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APPROXIMATE CONTROLLABILITY FOR SOME INTEGRODIFFERENTIAL MEASURE DRIVEN SYSTEMS WITH NONLOCAL CONDITIONS

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This paper investigates approximate controllability of semilinear measure driven equations in Hilbert spaces. We focus on a specific category of nonlocal integrodifferential equations. We apply the theory of the resolvent operator in the sense of Grimmer, as well as the fixed point strategy and the theory of the Lebesgue-Stieljes integral, in the context of the space of regulated functions. In light of this, the prevalence of our findings is greater than that which is found in the literature. At last, an example is comprised that exhibits the significance of developed theory.

1. Introduction

Controllability was first assessed by Kalman et al. 1962 [19] as one of the key properties that determine system behavior. It has attracted the attention of many mathematicians and engineers as it plays a key role in control theory, physics, and engineering and has very important applications in these fields. In recent years, the concept of controllability has become widespread in many areas of science and technology. For more details on this topic we refer to [1, 2, 4] and the references therein.

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The ideas of exact and approximate controllability are the most important for understanding differential equations with infinite dimensions. Exact controllability makes it possible to control the system from an initial state to any final state. To solve the problem of exact controllability, the induced inverse of the control operator must be present in the corresponding space. On the other hand, Triggiani[30] showed that the control operator has no inverse if the C_0 semigroup is compact in an infinite dimensional space. Therefore, the concept of precise controllability has gained ground. As a result, researchers are growing interested in the study of approximately controllable systems. When we say that a system is approximately controllable, we mean that there is a control that can bring it from a certain initial state to a small range that comes relatively close to the final state in a limited time. Approximately controllable systems are more widespread, and approximate controllability is often sufficient in applications. Hence it is important to examine this concept. Recently the authors discussed the approximate controllability of the nonlinear evolution of systems under different conditions. Some references for the approximate controllability of the nonlinear evolution of systems are [11, 20].

On the other hand, the theory of impulsive differential equations is mature (see [5, 23] and the references contained therein). However, these systems only allow several discontinuities within a limited area. As a result, there may be some complex phenomena, such as the behavior of Zenos that the Impulsive differential equations (IDE) does not simulate. The dynamical system with discontinuous trajectory is modeled by a measure differential system or measure driven system, which is an ordinary differential equation (ODEs) with impulsive inputs, see [6] where the MDEs are used to describe a system where the control input is impulsive, see also [8, 22, 26] for more about measure differential systems (MDSs)).

Nonlocal initial conditions initiated by Byszewski [7] have a better effect on physical problems than the classical initial condition $u(0) = u_0$. For example, Deng et al.[13] have used the nonlocal conditions $u(0) = \sum_{k=1}^{n} c_k u(t_k)$ to obtain better results about the diffusion phenomenon of a small amount of gas in a transparent tube. That is, using the initial condition $u(0) = \sum_{k=1}^{m} c_k u(t_k)$ allow additional measurements at $t_k, k = 1, 2, ..., m$ which is more precise than the measurement at $t = 0$ only.

Haide Gou et al. [17] investigated existence and approximate controllability of semilinear measure driven systems with nonlocal conditions. Cao et al.[10] studied complete controllability of semilinear measure driven differential systems with nonlocal conditions of the form

$$\begin{cases} \mathrm{d}v(t) = Av(t) + Cu(t)\mathrm{d}t + f(t, v(t))\mathrm{d}\lambda(t), \ t \in J = [0, b], \\ v(0) + g(v) = v_0. \end{cases}$$

They did it without assuming the compactness of the evolution system related to the linear part of the measure system, some sufficient conditions for controllability are then established by using the measure of noncompactness and the Mönch fixed point theorem.

There are no available results on the approximate controllability of measure driven evolution systems governed by a nonlocal integrodifferential equation of the form (1). This paper will fill the gap by constructing controls over the conjugate problems and discussing the approximate controllability of abstract semilinear measure driven evolution of integrodifferential equations. Specifically, the paper will focus on the evolution of the integrodifferential equations.

To the best of our knowledge, most of the works are devoted to study the approximate controllability of evolution systems with the semigroup approach, There have not been any results concerning the approximate controllability of measure driven evolution systems of nonlocal integrodifferential equation in the form (1). This paper will fill the gap and discuss approximate controllability of abstract semilinear measure driven evolution of integrodifferential equations by constructing the controls via the conjugate problems. Since the impulses depending on the measure λ in measure-driven integrodifferential equations are intrinsic and possibly not prefixed, we must study controllability in the space of regulated functions, which is different from the space of continuous functions and piecewise continuous functions. Haide Gou and Yongxiang Li [17] investigated the existence and approximate controllability of semilinear measure-driven systems in nonlocal circumstances.

Motivated by the above considerations, in this work we investigate the existence of mild solutions and approximate controllability for a measure driven integrodifferential evolution system of with nonlocal conditions of the following form

$$\begin{cases} \mathrm{d}v(t) = \left[Av(t) + \int_0^t \Gamma(t-s)v(s)\mathrm{d}s + \mathcal{C}u(t)\right]\mathrm{d}t + \Delta_1(t, v(t))\mathrm{d}\lambda(t), \ t \in J = [0, b], \\ v(0) = v_0 + g(v), \end{cases} \tag{1}$$

where $J = [0, b]$ with $b > 0$. The state $v(\cdot)$ takes values in a Hilbert space Σ. The operator $A : D(A) \subseteq \Sigma \to \Sigma$ is the infinitesimal generator of a strongly continuous semigroup $T(t)$, $t \geq 0$; $(\Gamma(t))_{t\geq 0}$ is closed linear operators on Σ with domain $D(\Gamma) \supset D(A)$ which is independent of t. $u(\cdot) \in L^2(J, \mathcal{V})$ is the control variable, $\mathcal{V}$ is another Hilbert space; $\mathcal{C} : \mathcal{V} \to \Sigma$ is a continuous linear operator; $\Delta_1 : J \times \Sigma \to \Sigma$ and $g : \mathcal{R}_f(J, \Sigma) \to \Sigma$ are suitable functions; λ is continuous from the left and nondecreasing and has the distributional derivative $d\lambda$.

Further down the page, the most important findings of this work are given:

• A new set of sufficient conditions has been constructed to ensure the existence of a solution and the approximate controllability of the system (1).

• The fact that MDEs are not continuous like ODEs presents the primary challenge associated with dealing with them.

• To obtain the results, both resolvent operator theory in the sense of Grimmer and fixed point techniques were used.

• To illustrate our results, we also constructed an example.

The rest of this work is organized as follows. Section 2 introduces some notions and recalls some basic known results about Lebesgue-Stielje's integral and regulated functions. Section 3 discusses mild solutions for the system (1). Section 4 presents an approximate controllability result for our system (1). Section 5 gives an example to illustrate the feasibility of our obtained results.

2. Preliminaries

This section introduces some notations, definitions, and preliminary results used in this paper. Let X and $\mathcal{V}$ be two real Banach spaces with the norms $||\cdot||$ and $||\cdot||_{\mathcal{V}}$, respectively. $J = [0, b]$ is a closed interval of the real line. We go further by considering regulated functions.

2.1. Regulated and equiregulated functions

In this part, we recall some notions and definitions about regulated function.

Definition 2.1. [24] A function $f : J \to X$ is called regulated on J, if its limits left and right given by

$$\lim_{s\to t^-} f(s) = f(t^-),\ t \in (0, b] \ \text{ and } \ \lim_{s\to t^+} f(s) = f(t^+),\ t \in [0, b)$$

exist and are finite.

