1 and Corollaries 2.1 and 2.2, the condition imposed on the function g is very mild. From the proof of Theorem 2.1 given below, one can see that the only condition needed on g is to guarantee that the functional Ψ defined in (2.2) is a C^1 functional with compact derivative and that it suffices to assume that $g \in C(\mathbb{R},\mathbb{R})$ to achieve this goal. Therefore, our results are applicable to problem (1.2). As an illustration, we present some applications of Corollary 2.2 below.

Corollary 2.3. Assume that (H1) and (2.9) hold. Then there exist $r > 0$ and $\eta > 1$ such that for any $\nu \in [0,\eta]$, problem (1.2) has at least three solutions in X whose norms are less than r.

Corollary 2.4. Assume that (H1) and (2.9) hold. Then there exist $r > 0$ such that the problem

$$\begin{cases} -\Delta^2 u(k-1) = f(k,u(k)),\ k \in [1,N]_{\mathbb{Z}}, \\ u(0) = 0,\ u(N+1) = u(N) \end{cases}$$

has at least three solutions in X whose norms are less than r.

We now provide two examples to apply our results.

Example 2.1. We claim that, for each compact interval $[a,b] \subset (2.6039,\infty)$, there exists $r > 0$ with the following property: for every $\lambda \in [a,b]$ and $g \in C(\mathbb{R},\mathbb{R})$, there exists $\delta > 0$ such that for any $\mu \in [0,\delta]$, the problem

$$\begin{cases} -\Delta^2 u(k-1) = \lambda \sin\left(\frac{\pi k}{8}\right) \frac{4u^3(k)}{1+u^8(k)} \cos(\arctan(u^4(k))),\ k \in [1,4]_{\mathbb{Z}}, \\ u(0) = 0,\ \Delta u(4) = \mu g(u(5)) \end{cases} \tag{2.10}$$

has at least three solutions in X whose norms are less than r. Moreover, if $g(0) \neq 0$, none of the three solutions is trivial.

In fact, first notice that problem (2.10) is of the form of (1.1) with $N = 4$ and

$$f(k,x) = \sin\left(\frac{\pi k}{8}\right) \frac{4x^3}{1+x^8} \cos\left(\arctan(x^4)\right) \quad \text{for all } (k,x) \in [1,4]_{\mathbb{Z}} \times \mathbb{R}.$$

Moreover, from (2.3), we have

$$F(k,x) = \sin\left(\frac{\pi k}{8}\right)\sin(\arctan(x^4)) \quad \text{for all } (k,x) \in [1,4]_{\mathbb{Z}} \times \mathbb{R}.$$

Then,

$$\limsup_{x\to 0} \frac{\max_{k\in[1,4]_{\mathbb{Z}}} F(k,x)}{x^2} = 0 \quad \text{and} \quad \limsup_{|x|\to\infty} \frac{\max_{k\in[1,4]_{\mathbb{Z}}} F(k,x)}{x^2} = 0.$$

Thus, for ρ defined in (H1), we have $\rho = 0$. Now, let

$$w(k) = \begin{cases} 1, & k = 4, \\ 0, & k \in [0,5]_{\mathbb{Z}} \setminus \{4\}. \end{cases}$$

Then, $w \in X$, $\|w\| = 1$, and $\sum_{k=1}^{4} F(k,w(k)) = \frac{\sqrt{2}}{2} > 0$. Thus, all the conditions of Corollary 2.1 are satisfied. For the matrix A defined by (2.7) with $N = 4$, using Matlab, we find that its largest eigenvalues λ_6 is given by $\lambda_5 \approx 3.6825$. Then, $\frac{\lambda_5\|w\|^2}{2\sum_{k=1}^{4} F(k,w(k))} \approx 2.6039$. Hence, the first part of the claim readily follows from Corollary 2.1. The "moreover" part of the claim is obviously true.

