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ON THE COMPLEMENTS OF UNION OF OPEN BALLS OF FIXED RADIUS IN THE EUCLIDEAN SPACE

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Let an R-body be the complement of the union of open balls of radius R in $\mathbb{E}^d$. The R-hulloid of a closed not empty set A, the minimal R-body containing A, is investigated; if A is the set of the vertices of a simplex, the R-hulloid of A is completely described (if $d = 2$) and if $d > 2$ special examples are studied. The class of R-bodies is compact in the Hausdorff metric if $d = 2$, but not compact if $d > 2$.

1. Introduction

Given a closed set $E \subset \mathbb{E}^d$ ($d \geq 2$), the convex hull of E is the intersection of all closed half spaces containing E; the convex hull can be considered as a regularization of E. Given $R > 0$, a different hull of E could be the intersection of all closed sets, containing E, complement of open balls of radius R not intersecting E. Let us call this set the R-hulloid of E, denoted as $co_R(E)$; the R-bodies are the sets coinciding with their R-hulloids. R-bodies are called $2R$-convex sets in [10].

The R-hulloid $co_R(E)$ has been introduced by Perkal [10] as a regularization of E, hinting that $co_R(E)$ is a mild regularization of a closed set. Mani-Levitska [8] hinted that the R-bodies cannot be too irregular.

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In our work it is shown that this may not be true: in Theorem 5.7 an example of a connected set is constructed with disconnected R-hulloid. A deeper study gave us the possibility to add new properties to the R-bodies: a representation of $co_R(E)$ is given in Theorem 3.4 and new properties of $\partial co_R(E)$ are proved in Theorem 3.5, Theorem 3.6 and Corollary 3.7. Moreover contrasting results on regularity are found: every closed set contained in an hyperplane or in a sphere of radius $r \geq R$ is an R-body (Theorems 3.10 and 3.11). As a consequence a problem of Borsuk, quoted by Perkal [10], has a negative answer (Remark 3.10). In § 4 it is shown that the R-body regularity heavily depends on the dimension. A definition (Definition 4.3) similar to the classic convexity is given for the class of planar R-bodies, namely (Theorem 4.5):

$$A \quad \text{is an } R\text{-body iff} \quad co_R(\{a_1,a_2,a_3\}) \subset A \quad \forall a_1,a_2,a_3 \in A.$$

As consequence, if $d = 2$: a sequence of compact R-bodies converges in the Hausdorff metric to an R-body (Corollary 4.7). If $d > 2$, in Theorem 3.16 it is proved that a sequence of compact R-bodies converges to an $(R-\varepsilon)$-body, for every $0 < \varepsilon < R$; however, the limit body may not be an R-body as an example in § 5 shows. If E is connected, properties of connectivity of $co_R(E)$ are investigated in § 4.3.

In [7, Definition 2.1] V. Golubyatnikov and V. Rovenski introduced the class $\mathcal{K}_2^{1/R}$. In Theorem 6.1 it is proved that the class of R-bodies is strictly included in $\mathcal{K}_2^{1/R}$. If $d = 2$, under additional assumptions, it is also proved that the two classes coincide.

2. Definitions and Preliminaries

Let $\mathbb{E}^d, d \geq 2$, be the linear Euclidean Space with unit sphere $\mathcal{S}^{d-1}$; $A \subset \mathbb{E}^d$ will be called a **body** if A is non empty and closed. The minimal affine space containing A will be $Lin(A)$. The convex hull of A will be $co(A)$; for notations and results of convex bodies, let us refer to [13].

Definition 2.1. Let A be a not empty set.

$A_\varepsilon := \{x \in \mathbb{E}^d : \text{dist}(A,x) < \varepsilon;\}$; $A'_\varepsilon := \{x \in \mathbb{E}^d : \text{dist}(A,x) \geq \varepsilon\}$; $A^- := A \cup \partial A$; $A^c := \mathbb{E}^d \setminus A$; $Int(A) = A^- \setminus \partial A$.

$B(x,r)$ will be the open ball of center $x \in \mathbb{E}^d$ and radius $r > 0$; a sphere of radius r is $\partial B(x,r)$.

Let us recall the following facts for reference.

Proposition 2.2. Let A be a not empty set.