We denote the space of all regulated functions $f : J \to X$ by $\mathcal{R}_f(J,X)$. It is well known that the set of discontinuities of a regulated function is at most countable and that the space $\mathcal{R}_f(J,X)$ is a Banach space endowed with the norm $||f||_\infty = \sup_{t\in J} ||f(t)||$ (see [28]) and by $\mathcal{L}(X)$ the Banach space of all linear bounded operators on X endowed with the topology defined by the operator norm. Let $L^2(J,\mathcal{V})$ be the Banach space of all $\mathcal{V}-$valued Bochner square integrable functions defined on J endowed of the norm

$$||v||_2 = \left(\int_0^b ||v(t)||^2_{\mathcal{V}} \mathrm{d}t \right)^{1/2}, v \in L^2(J,\mathcal{V}).$$

Denote by $\mathcal{LS}_\lambda(J,X)$ the space of all functions $f_1 : J \to X$ that are Lebesgue-Stieltjes integrable with respect to λ. Let τ_λ be the Lebesgue-Stieltjes measure on J induced by λ.

Lemma 2.2. *[29] Let the functions $f : J \to X$ and $\lambda : J \to \mathbb{R}$ be such that λ is regulated and $\int_0^b f \mathrm{d}\lambda$ exists. Then for every $t_0 \in [0,b]$, the function*

$$k(t) = \int_{t_0}^t f \mathrm{d}\lambda, t \in [0,b],$$

is regulated and satisfies

$$k(t^+) = k(t) + f(t)\delta^+\lambda(t), t \in [0,b),$$
$$k(t^-) = k(t) - f(t)\delta^-\lambda(t), t \in (0,b],$$

where $\delta^+\lambda(t) = \lambda(t^+) - \lambda(t)$ and $\delta^-\lambda(t) = \lambda(t) - \lambda(t^-)$.

Definition 2.3. [24] A set $\mathcal{P} \subset \mathcal{R}_f(J,X)$ is called equiregulated, if for every $\varepsilon > 0$ and $t_0 \in J$, there is a $\sigma > 0$ such that :

(i) if $x \in \mathcal{P}, t \in J$ and $t_0 - \sigma < t < t_0$, then $\|x(t_0^-) - x(t)\| < \varepsilon$.

(ii) if $x \in \mathcal{P}, t \in J$ and $t_0 < t < t_0 + \sigma$, then $||x(t) - x(t_0^+)|| < \varepsilon$.

Lemma 2.4. *[24] Let $\{v_n\}_{n=1}^\infty$ be a sequence of functions from J to X. If v_n converges pointwisely to v_0 as $n \to \infty$ and the sequence $\{v_n\}_{n=1}^\infty$ is equiregulated, then v_n converges uniformly to v_0.*

Lemma 2.5. *[24] Let X be a Banach space. Assume that $\mathcal{P} \subset \mathcal{R}_f(J,X)$ is equiregulated and, for every $t \in J$, the set $\{v(t) : v \in \mathcal{P}\}$ is relatively compact in X. Then the set $\mathcal{P}$ is relatively compact in $\mathcal{R}_f(J,X)$.*

Lemma 2.6. *[15, Corollary 2.6] Let X be the Banach space and $L^1_\kappa(J,X)$ be the set of all integrable functions from J to X with respect to the measure κ. Assume that $\mathcal{W} \subset L^1_\kappa(J,X)$ is a bounded set and that there exists a positive function $q \in L^1_\kappa(J,\mathbb{R})$ such that $||w(t)|| \le q(t)$ $\kappa-$ a.e. $t \in J$ for all $w \in \mathcal{W}$. If for $w \in \mathcal{W}$, one has $w(t) \in G(t)$ for $\kappa-$ a.e. $t \in J$ where, for $t \in J$, $G(t) \subset X$ is weakly relatively compact, then $\mathcal{W}$ is weakly relatively compact in $L^1_\kappa(J,X)$.*

2.2. Integrodifferential equations in Banach spaces

In this part, we introduce some basic notions about resolvent operators that will be used to develop the main results of this work.
Let $\mathcal{X}_1$ be a Banach space, A and $\Gamma(t)$ are closed linear operators on $\mathcal{X}_1$. $\mathcal{X}_2$ represents the Banach space $D(A)$ equipped with the graph norm defined by

$$|y|_{\mathcal{X}_2} := |Ay| + |y| \text{ for } y \in \mathcal{X}_2.$$

The notations $C([0,+\infty);\mathcal{X}_2), B(\mathcal{X}_2,\mathcal{X}_1)$ stand for the space of all continuous functions from $[0,+\infty)$ into $\mathcal{X}_2$, the set of all bounded linear operators from $\mathcal{X}_2$ into $\mathcal{X}_1$, respectively. We consider the following Cauchy problem

$$\begin{cases} v'(t) = Av(t) + \int_0^t \Gamma(t-s)v(s)\mathrm{d}s \text{ for } t \geq 0, \\ v(0) = v_0 \in \mathcal{X}_1, \end{cases} \tag{2}$$

where A and Γ are closed linear operator on a Banach space $\mathcal{X}_1$.

Definition 2.7. [18] A resolvent for Eq. (2) is a bounded linear operator valued function $R(t) \in B(\mathcal{X}_1)$ for $t \geq 0$, having the following properties :

(a) $R(0) = I$ and $||R(t)|| \leq Me^{\beta t}$ for some constants $M > 0$ and $\beta \in \mathbb{R}$.

(b) For each $v \in \mathcal{X}_1, R(t)v$ is continuous for $t \geq 0$.

(c) $R(t) \in B(\mathcal{X}_2)$ for $v \in \mathcal{X}_2, R(\cdot)v \in C^1([0,+\infty);\mathcal{X}_1) \cap C([0;+\infty);\mathcal{X}_2)$ and

$$\begin{aligned} R'(t)v &= AR(t)v + \int_0^t \Gamma(t-s)R(s)v\mathrm{d}s \\ &= R(t)Av + \int_0^t R(t-s)\Gamma(s)v\mathrm{d}s, \ t \geq 0, \end{aligned}$$

The existence of a resolvent operator has been discussed in [18]. In what follows, we suppose the following assumptions:

(**H1**) A is the infinitesimal generator of a c_0-semigroup $(T(t))_{t\geq 0}$ on $\mathcal{X}_1$.

(**H2**) For all $t \geq 0$, $\Gamma(t)$ is a closed linear operator from $D(A)$ to $\mathcal{X}_1$, and $\Gamma(t) \in B(\mathcal{X}_2,\mathcal{X}_1)$. For any $w \in \mathcal{X}_2$, the map $t \to \Gamma(t)w$ is bounded, differentiable and the derivative $t \to \Gamma(t)'w$ is bounded uniformly continuous on $\mathbb{R}^+$.

The resolvent operator for Eq. (2) exists under the following theorem.

Theorem 2.8. *[18] Assume that* (***H1***) $-$ (***H2***) *hold. Then, there exists a unique resolvent operator for the Cauchy problem* (1).

In the following, we give some results for the existence of solutions for the following integrodifferential equation:

$$\begin{cases} v'(t) = Av(t) + \int_0^t \Gamma(t-s)v(s)\mathrm{d}s + q(t) \quad for\, t \geq 0 \\ v(0) = v_0 \in \mathcal{X}_1, \end{cases} \tag{3}$$

where $q : \mathbb{R}^+ \to \mathcal{X}_1$ is a continuous function.

Definition 2.9. [18] A continuous function $v : \mathbb{R}^+ \to \mathcal{X}_1$ is a strict solution of equation (3) if :

1. $v \in C^1(\mathbb{R}^+; \mathcal{X}_1) \bigcap C(\mathbb{R}^+; \mathcal{X}_2)$ and
2. v satisfies equation (3).