Example 2.2. We claim that there exist $r > 0$ and $\eta > 1$ such that for any $\nu \in [0,\eta]$, the problem

$$\begin{cases} -\Delta^2 u(k-1) = f(k,u(k)), \ k \in [1,5]_{\mathbb{Z}}, \\ u(0) = 0, \ u(6) = \nu u(5) \end{cases} \tag{2.11}$$

has at least three solutions in X whose norms are less than r, where

$$f(k,x) = \begin{cases} 4k & \text{if } \ x > 1, \\ 4kx^3 & \text{if } \ |x| \le 1, \\ -4k & \text{if } \ x < -1, \end{cases} \qquad \text{for all } (k,x) \in [1,5]_{\mathbb{Z}} \times \mathbb{R}. \tag{2.12}$$

In fact, first notice that problem (2.11) is of the form of (1.2) with $N = 5$. Moreover, from (2.12) we see that

$$F(k,x) = \begin{cases} k(4x-3) & \text{if } \ x > 1, \\ kx^4 & \text{if } \ |x| \le 1, \\ -k(4x+3) & \text{if } \ x < -1, \end{cases} \qquad \text{for all } (k,x) \in [1,5]_{\mathbb{Z}} \times \mathbb{R}.$$

Then,

$$\limsup_{x\to 0} \frac{\max_{k\in[1,5]_{\mathbb{Z}}} F(k,x)}{x^2} = 0 \quad \text{and} \quad \limsup_{|x|\to\infty} \frac{\max_{k\in[1,5]_{\mathbb{Z}}} F(k,x)}{x^2} = 0.$$

Thus, for ρ defined in (H1), we have $\rho = 0$. Now, let

$$w(k) = \begin{cases} 1, & k = 5, \\ 0, & k \in [0,6]_{\mathbb{Z}} \setminus \{5\}. \end{cases}$$

Then, $w \in X$, $\|w\| = 1$, and $\sum_{k=1}^{5} F(k, w(k)) = 5$. For the matrix A defined by (2.7) with $N = 5$, using Matlab, we find that its smallest and the largest eigenvalues λ_1 and λ_6 is given by $\lambda_1 \approx 0.0581$ and $\lambda_6 \approx 3.7709$. Hence,

$$\frac{\lambda_6 \|w\|^2}{\sum_{k=1}^{5} F(k, w(k))} \approx 0.7542 < 2 < \frac{\lambda_1}{\rho} = \infty.$$

So (2.9) holds. Thus, all the conditions of Corollary 2.3 are satisfied. The claim then readily follows from Corollary 2.3.

Remark 2.2. As illustrated in Examples 2.1 and 2.2, the hypotheses of the theorems in this paper are simple and easily verifiable, while the conclusions are very rich and precise. We comment that the results in [9] cannot be applied to these examples. This is because of the following:

(a) In Example 2.1, problem (2.10) contains a very general BC $\Delta u(4) = \mu g(u(5))$, which is not covered by the results in [9]. Moreover, even if for the special case of the BC (i.e., when $g(u(5)) \equiv u(5)$ in the BC), we still cannot use the results in [9] to obtain the conclusion as given in Example 2.1. The assumptions of the results in [9] heavily depend on the value of the parameter μ in the BC, while the assumptions of the theorems in this paper are independent of the values of μ. This enables us to obtain the existence of multiple solutions of problem (2.10) for all $\mu \in [0, \delta]$. Such a general conclusion cannot be obtained using the results in [9].

(b) A comment similar to part (a) also applies to Example 2.2: The assumptions of the results in [9] depend on a fixed value of the parameter in the BC and cannot be applied to problem (2.11) to obtain the existence of multiple solutions when the parameter ν in the BC $u(6) = \nu u(5)$ is changing in the interval $[0, \eta]$ as obtained in Example 2.2. Moreover, even for a fixed ν, the nonlinear function f in problem (2.11) does not satisfy the assumptions of [9, Theorem 3.2] and all its corollaries.