- **1** A_ε is open; $A_\varepsilon = (A^-)_\varepsilon \subset (A_\varepsilon)^-$.

- **2** $A_\varepsilon = \{x \in \mathbb{E}^d : \exists a \in A, \text{ for which } x \in B(a,\varepsilon)\} = \{x \in \mathbb{E}^d : B(x,\varepsilon) \cap A \neq \emptyset\}$
$$= \cup_{a\in A} B(a,\varepsilon) = A + B(0,\varepsilon).$$
- **3** $A'_\varepsilon = \{x \in \mathbb{E}^d : \forall a \in A, x \notin B(a,\varepsilon)\} = \{x \in \mathbb{E}^d : B(x,\varepsilon) \cap A = \emptyset\} = \cap_{a\in A} B(a,\varepsilon)^c$.
- **4** Let $A_i, i = 1,2$ be non empty sets. Then
$$A^1 \subset A^2 \Rightarrow (A^1)_\varepsilon \subset (A^2)_\varepsilon.$$
- **5** If E is non empty, then $E \subset (E'_R)'_R \subset E_R$, see [1, lemma 4.3].

Definition 2.3. ([3]) If $A \subset \mathbb{E}^d$, $a \in A$, then $reach(A,a)$ is the supremum of all numbers ρ such that for every $x \in B(a,\rho)$ there exists a unique point $b \in A$ satisfying $|b-x| = \text{dist }(x,A)$. Also:

$$reach(A) := \inf\{reach(A,a) : a \in A\}.$$

Let $b_1, b_2 \in \mathbb{E}^d, |b_1 - b_2| < 2R$ and let $\mathfrak{h}(b_1,b_2)$ be the intersection of all closed balls of radius R containing b_1, b_2.

Proposition 2.4. ([1, Theorem 3.8], [11]) The body A has reach $\geq R$ if and only if $A \cap \mathfrak{h}(b_1,b_2)$ is connected for every $b_1, b_2 \in A, 0 < |b_1 - b_2| < 2R$.

Remark 2.1. The R-hull of a set E was introduced in [1, Definition 4.1] as the minimal set $\hat{E}$ of reach $\geq R$ containing E. Therefore if $reach(A) \geq R$, then A coincides with its R-hull. The R-hull of a set E may not exist, see [1, Example 2].

Proposition 2.5. [1, Theorem 4.4] Let $A \subset \mathbb{E}^d$. If $reach(A'_R) \geq R$ then A admits R-hull $\hat{A}$ and
$$\hat{A} = (A'_R)'_R.$$

Proposition 2.6. [1, Theorem 4.8] If $A \subset \mathbb{E}^2$ is a connected subset of an open ball of radius R, then A admits R-hull.

Let us also recall the following result:

Proposition 2.7. [1, Theorem 3.10], [12]) Let $A \subset \mathbb{E}^d$ be a closed set such that $reach(A) \geq R > 0$. If $D \subset \mathbb{E}^d$ is a closed set such that for every $a, b \in D$, $\mathfrak{h}(a,b) \subset D$ and $A \cap D \neq \emptyset$, then $reach(A \cap D) \geq R$.

3. R-bodies

Let R be a fixed positive real number. B will be any open ball of radius R. $B(x)$ will be the open ball of center $x \in \mathbb{E}^d$ and radius R. Next definitions have been introduced in [10].

Definition 3.1. Let A be a body, A will be called an R-body if $\forall y \in A^c$, there exists an open ball B in E^d (of radius R) satisfying $y \in B \subset A^c$. This is equivalent to say

$$A^c = \cup\{B : B \cap A = \emptyset\};$$

that is

$$A = \cap\{B^c : B \cap A = \emptyset\}.$$

Let us notice that for any $r \geq R$ and for every x, the body $(B(x,r))^c$ is an R-body.

Definition 3.2. Let $E \subset \mathbb{E}^d$ be a non empty set. The set

$$co_R(E) := \cap\{B^c : B \cap E = \emptyset\}$$

will be called the **R-hulloid** of E. Let $co_R(E) = \mathbb{E}^d$ if there are no balls $B \subset E^c$.