Theorem 2.10. *[18] Assume that* ***(H1)****-****(H2)*** *hold. If v is a strict solution of equation* (3)*, then*

$$v(t) = R(t)v_0 + \int_0^t R(t-s)q(s)\mathrm{d}s \quad for\, t \geq 0.$$

Lemma 2.11. *[14] Assume that* ***(H1)****-****(H2)*** *hold. Then, the resolvent operator $R(t))_{t\geq 0}$ is compact for $t > 0$ if and only if the semigroup $(T(t))_{t\geq 0}$ is compact for $t > 0$.*

Lemma 2.12. *[21] Assume that* ***(H1)****-****(H2)*** *hold. If the resolvent operator $(R(t))_{t\geq 0}$ is compact for, $t > 0$ then it is norm continuous (or continuous in the uniform operator topology) for $t > 0$.*

Lemma 2.13. *[21] For any $b > 0$, there exists a constant $\gamma = \gamma(b)$ such that*

$$||R(t+\varepsilon) - R(\varepsilon)R(t)||_{\mathcal{X}_1} \leq \gamma\varepsilon \quad for\, 0 \leq \varepsilon \leq t \leq b.$$

Therefore, we have the following definitions

Definition 2.14. A function $v \in R_f(J, \Sigma)$ is called a mild solution of the system (1) on J if for any $u \in L^2(J, \mathcal{V})$ the following measure integral equation

$$v(t) = R(t)[v_0 + g(v)] + \int_0^t R(t-s)\mathcal{C}u(s)\mathrm{d}s + \int_0^t R(t-s)\Delta_1(s, v(s))\mathrm{d}\lambda(s),\ t \in J.$$

is satisfied.

Definition 2.15. Let $v(t, \Delta_1, u)$ be a mild solution of the system (1) associated with nonlinear term Δ_1 and control $u \in L^2(J, \mathcal{V})$ at the time t. Then

$$N_b(\Delta_1) = \left\{ v(b, \Delta_1, u) : u \in L^2(J, \mathcal{V}) \right\},$$

is a nonempty subset of Σ consisting of all terminal states of (1) called the reachable set of the system (1) at the time b.

Definition 2.16. The system (1) is said to be approximately controllable on the interval J if $N_b(\Delta_1)$ is dense in Σ, means $\overline{N_b(\Delta_1)} = \Sigma$. That is, for any $\varepsilon > 0$ and every desired final state $v_b \in \Sigma$, there exists a control $u \in L^2(J, \mathcal{V})$ such that v satisfies $\|v(b) - v_b\| < \varepsilon$.

To discuss the approximate controllability of system (1) we introduce the following operators.

1. The controllability Grammian Ξ_0^b is defined by :

$$\Xi_0^b = \int_0^b R(b-s)\mathcal{C}\mathcal{C}^* R^*(b-s)\mathrm{d}s,$$

   where $\mathcal{C}^*$ and $R^*(t)$ denote the adjoint operators of $\mathcal{C}$ and $R(t)$, respectively.

2. $S(\kappa, \Xi_0^b) = (\kappa I + \Xi_0^b)^{-1}$.

In the sequel we assume that the operator $S(\kappa, \Xi_0^b)$ satisfies

(A0) $\kappa S(\kappa, \Xi_0^b) \to 0$ as $\kappa \to 0^+$ in the strong operator topology.

From [3], the above condition **(A0)** is equivalent to the approximate controllability of the linear system

$$\begin{cases} v'(t) = Av(t) + \displaystyle\int_0^t \Gamma(t-s)v(s)\mathrm{d}s + \mathcal{C}u(t), \ t \in J, \\ v(0) = v_0 + g(v), \end{cases} \tag{4}$$

In fact, we have that

Theorem 2.17. *[3, 12] The following statements are equivalent:*

(i) The control system (1) is approximately controllable on $[0,b]$.

(ii) $\mathcal{C}^* R^*(t)v = 0$ *for all* $t \in [0,b]$ *imply* $v = 0$.

*(iii) The condition (**A0**) holds.*

Now, we give the following lemma and the Schauder fixed-point theorem appeared in [31].

Lemma 2.18. *[16, Lemma 12] Let Z be a Banach space and $(T_n)_{n\geq 1}$ be a sequence of bounded linear maps on $\mathcal{B}$ converging pointwisely to $T \in \mathcal{B}(Z)$. Then for any compact set $\mathbb{K}$ in Z, T_n converges to T uniformly in $\mathbb{K}$, namely,*

$$\sup_{x \in \mathbb{K}} \|T_n(x) - T(x)\| \to 0, \ as \ n \to +\infty.$$

Theorem 2.19. *[31, Theorem 2.A] Let $\mathcal{B}$ be a Banach space and let $\mathcal{D} \subseteq \mathcal{B}$ be a bounded, closed and convex set. If the operator $\mathcal{P} : \mathcal{D} \to \mathcal{D}$ is completely continuous, then $\mathcal{P}$ has a fixed point in $\mathcal{D}$.*

3. Existence of mild solutions of system (1)

In this section, we will present and prove the existence of mild solutions of the system (1) using Schauder's fixed point Theorem. We first make the following assumptions:

(A1) $T(t),\ t > 0$ is compact operator on Σ.

(A2) Let τ_λ be the Lebesgue-Stieljes measure on J induced by λ.

(i) For $\tau_\lambda-$ a.e. $t \in J$, the function $\Delta_1(t,\cdot) : \Sigma \to \Sigma$ is continuous and for all $v \in \Sigma$ and $\Delta_1(\cdot, v) : J \to \Sigma$ is $\tau_\lambda-$ measurable.

(ii) There exist a function $L_{\Delta_1} \in \mathcal{LS}_\lambda(J,\mathbb{R}_+)$ and a non-decreasing continuous function $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$such that

$$||\Delta_1(t,v)|| \le L_{\Delta_1}(t)\Psi(||v||) \text{ for } v \in \Sigma,\ \ \tau_\lambda \ a.e.\ t \in J$$
$$\text{and } \lim_{y\to+\infty} \inf \frac{\Psi(y)}{y} = 0.$$

(A3) The function $g : \mathcal{R}_f(J,\Sigma) \to \Sigma$ is continuous, compact and satisfies

$$\lim_{y\to+\infty} \inf \frac{g_y}{y} = 0$$

where $g_y = \sup\{||g(v)|| \ : \ ||v|| \le y\}$.

Let $M_b = \sup_{0\le t\le b} \|R(t)\|$ and $M_c = \|\mathcal{C}\|$.

Theorem 3.1. *Assume that the hypotheses **(H1)**, **(H2)**, **(A0)**-**(A3)** are satisfied, then the nonlocal system (1) possesses at least one mild solution on J.*

Proof. For any $\kappa > 0$ and $v_b \in \Sigma$, we define the control function $u_\kappa(t,v)$ as follows

$$u_\kappa(t,v)$$
$$= \mathcal{C}^*R^*(b-t)S(\kappa,\Xi_0^b)\Big(v_b - R(b)[v_0 + g(v)] - \int_0^b R(b-s)\Delta_1(s,v(s))\mathrm{d}\lambda(s)\Big),$$

and define the operator $G_\kappa : \mathcal{R}_f(J,\Sigma) \to \mathcal{R}_f(J,\Sigma)$ by

$$(G_\kappa v)(t) = R(t)[v_0 + g(v)] + \int_0^t R(t-s)\mathcal{C}u_\kappa(s,v)\mathrm{d}s + \int_0^t R(t-s)\Delta_1(s,v(s))\mathrm{d}\lambda(s),\ t \in J.$$

As a result of the hypotheses **(A1)** and **(A2)**, the integrals in the above formula are well defined. Next it will be shown that for all $\kappa > 0$, the operator $G_\kappa v$ has a fixed point and the proof will be divided into several steps.