In the remainder of this section, we prove our results. We first recall a three critical points theorem due Ricceri [16]. Let X be a real Banach space with the norm $\|\cdot\|$. As in [16], let $\mathcal{W}_X$ be the class of all functionals $\Phi : X \to \mathbb{R}$ with the property: if $\{u_n\}$ is a sequence in X converging weakly to u and

$\liminf_{n\to\infty} \Phi(u_n) \le \Phi(u)$, then $\{u_n\}$ has a subsequence converging strongly to u.

Lemma 2.2 below is taken from [16, Theorem 2] and plays a key role in the proof of our main theorem. For some related results and a small sample of recent applications of the lemma to other problems, see [2, 4, 6, 13–15] and the references therein.

Lemma 2.2. *Let X be a separable and reflexive real Banach space with the norm $\|\cdot\|$. Let $\Phi : X \to \mathbb{R}$ be a coercive, sequentially weakly lower semicontinuous C^1 functional, belonging to $\mathcal{W}_X$, bounded on each bounded subset of X and whose derivative admits a continuous inverse on X^*, and $J : X \to \mathbb{R}$ be a C^1 functional with compact derivative. Assume that Φ has a strict local minimum u_0 with $\Phi(u_0) = J(u_0) = 0$. Finally, let*

$$\alpha = \max\left\{0,\ \limsup_{\|u\|\to\infty} \frac{J(u)}{\Phi(u)},\ \limsup_{u\to u_0} \frac{J(u)}{\Phi(u)}\right\}, \tag{2.13}$$

$$\beta = \sup_{u\in\Phi^{-1}(0,\infty)} \frac{J(u)}{\Phi(u)}, \tag{2.14}$$

and assume that $\alpha < \beta$. Then, for each compact interval $[a,b] \subset (1/\beta, 1/\alpha)$ (with the conventions $1/0 = \infty$ and $1/\infty = 0$), there exists $r > 0$ with the following property: for every $\lambda \in [a,b]$ and every C^1 functional $\Psi : X \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that for each $\mu \in [0,\delta]$, the equation $\Phi'(u) = \lambda J'(u + \mu\Psi'(u)$ has at least three solutions in X whose norms are less than r.

Proof of Theorem 2.1. Let the Banach space X be defined by (2.1) and let the continuously differentiable functionals $\Phi, \Psi, J : X \to \mathbb{R}$ be given as in (2.2). Then, from Lemma 2.1, a critical point of $\Phi - \lambda J - \mu\Psi$ is a solution of problem (1.1). Below, we will apply Lemma 2.2 to prove Theorem 2.1.

First, it is trivial to check that Φ is a coercive and sequentially weakly lower semicontinuous functional and is bounded on each bounded subset of X, and $\Phi \in \mathcal{W}_X$. For all $u \in X$, in view of (2.2) and (2.4), we have $\Phi'(u)(u) = \sum_{k=1}^{N+1} |\Delta u(k-1)|^2 = \frac{1}{2}\Phi(u)$. This, together with (2.8), implies that

$$\frac{1}{4}\lambda_1\|u\|^2 \le \Phi'(u)(u) \le \frac{1}{4}\lambda_{N+1}\|u\|^2.$$

Thus, $\lim_{\|u\|\to\infty} \Phi'(u)(u)/\|u\| = \infty$. This shows that Φ' is coercive. Moreover, for any $u, v \in X$, again by (2.2) and (2.4), we have

$$(\Phi'(u) - \Phi'(v))(u-v) = \sum_{k=1}^{N+1} |\Delta(u-v)(k-1)|^2 = \frac{1}{2}\Phi(u-v).$$

Then, using (2.8), we obtain that

$$\frac{1}{4}\lambda_1\|u-v\|^2 \le (\Phi'(u)-\Phi'(v))(u-v) \le \frac{1}{4}\lambda_{N+1}\|u-v\|^2.$$