For any $\omega_0 > 0$, we define the following set

$$B_{\omega_0} := \left\{ v \in \mathcal{R}_f(J,\Sigma) : ||v||_\infty \le \omega_0 \right\}.$$

B_{ω_0} is clearly a bounded closed convex set in $\mathcal{R}_f(J,\Sigma)$. Write $G_\kappa(B_{\omega_0}) = \{G_\kappa v : v(\cdot) \in B_{\omega_0}\}$. The proof will be given in four steps.

Step 1. We claim that there exists $\omega_0 > 0$ such that $G_\kappa(B_{\omega_0}) \subseteq B_{\omega_0}$. Suppose on the contrary that this is not true. Then for each positive number ω_0, there exist a function $v_{\omega_0} \in B_{\omega_0}$ such that $G_\kappa(v_{\omega_0}) \notin B_{\omega_0}$, i.e., $||(G_\kappa v_{\omega_0})(\tau)|| > \omega_0$, for some $\tau = \tau(\omega_0) \in J$. Now

$$\frac{||(G_\kappa v_{\omega_0})(\tau)||}{\omega_0} > 1 \text{ implies that } \lim_{\omega_0 \to +\infty} \inf \frac{||(G_\kappa v_{\omega_0})(\tau)||}{\omega_0} \ge 1. \tag{5}$$

Note that

$$\begin{aligned}
&\|u_\kappa(t,v)\| \\
&= \|\mathcal{C}^* R^*(b-t)S(\kappa,\Xi_0^b)\Big[v_b - R(b)[v_0+g(v)] - \int_0^b R(b-s)\Delta_1(s,v(s))\mathrm{d}\lambda(s)\Big]\| \\
&\le M_c M_b \frac{1}{\kappa}\Big[\|v_b\| + M_b(\|v_0\| + g_{\omega_0}) + M_b \Psi(\omega_0)\int_0^b L_{\Delta_1}(s)\mathrm{d}\lambda(s)\Big].
\end{aligned}$$

Here, we have supposed w.l.o.g by condition (**A0**) that there exists a positive constant $\delta > 0$ such that $\|S(\kappa,\Xi_0^b)\| \le \frac{1}{\kappa}$, for all $\kappa \in (0,\delta)$. Then we have

$$\begin{aligned}
||(G_\kappa v_{\omega_0})(t)|| &\le ||R(t)[v_0 + g(v_{\omega_0})]|| + ||\int_0^t R(t-s)\mathcal{C}u_\kappa(s,v_{\omega_0})\mathrm{d}s|| \\
&\quad + ||\int_0^t R(t-s)\Delta_1(s,v_{\omega_0}(s))\mathrm{d}\lambda(s)|| \\
&\le M_b||v_0|| + M_b||g(v_{\omega_0})|| + M_b M_c \int_0^b ||u_\kappa(s,v_{\omega_0})||\mathrm{d}s \\
&\quad + M_b \int_0^b ||\Delta_1(s,v_{\omega_0}(s))||\mathrm{d}\lambda(s)
\end{aligned}$$

$$
\begin{aligned}
&\leq M_b||v_0|| + M_b g_{\omega_0} + M_b^2 M_c^2 \frac{b}{\kappa}\Big[\|v_b\| + M_b(\|v_0\| + g_{\omega_0}) \\
&+ M_b \Psi(\omega_0)\int_0^b L_{\Delta_1}(s)\mathrm{d}\lambda(s)\Big] + M_b\Psi(\omega_0)\int_0^b L_{\Delta_1}(s)\mathrm{d}\lambda(s) \\
&\leq \delta_{\omega_0} \;:=\; M_b^2 M_c^2 \frac{b}{\kappa}\|v_b\| \\
&+ \Big[M_b + M_b^3 M_c^2 \frac{b}{\kappa}\Big]\Big[\|v_0\| + g_{\omega_0} + \int_0^b L_{\Delta_1}(s)\mathrm{d}\lambda(s)\Psi(\omega_0)\Big].
\end{aligned}
$$

It follows now that

$$\lim_{\omega_0 \to +\infty} \inf \frac{||(G_\kappa v_{\omega_0})(\tau)||}{\omega_0} = 0$$

since

$$\lim_{\omega_0 \to +\infty} \inf \frac{\delta_{\omega_0}}{\omega_0} = 0 = \lim_{\omega_0 \to +\infty} \inf \frac{g_{\omega_0}}{\omega_0}.$$

This is clearly a contradiction to (5). Consequently, there exists $\omega_0 > 0$ such that $G_\kappa(B_{\omega_0}) \subset B_{\omega_0}$.
Step 2. $G_\kappa(B_{\omega_0})$ is an equiregulated family of functions on J.
For $t_0 \in [0,b)$, we have

$$
\begin{aligned}
&||(G_\kappa v)(t) - (G_\kappa v)(t_0^+)|| \\
&\leq ||R(t)v_0 - R(t_0)v_0|| + ||R(t)g(v) - R(t_0^+)g(v)|| \\
&+ ||\int_0^t R(t-s)\mathcal{C}u_\kappa(s,v)\mathrm{d}s - \int_0^{t_0^+} R(t_0^+ - s)\mathcal{C}u_\kappa(s,v)\mathrm{d}s|| \\
&+ ||\int_0^t R(t-s)\Delta_1(s,v(s))\mathrm{d}\lambda(s) - \int_0^{t_0^+} R(t_0^+ - s)\Delta_1(s,v(s))\mathrm{d}\lambda(s)|| \\
&:= I_1 + I_2 + I_3 + I_4,
\end{aligned}
$$

where,

$$
\begin{aligned}
&I_1 = ||R(t)v_0 - R(t_0^+)v_0||, \\
&I_2 = ||R(t)g(v) - R(t_0^+)g(v)||, \\
&I_3 = ||\int_0^t R(t-s)\mathcal{C}u_\kappa(s,v)\mathrm{d}s - \int_0^{t_0^+} R(t_0^+ - s)\mathcal{C}u_\kappa(s,v)\mathrm{d}s||, \\
&I_4 = ||\int_0^t R(t-s)\Delta_1(s,v(s))\mathrm{d}\lambda(s) - \int_0^{t_0^+} R(t_0^+ - s)\Delta_1(s,v(s))\mathrm{d}\lambda(s)||.
\end{aligned}
$$

Therefore, we need to show that I_i tends to 0 independently of $v \in B_{\omega_0}$ when $t \to t_0, i = 1,2,3,4$.

Firstly for $t_0>0$. By *(b)* of definition 2.7, I_1 tends to 0 as $|t\to t_0^+|\to\ 0$. That is

$$||R(t)v_0-R(t_0^+)v_0||\to 0\ \text{ as }\ |t\to t_0^+|\to\ 0.$$

Furthermore we have that

$$\begin{aligned}I_2&\leq\|R(t)-R(t_0^+)\|\|g(v)\|\\&\leq\|R(t)-R(t_0^+)\|g_{\omega_0},\end{aligned}$$

where $g_{\omega_0}=\sup\{\|g(v)\|:\|v\|\leq\omega_0\}$. And so $\|R(t)-R(t_0)\|\to 0$ as $|t\to t_0^+|\to 0$, by the continuity of $(R(t))_{t\geq 0}$ for $t>0$ in the operator-norm topology which shows that I_2 tends to 0 as $|t\to t_0^+|\to\ 0$

Now for $t_0^+=0$, we have that

$\|R(t)g(v)-g(v)\|\leq\sup_{\omega\in\overline{g(B_{\omega_0})}}\|R(t)\omega-\omega\|\to 0,\text{as}\,\omega\to 0^+$, by Lemma 2.18 since $\overline{g(B_{\omega_0})}$ is compact. Therefore I_2 tends to 0 as $|t\to t_0^+|\to\ 0$.
For I_3 we have that

$$\begin{aligned}I_3\leq{}& M_c\int_{t_0^+}^{t}\|u_\kappa(s,v)\|\mathrm{d}s\\&+\int_0^{t_0^+}||R(t-s)-R(t_0^+-s)||||u_\kappa(s,v)\|\mathrm{d}s\\&\leq\Big[M_cM_b\frac{1}{\kappa}\Big[\|v_b\|+M_b(\|v_0\|+g_{\omega_0})+M_b\Psi(\omega_0)\int_0^b L_{\Delta_1}(s)\mathrm{d}\lambda(s)\Big]\Big]\\&\Big[M_c(t-t_0^+)+\int_0^{t_0^+}||R(t-s)-R(t_0^+-s)||\mathrm{d}s\Big]\\&\to 0\ as\ t\to t_0^+.\end{aligned}$$

Also we have that

$$\begin{aligned}I_4\leq{}&\int_{t_0^+}^{t}||R(t-s)||||\Delta_1(s,v(s))||\mathrm{d}\lambda(s)\\&+\int_0^{t_0^+}||R(t-s)-R(t_0^+-s)||||\Delta_1(s,v(s))||\mathrm{d}\lambda(s)\\&\leq M_b\Psi(\omega_0)\int_{t_0^+}^{t}L_{\Delta_1}(s)\mathrm{d}\lambda(s)\\&+\Psi(\omega_0)\int_0^{t_0^+}||R(t-s)-R(t_0^+-s)||||L_{\Delta_1}(s)||\mathrm{d}\lambda(s)\\&:=I_{41}+I_{42}.\end{aligned}$$

Let $\mathcal{K}(t)=\int_0^t L_{\Delta_1}(s)\mathrm{d}\lambda(s)$, from the Lemma 2.2, $\mathcal{K}(t)$ is a regulated function on J. Thus,

$$I_{41}\le M_b\Psi(\omega_0)\int_{t_0^+}^t L_{\Delta_1}(s)\mathrm{d}\lambda(s)=M_b\Psi(\omega_0)(\mathcal{K}(t)-\mathcal{K}(t_0^+))\to 0\ as\ t\to t_0^+,$$

also independently of $v(\cdot)$. By the Lebesgue dominated convergence theorem we can see that $I_{42}\to 0$ as $t\to t_0^+$.
A similar method can be used to show $||(G_\kappa v)(t)-(G_\kappa v)(t_0^-)||\to 0$ as $t\to t_0^-$ for each $t_0\in(0,b]$. Therefore, $Q(B_{\omega_0})$ is equiregulated on J from Definition 2.3.
Step 3. The operator $G_\kappa : B_{\omega_0}\to B_{\omega_0}$ is continuous. To this end, let $\{v_n\}_{n=1}^\infty\subset B_{\omega_0}\subset R_f(J,\Sigma)$ be a sequence such that $\lim_{n\to\infty}v_n=v$ in B_{ω_0}. By the hypothesis **(A2)**, we have

$$\lim_{n\to+\infty}\Delta_1(t,v_n(t))=\Delta_1(t,v(t)),\ \text{for}\ \forall t\in J.$$

Further note that

$$||\Delta_1(t,v_n(t))-\Delta_1(t,v(t))||\le 2\Psi(\omega_0)L_{\Delta_1}(t),\ \tau_\lambda-a.e.t\in J,$$

$$\begin{aligned}&||u_\kappa(t,v_n(t))-u_\kappa(t,v(t))||\\&\le M_cM_b^2\frac{2}{\kappa}\Big[||g(v_n)-g(v)]||+\int_0^b||\Delta_1(t,v_n(t))-\Delta_1(t,v(t))||\mathrm{d}\lambda(s)\Big],\ s\in J.\end{aligned}$$

Using the fact the function $t\to L_{\Delta_1}(t)$ is Lebesgue-Stieljes integrable with respect to λ and by the dominated convergence Theorem, we have

$$\begin{aligned}||(G_\kappa v_n)(t)-(G_\kappa v)(t)||\le{}&||R(t)[g(v_n)-g(v)]||\\&+\int_0^t||R(t-s)||||u_\kappa(s,v_n(s))-u_\kappa(s,v(s))||\mathrm{d}s\\&+\int_0^t||R(t-s)||.||\Delta_1(s,v_n(s))-\Delta_1(s,v(s))||\mathrm{d}\lambda(s)\\&\le M_b||g(v_n)-g(v)||+M_cM_b^2\frac{2}{\kappa}\Big[||g(v_n)-g(v)]||\Big]\\&+M_cM_b^2\frac{2}{\kappa}\Big[\int_0^b||\Delta_1(t,v_n(t))-\Delta_1(t,v(t))||\mathrm{d}\lambda(s)\Big]\\&+M_b\int_0^t||\Delta_1(s,v_n(s))-\Delta_1(s,v(s))||\mathrm{d}\lambda(s)\\&\to 0\ \text{ as }\ n\to\infty,\end{aligned}$$

In addition, the same analysis as in **Step 2** demonstrates that $\{G_\kappa v_n\}_{n=1}^{\infty}$ is equiregulated on J. This property combined with Lemma 2.4 implies that $\{G_\kappa v_n\}$ converges uniformly to $\{G_\kappa v\}$ as $n \to \infty$, namely,

$$\begin{aligned} ||(G_\kappa v_n) - (G_\kappa v)||_\infty &= \sup_{t \in J} ||(G_\kappa v_n)(t) - (G_\kappa v)(t)||_\infty \\ &= \sup_{t \in J} ||(G_\kappa v_n)(t) - (G_\kappa v)(t)|| \to 0, \text{ as } n \to \infty. \end{aligned}$$

Therefore, the operator $G_\kappa : B_{\omega_0} \to B_{\omega_0}$ is a continuous operator.
Step 4. Finally, we demonstrate that the operator $G_\kappa : B_{\omega_0} \to B_{\omega_0}$ is compact. To prove this, we show that $\{(G_\kappa v)(t) : v \in B_{\omega_0}\}$ is relatively compact in Σ, for every $t \in J$. We have that for $t = 0$, the set

$$\{(G_\kappa v)(0) : v \in B_{\omega_0}\} = \{v_0 + g(v) : v \in B_{\omega_0}\} = v_0 + g(B_{\omega_0})$$

is relatively compact in Σ. Since g is compact, it follows that $\overline{g(B_{\omega_0})}$ is compact also.
Let $t \in (0,b]$ be given, $0 < \varepsilon < t$ and $v \in B_{\omega_0}$, we define the operators

$$\begin{aligned} (G_\kappa^\varepsilon v)(t) = R(\varepsilon)[v_0 + g(v)] + R(\varepsilon) \int_0^{t-\varepsilon} R(t-\varepsilon-s)\mathcal{C}u_\kappa(s,v)\mathrm{d}s \\ + R(\varepsilon) \int_0^{t-\varepsilon} R(t-\varepsilon-s)\Delta_1(s,v(s))\mathrm{d}\lambda(s) \end{aligned}$$

and

$$\begin{aligned} &(\tilde{G}_\kappa^\varepsilon v)(t) \\ &= R(\varepsilon)[v_0 + g(v)] + \int_0^{t-\varepsilon} R(t-s)\mathcal{C}u_\kappa(s,v)\mathrm{d}s + \int_0^{t-\varepsilon} R(t-s)\Delta_1(s,v(s))\mathrm{d}\lambda(s). \end{aligned}$$