Therefore, Φ' is uniformly monotone. Hence, by [19, Theorem 26.A (d)], $(\Phi')^{-1} : X^* \to X$ exists and is continuous. Suppose that $u_n \rightharpoonup u \in X$. Then, $u_n \to u$ by the fact that X is a finite dimensional space. Thus, in virtue of the assumptions that $f \in C([1,T]_{\mathbb{Z}} \times \mathbb{R}, \mathbb{R})$ and $g \in C(\mathbb{R}, \mathbb{R})$, we have $J'(u_n) \to J'(u)$ and $\Psi'(u_n) \to \Psi'(u)$, i.e., J' and Ψ' are strongly continuous. Thus, from [19, Proposition 26.2], $J'$and Ψ' are compact operators. Let $u_0 \equiv 0 \in X$. Then, Φ has a strict local minimum u_0 and $\Phi(u_0) = J(u_0) = 0$.

Now, to apply Lemma 2.2, the only condition we need to verify is that $\alpha < \beta$, where α and β are defined by (2.13) and (2.14), respectively. To this end, for any $\varepsilon > 0$, from (H1), there exist $0 < c_1 < c_2 < \infty$ such that

$$F(k,y) \le (\rho+\varepsilon)|y|^2 \quad \text{for all } k \in [1,N]_{\mathbb{Z}} \text{ and } |y| \in [0,c_1) \cup (c_2,\infty). \tag{2.15}$$

Then, from the definition of J in (2.2), we have

$$J(u) \le (\rho+\varepsilon)\sum_{k=1}^{N}|u(k)|^2 = (\rho+\varepsilon)\|u\|^2 \quad \text{for all } u \in X \text{ with } \|u\| < c_1.$$

Hence, from (2.8), it follows that

$$\limsup_{u\to u_0}\frac{J(u)}{\Phi(u)} \le \frac{2(\rho+\varepsilon)}{\lambda_1}. \tag{2.16}$$

In view of the continuity of f and (2.15), there exists $c_3 > 0$ large enough so that

$$F(k,y) \le (\rho+\varepsilon)|y|^2 + c_3 \quad \text{for all } k \in [1,N]_{\mathbb{Z}} \text{ and } y \in \mathbb{R}.$$

Then,

$$J(u) \le (\rho+\varepsilon)\sum_{k=1}^{N}|u(k)|^2 + c_3 = (\rho+\varepsilon)\|u\|^2 + c_3 \quad \text{for all } u \in X.$$

Thus, in virtue of (2.8), we have

$$\limsup_{\|u\|\to\infty}\frac{J(u)}{\Phi(u)} \le \frac{2(\rho+\varepsilon)}{\lambda_1}. \tag{2.17}$$

From (2.13), (2.16), (2.17), and the arbitrariness of $\varepsilon > 0$, we reach that (by letting $\varepsilon \to 0$)

$$\alpha = \max\left\{0,\ \limsup_{\|u\|\to\infty}\frac{J(u)}{\Phi(u)},\ \limsup_{u\to u_0}\frac{J(u)}{\Phi(u)}\right\} \le \frac{2\rho}{\lambda_1}. \tag{2.18}$$

By (2.2), (2.8), and (2.14), we derive that

$$\beta = \sup_{u\in\Phi^{-1}(0,\infty)} \frac{J(u)}{\Phi(u)} \geq \frac{2}{\lambda_{N+1}} \sup_{u\in\Phi^{-1}(0,\infty)} \frac{\sum_{k=1}^{N} F(k,u(k))}{\|u\|^2}. \tag{2.19}$$

Then, from (H2), (2.18), and (2.19), we have $\alpha < \beta$. Hence, we have verified all the assumptions of Lemma 2.2. The conclusion then follows from Lemma 2.2. This completes the proof of the theorem. □

Finally, the proofs of Corollaries 2.1–2.4 are straightforward and hence are omitted.

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