By Lemma 2.13 and the compactness of the operator $R(\varepsilon)$, the set $\Omega^\varepsilon(t) = \{(G^\varepsilon v)(t) : v \in B_{\omega_0}\}$ is relatively compact in Σ. Moreover, also by Lemma 2.13, for each $v \in B_{\omega_0}$, we obtain

$$\begin{aligned} &||(G_\kappa^\varepsilon v)(t) - (\tilde{G}_\kappa^\varepsilon)(t)|| \\ &\leq || \int_0^{t-\varepsilon} [R(\varepsilon)R(t-\varepsilon-s) - R(t-s)]\mathcal{C}u_\kappa(s,v)\mathrm{d}s|| \\ &+ || \int_0^{t-\varepsilon} [R(\varepsilon)R(t-\varepsilon-s) - R(t-s)]\Delta_1(s,v(s))\mathrm{d}\lambda(s)|| \\ &\leq (\gamma\varepsilon) \int_0^{t-\varepsilon} ||\mathcal{C}u_\kappa(s,v)||\mathrm{d}s + (\gamma\varepsilon) \int_0^{t-\varepsilon} ||\Delta_1(s,v(s))||\mathrm{d}\lambda(s) \\ &\leq \gamma\varepsilon \left(\int_0^{t-\varepsilon} ||\mathcal{C}u_\kappa(s,v)||\mathrm{d}s + \Psi(\omega_0) \int_0^{t-\varepsilon} L_{\Delta_1}(s)\mathrm{d}\lambda(s) \right) \end{aligned}$$

$$\to 0 \ as \ \varepsilon \to 0.$$

So the set $\tilde{\Omega}^{\varepsilon}(t) = \left\{(\tilde{G}^{\varepsilon}_{\kappa}v)(t) : v \in B_{\omega_0}\right\}$ is precompact in Σ by using the total boundedness.

Applying this idea again, we obtain

$$\begin{aligned}
&||(G_{\kappa}v)(t) - (\tilde{G}^{\varepsilon}_{\kappa})(t)|| \\
&\leq || \int_{t-\varepsilon}^{t} R(t-s)\mathcal{C}u_{\kappa}(s,v)\mathrm{d}s|| + || \int_{t-\varepsilon}^{t} R(t-s)\Delta_1(s,v(s))\mathrm{d}\lambda(s)|| \\
&\leq M_b \left[\int_{t-\varepsilon}^{t} ||\mathcal{C}u_{\kappa}(s,v)||\mathrm{d}s + \Psi(\omega_0) \int_{0}^{t-\varepsilon} L_{\Delta_1}(s)\mathrm{d}\lambda(s) \right] \\
&\to 0 \ as \ \varepsilon \to 0,
\end{aligned}$$

and there are precompact sets arbitrarily close to $\Omega(t) = \{(G_{\kappa}v)(t) : v \in B_{\omega_0}\}$. Thus, the set $\Omega(t) = \{(G_{\kappa}v)(t) : v \in B_{\omega_0}\}$ is precompact in Σ. Therefore, by Lemma 2.5, we see that $G_{\kappa} : B_{\omega_0} \to B_{\omega_0}$ is a compact operator.

Finally, by Schauder's fixed-point theorem, we obtain that the operator G has at least one fixed point $v \in B_{\omega_0}$, which is a mild solution of system (1) on J. This completes the proof. $\square$

4. Approximate Controllability

In this section, we establish the sufficient conditions for the approximate controllability of system (1).

Theorem 4.1. *Assume that the assumptions* ***(H1)**, **(H2)** and **(A0)-(A3)** are valid. Then the nonlocal integrodifferential system (1) is approximately controllable on J.*

Proof. For $\kappa > 0$, define the operator G_{κ} on $\mathcal{R}_f(J,\Sigma)$ as following,

$$\begin{aligned}
&(G_{\kappa}v_{\kappa})(b) \\
&= R(b)[v_0 + g(v_{\kappa})] + \int_0^b R(b-s)\mathcal{C}u_{\kappa}(s,v_{\kappa})\mathrm{d}s + \int_0^b R(b-s)\Delta_1(s,v_{\kappa}(s))\mathrm{d}\lambda(s),
\end{aligned}$$

where

$$\begin{aligned}
u_{\kappa}(t,v_{\kappa}) &= \mathcal{C}^* R^*(b-t)S(\kappa,\Xi_0^b)z(v_{\kappa}(\cdot)), \\
z(v_{\kappa}(\cdot)) &= v_b - R(b)[v_0 + g(v_{\kappa})] - \int_0^b R(b-s)\Delta_1(s,v_{\kappa}(s))\mathrm{d}\lambda(s).
\end{aligned}$$

Let $v_\kappa(.)$ be the fixed point of G_κ in B_{ω_0}. By Theorem 3.1, any fixed point of G_κ is a mild solution of the control system (1) under the control

$$u_\kappa(t,v_\kappa) = \mathcal{C}^* R^*(b-t)S(\kappa,\Xi_0^b)z(v_\kappa(\cdot)),$$

and satisfies

$$\begin{aligned}
&v_\kappa(b)\\
&= R(b)[v_0+g(v_\kappa)] + \int_0^b R(b-s)\mathcal{C}u_\kappa(s,v_\kappa)\mathrm{d}s + \int_0^b R(b-s)\Delta_1(s,v_\kappa(s))\mathrm{d}\lambda(s).
\end{aligned}$$

From definition of Ξ_0^b, it follows that

$$\begin{aligned}
&v_\kappa(b)\\
&= R(b)[v_0+g(v_\kappa)] + v_b - R(b)[v_0+g(v_\kappa)] - z(v_\kappa(\cdot)) + \Xi_0^b S(\kappa,\Xi_0^b)z(v_\kappa(\cdot))\\
&= v_b - z(v_\kappa(\cdot)) + \Xi_0^b S(\kappa,\Delta_0^b)z(v_\kappa(\cdot))\\
&= v_b - (\Delta_0^b S(\kappa,\Xi_0^b) - Id)z(v_\kappa(\cdot))\\
&= v_b - \kappa S(\kappa,\Xi_0^b)\Big[v_b - R(b)[v_0+g(v_\kappa)]\\
&- \int_0^b R(b-s)\Delta_1(s,v_\kappa(s))\mathrm{d}\lambda(s)\Big].
\end{aligned}$$

By hypothesis **(A2)** and Lemma 2.6, the set $\{\Delta_1(\cdot,v_\kappa(\cdot))\}$ is weakly relatively compact in $\mathcal{LS}_\lambda(J,\Sigma)$. Then it could be taken a sequence $\kappa_n \to 0$ and to extract a subsequence from $\{\Delta_1(\cdot,v_{\kappa_n}(\cdot))\}$, still denoted by $\{\Delta_1(\cdot,v_{\kappa_n}(\cdot))\}$, which converges weakly to some $\mathcal{M}(\cdot) \in \mathcal{LS}_\lambda(J,\Sigma)$. Let

$$\tau := v_b - R(b)[v_0+g(v_\kappa)] - \int_0^b R(b-s)\mathcal{M}(s)\mathrm{d}\lambda(s).$$

Thus, we have

$$||z(v_\kappa(\cdot)) - \tau|| \leq ||\int_0^b R(b-s)[\Delta_1(s,v_{\kappa_n}(s)) - \mathcal{M}(s)]\mathrm{d}\lambda(s)||. \tag{6}$$

From the fact that $T(t)$ is a compact operator for $t>0$, by Lemma 2.11, we get that $R(t)$ is compact for $t>0$. And similarly to the proof of compactness of the operator G_κ in Theorem 3.1, one can easily verify that the mapping

$$x(\cdot) \to \int_0^{\cdot} R(\cdot - s)x(s)\mathrm{d}\lambda(s)$$

defined from $\mathcal{LS}_\lambda(J,\Sigma)$ to $\mathcal{R}_f(J,\Sigma)$ is compact. Therefore, we have

$$\int_0^b R(b-s)[\Delta_1(s,v_{\kappa_n}(s)) - \mathcal{M}(s)]\mathrm{d}\lambda(s) \to \ 0 \text{ as } \kappa_n \to 0^+ \tag{7}$$

Thus, from (6) and (7), we have

$$||z(v_\kappa(\cdot))-\tau|| \to 0 \ as \ \kappa \to 0^+. \tag{8}$$

Finally, from (8) and the assumption **(A0)**, we have

$$\begin{aligned}||v_\kappa(b)-v_b|| &\le ||\kappa S(\kappa,\Xi_0^b)z(v_\kappa(\cdot))|| \\ &\le ||\kappa S(\kappa,\Xi_0^b)(\tau)|| + ||\kappa S(\kappa,\Xi_0^b)|| \cdot ||z(v_\kappa(\cdot))-\tau|| \\ &\to 0 \ as \ \kappa \to 0^+.\end{aligned}$$

This concludes that the control system (1) is approximately controllable on J. This completes the proof of Theorem 4.1. □

5. Application

We consider the following partial integro-differential equation of evolution type with nonlocal conditions of the form

$$\begin{cases} \frac{\partial}{\partial t}y(t,z) = \dfrac{\partial^2}{\partial z^2}y(t,z)+\alpha y(t,z)+\displaystyle\int_0^t \zeta(t-s)\frac{\partial^2}{\partial z^2}y(s,z)\mathrm{d}s \\ \qquad\qquad +\delta_1(t,y(t,z))\mathrm{d}\lambda(t)+\mathcal{C}\mu(t,z), \ t\in[0,1], z\in[0,\pi], \\ y(t,0)=y(t,\pi)=0, \\ y(0,z)=y_0(z)-\displaystyle\int_0^1 \rho(s)\log(1+|y(s,z)|)\mathrm{d}s, \ z\in[0,\pi], \end{cases} \tag{9}$$

where $\alpha>0$ is a constant, $\delta_1 : J\times\mathbb{R}\to\mathbb{R}$, $u\in L^2(J,L^2([0,\pi];\mathbb{R}))$ and $\rho\in L^2([0,1],\mathbb{R})$.
Let $\Sigma=\mathcal{V}=L^2([0,\pi];\mathbb{R})$ with the norm $||\cdot||$ and inner product $\langle\cdot,\cdot\rangle$. Consider the linear operator Q in Hilbert space Σ defined by

$$Qy := \frac{\partial^2}{\partial z^2}y, \quad y\in D(Q),$$

where

$$D(Q)=H^2([0,\pi])\cap H_0^1([0,\pi]).$$

By Lemma 2.5 in [27], the operator Q generates a compact c_0-semigroup $T(.)$ in Σ. Furthermore, Q has a discrete spectrum, and its eigenvalues are $-n^2, n\in\mathbb{N}^+$ with the corresponding normalized eigenvectors $e_n(z)=\sqrt{\frac{2}{\pi}}\sin(nz), 0\le z\le \pi, n=1,2,\cdots$
Define the operator $Ay=Qy+\alpha y$ on $L^2([0,\pi];\mathbb{R})$ with domain $D(A)=D(Q)$.

It follows from ([27] and Lemma 2.6) that the operator A generates a strongly continuous semigroup $T(t)$ defined by,

$$T(t)y = \sum_{n=1}^{\infty} e^{(-n^2+\alpha)t} \langle y, e_n \rangle e_n, \; y \in \Sigma,$$

then (**H1**) is satisfied.

Let $\Gamma : D(A) \subset \Sigma \to \Sigma$ be the operator defined by $\Gamma(t)(z) = \zeta(t)Az$ for $t \geq 0$ and $z \in D(A)$
Define the bounded linear operator $\mathcal{C} : \mathcal{V} \to \Sigma$ as

$$(\mathcal{C}u)(z) := u(t)(z) = \mu(t,z), \; t \in [0,1], z \in [0,\pi].$$

Let

$$\lambda(t) = \begin{cases} 1 - \frac{1}{2}, & 0 \leq t \leq 1 - \frac{1}{2}, \\ \cdots & \\ 1 - \frac{1}{n}, & 0 \leq t \leq 1 - \frac{1}{n}, \text{ for } n > 2 \text{ and and } n \in \mathbb{N}, \\ \cdots & \\ 1, \; t = 1. & \end{cases} \tag{10}$$

It is easy to see that $\lambda : J \to \mathbb{R}$ is nondecreasing left continuous function on J.

In order to reformulate the partial differential system(9) as the abstract problem (1), we introduce the following notations

$$\begin{cases} v(t)(z) = y(t,z) & \text{for } t \in [0,1] \text{ and } z \in [0,\pi], \\ v(0)(z) = y(0,z) & \text{for } z \in [0,\pi]. \end{cases}$$

We suppose the following assumptions

(F1) $\delta_1(\cdot, v) : [0,1] \to \Sigma$ is τ_λ-measurable for all $v \in \Sigma$ and the function $\delta_1(t,\cdot) : \Sigma \to \Sigma$ is continuous for $\tau_\lambda - a.e. t \in [0,1]$.

(F2) $||\delta_1(t,v)|| \leq \psi(l)$ for all $v \in \Sigma$, where $\psi(\cdot) \in \mathcal{LS}_g(J,\mathbb{R}^+)$.

Let us introduce the operators Δ_1 and g defined by

$$\Delta_1(t,v)(z) = \delta_1(t, v(t,z)), \quad \text{for } t \in [0,1] \text{ and } z \in [0,\pi],$$
$$g(v)(z) = \int_0^1 \rho(s)\log(1+|v(s,z)|)\mathrm{d}s, \qquad z \in [0,\pi], \; s \in [0,1].$$

Then the system (9) can be transformed into the abstract form of system (1) :

$$\begin{cases} \mathrm{d}v(t) = \left[Av(t) + \int_0^t \Gamma(t-s)v(s)\mathrm{d}s + \mathcal{C}u(t)\right]\mathrm{d}t + \Delta_1(t,v(t))\mathrm{d}\lambda(t),\ t \in J, \\ v(0) = v_0 + g(v). \end{cases} \tag{11}$$

Moreover, we suppose that ζ is a bounded and C^1 function such that ζ' is bounded and uniformly continuous, then (**H2**) is satisfied, hence by theorem 2.8 equation (2) has resolvent operator $(R(t))_{t\geq 0}$ on Σ.

Lemma 5.1. *The map $g : \mathcal{R}_f([0,1],\Sigma) \to \Sigma$ defined by*

$$g(v)(z) = \int_0^1 \rho(s)\log(1+|v(s,z)|)\mathrm{d}s,\ for\ z \in [0,\pi]\ and\ v \in R_f([0,1],\Sigma),$$

is compact.

Proof. With $v(s)z = v(s,z)$. By using the same proof as in the article [9] we show that g is a continuous and compact operator, and we have

$$||g(v)||_\Sigma \leq \left(\int_0^1 |\rho(s)|^2\mathrm{d}s\right)^{1/2}||v||_\infty = ||\rho||_{L^2}||v||_\infty.$$

Thus the hypothesis **(A4)** in Theorem 3.1 is satisfied. □

Theorem 5.2. *Assume that the conditions **(F1)** and **(F2)** are satisfied. The systems (10) has at least one mild solution $v \in C([0,\pi]\times J,\Sigma)$.*

Proof. From the definition of nonlinear term Δ_1 and bounded linear operator $\mathcal{C}$ combined with the above discussion, we can easily guarantee that assumptions of Theorem 3.1 hold. This completes the proof of Theorem 5.2. □

To obtain the approximate controllability for system (9), it suffices for us to verify that hypothesis (**A0**) is satisfied. To this end, we have the following result :

Lemma 5.3. *[25] Let $\theta(t) \in L^1(\mathbb{R}^+) \cap C^1(\mathbb{R}^+)$ with primitive $B(t) \in L^1_{loc}(\mathbb{R}^+)$ such that $\Gamma(t)$ is non-positive, non-decreasing and $B(0) = -1$. If the operator A is self-adjoint and positive semi-definite, the resolvent operator $R(t)$ associated to system (2) is self-adjoint as well.*

From Lemma 5.3 above, the resolvent operator $R(t)$ of (9) if self-adjoint. Since $\mathcal{C} = I$ in the system (9) then we have $\mathcal{C}^*R^*(t)v = R(t)v$, for any $v \in \Sigma$. Now, let $\mathcal{C}^*R^*(t)v = 0$, for all $t \in [0,b]$, then $R(t)v = 0, \forall t \in [0,b]$. Since $R(0) = Id$ for $t = 0$, we get $v = 0$. So from [12](Theorem 4.1.7), it follows that the linear control system corresponding to (9) is approximately controllable on J, which mean that the hypothesis (**A0**) is satisfied. Therefore, by Theorems 3.1 and 4.1, the integrodifferential equation (9) is approximately controllable on J.

6. Conclusion

The question of the approximate controllability of measure-driven semilinear control equations in Hilbert spaces has been addressed in this article.These equations can model a large class of hybrid systems without imposing any restrictions on the Zeno behavior of the systems. The existence criteria of mild solutions for this kind of measure control system can be obtained using Schauder's fixed point theorem. The results of the approximate controllability are then presented. There are immediate concerns that require additional research to be conducted. We will extend and unify the existing results on evolution equations with discrete nonlocal initial conditions and state-dependent delays by introducing a new Green's function, which is very important in dealing with these types of problems.

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REFERENCES

[1] Arthi, G, Balachandran, K, *Controllability of damped second-order neutral functional differential systems with impulses*, Taiwanese Journal of Mathematics, 16(1), (2012), 89–106.

[2] Arthi, Ganesan, Balachandran, K, *Controllability results for damped second-order impulsive neutral integrodifferential systems with nonlocal conditions*, Journal of Control Theory and Applications, 11(2), (2013), 186–192.

[3] Bashirov, A. E., Mahmudov, N. I.: *On concepts of controllability for deterministic and stochastic systems*, SIAM Journal on Control and Optimization, 37(6), (1999), 1808–1821.

[4] Benchohra, M, Gorniewicz, L., Ntouyas, S., Ouahab, A, *Controllability results for impulsive functional differential inclusions* , Reports on Mathematical Physics, 54(2), (2004), 211–228.

[5] Benchohra, Mouffak, Litimein, S., Nieto, J. J, *Semilinear fractional differential equations with infinite delay and non-instantaneous impulses* , Journal of Fixed Point Theory and Applications, 21(1), (2019), 1–16.

[6] Brogliato, Bernard, *Nonsmooth Mechanics* , Models, Dynamics and Control. Springer(2018).

[7] Byszewski, L, *Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem*, Journal of Mathematical Analysis and Applications, 162(2), (1991), 494–505.

[8] Cao, Y., Sun, J, *Existence of solutions for semilinear measure driven equations* , Journal of Mathematical Analysis and Applications, 425(2), (2015), 621–631.

[9] Cao, Y., Sun, J, *Measure of noncompactness in spaces of regulated functions with application to semilinear measure driven equations* , Boundary Value Problems, (1), (2016), 1–17.

[10] Cao, Y., Sun, J, *Controllability of measure driven evolution systems with nonlocal conditions* , Applied Mathematics and Computation, 299, (2017), 119–126.

[11] Cao, Y., Sun, J, *Approximate controllability of semilinear measure driven systems* , Mathematische Nachrichten, 291(13), (2018), 1979–1988.

[12] Curtain, R. F., Zwart, H, *An introduction to infinite-dimensional linear systems theory* Springer Science and Business Media Vol.21, (2012).

[13] Deng, K, *Exponential decay of solutions of semilinear parabolic equations with nonlocal initial conditions* , Journal of Mathematical Analysis and Applications, 179(2), (1993), 630–637.

[14] Desch, W., Grimmer, R., Schappacher, W, *Some considerations for linear integrodifferential equations* , Journal of Mathematical Analysis and Applications, 104(1), (1984), 219–234.

[15] Diestel, J. et al, *On weak compactness in* $L^1(\mu,X)$, Proceedings of the American Mathematical Society, 118(2), (1993), 447–453.

[16] Ezzinbi, K., Degla, G., Ndambomve, P.: *Controllability for some partial functional integrodifferential equations with nonlocal conditions in Banach spaces* , Discussiones Mathematicae, Differential Inclusions, Control and Optimization, 35(1), (2015), 25–46.

[17] Gou, H., Li, Y, *Existence and Approximate Controllability of Semilinear Measure Driven Systems with Nonlocal Conditions* , Bulletin of the Iranian Mathematical Society, 48(2), (2022), 769–789.

[18] Grimmer, R. C, *Resolvent operators for integral equations in a Banach space* , Transactions of the American Mathematical Society, 273(1), (1982), 333–349.

[19] Kalman, R. E, *Canonical structure of linear dynamical systems* , Proceedings of the National Academy of Sciences, 48(4), (1962), 596–600.

[20] Kumar, S., Mohammad Abdal, S, *Approximate controllability of non-instantaneous impulsive semilinear measure driven control system with infinite delay via fundamental solution* , IMA Journal of Mathematical Control and Information, 38(2), (2021), 552–575.

[21] Liang, J., Liu, J. H., Xiao, T.-J, *Nonlocal problems for integrodifferential equations* , Dynamics of Continuous, Discrete & Impulsive Systems. Series A, 15(6), (2008), 815–824.

[22] López Pouso, R., Márquez Albes, I., Rodrıgguez-Lopez, J.: *Solvability of non-semicontinuous systems of Stieltjes differential inclusions and equations* , Ad-

vances in Difference Equations,(1), (2020), 1–14.

[23] Ma, L., Gu, H., Chen, Y, *Approximate Controllability of Neutral Measure Evolution Equations with Nonlocal Conditions* , Journal of Mathematics, (2021).

[24] Mesquita, J. G., others, *Measure functional differential equations and impulsive functional dynamic equations on time scales* , In PhD Thesis, Universidade de Sáo Paulo (2012).

[25] Mokkedem, F. Z., Fu, X, *Approximate controllability of semi-linear neutral integro-differential systems with finite delay* , Applied Mathematics and Computation, 242, (2014), 202–215.

[26] Muslim, M., Kumar, A., Fečkan, M, *Existence, uniqueness and stability of solutions to second order nonlinear differential equations with non-instantaneous impulses* , Journal of King Saud University-Science, 30(2), (2018), 204–213.

[27] Pazy, A, *Semigroups of linear operators and applications to partial differential equations* , Springer Science & Business Media, (Vol. 44), (2012).

[28] Prossdorf, S., Silbermann, B, Honig, CS, *Volterra Stieltjes-Integral Equations* , Amsterdam-Oxford(1979).

[29] Satco, B, *Regulated solutions for nonlinear measure driven equations* , Nonlinear Analysis: Hybrid Systems, 13, (2014), 22–31.

[30] Triggiani, R, *A note on the lack of exact controllability for mild solutions in Banach spaces* , SIAM Journal on Control and Optimization, 15(3), (2021), 407–411.

[31] Zeidler, E., Wadsack, P. R.: *Nonlinear Functional Analysis and Its Applications* , Fixed-point Theorems/Transl. by Peter R. Wadsack. Springer-Verlag (1993).

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