Remark 3.1. In [10] the sets defined in Definition 3.1 are called $2R$ convex sets and the sets defined in Definition 3.2 are called $2R$ convex hulls. On the other hand Meissner [9] and Valentine [15, pp. 99-101] use the names of R-convex sets and R-convex hulls for different families of sets. An s-convex set is also defined in [4, p. 42]. To avoid misunderstandings we decided to call R-bodies and R-hulloids the sets defined in Definition 3.1 and in Definition 3.2.

Remark 3.2. Let us notice that $co_R(E)$ is an R-body (by definition) and $E \subset co_R(E)$. Moreover A is an R-body if and only if $A = co_R(A)$. The R-hulloid always exists.

Clearly every convex body E is an R-body (for all positive R) and its convex hull $co(E) = E$ coincides with its R-hulloid.

Remark 3.3. It was noticed in [1, Corollary 4.7] and proved in [2, Proposition 1] that, when the R-hull exists, it coincides with the R-hulloid. If A has *reach* greater or equal than R, then (see remark 2.1) A has R-hull, which coincides with A and with its R-hulloid, then A is an R-body.

Proposition 3.3. Let E be a non empty set. The following facts have been proved in [10].

- **a** $co_R(E) = (E'_R)'_R$;

- **b** $E^- \subset co_R(E)$;
- **c** Let $E^1 \subset E^2$; then $co_R(E^1) \subset co_R(E^2)$;
- **d** $co_R(E^1) \cup co_R(E^2) \subset co_R(E^1 \cup E^2)$;
- **e** $co_R(co_R(E)) = co_R(E)$;
- **f** Let $A^{(\alpha)}, \alpha \in \mathcal{A}$ be R-bodies, then $\cap_{\alpha\in\mathcal{A}} A^{(\alpha)}$ is an R-body;
- **g** $\operatorname{diam} E = \operatorname{diam} co_R(E)$;
- **h** If A is an R-body then A is an r-body for $0 < r < R$;
- **i** $co_R(E) \subset co(E)$ for all $R > 0$.

Remark 3.4. Let E be a body. From **c** of Proposition 3.3 it follows that if A is an R-body and $A \supset E$, then $A \supset co_R(E)$ and $co_R(E)$ is the minimal R-body containing E.

Lemma 3.5. *A point $k \in co_R(E)$ if and only if there does not exist any open ball $B(x,l) \ni k$ with $l \geq R$, $B(x,l) \subset E^C$.*

Proof. As $(B(x,l))^c$ is an R-body, the set $co_R(E) \cap (B(x,l))^c \supset E$ would be an R-body strictly included in $co_R(E)$, which is the minimal R-body containing E. □

Lemma 3.6. *Let E be a body. Then*

$$co_R(E) \subset E_R. \tag{1}$$

Moreover there exists E such that $(E_R)^-$ is not an R-body.

Proof. By **5** of Proposition 2.2, $(E'_R)'_R \subset E_R$ and by **a** of Proposition 3.3, the inclusion (1) follows. Let $x_0 \in \mathbb{E}^d$, $R < \rho < 2R$ and let $E = (B(x_0,\rho))^c$. Then $(E_R)^-$ is $(B(x_0, \rho - R))^c$, not an R-body. □

Theorem 3.4. Let $E \subset \mathbb{E}^d$ be a body. Then

$$co_R(E) = E_R \cap \Big(\partial(E_R)\Big)'_R. \tag{2}$$

Proof. Formula (2) can also be written as:

$$(co_R(E))^c = E'_R \cup \Big(\partial(E_R)\Big)_R. \tag{3}$$

Let $\Omega = E'_R \cup \Big(\partial(E_R)\Big)_R$.

Inclusion (1) implies that $E'_R \subset (co_R(E))^c$. Let us notice that:

$$\Big(\partial(E_R)\Big)_R = \cup\{B(x) : x \in \partial(E_R)\} = \cup\{B(x) : \text{dist}\,(x,E) = R\},$$

then

$$\Big(\partial(E_R)\Big)_R \subset \cup\{B(x) : \text{dist}\,(x,E) \geq R\} = (co_R(E))^c. \tag{4}$$

Then from (4):

$$\Omega \subset (co_R(E))^c$$

holds too.

The open set $(co_R(E))^c$ is the union of the balls $B(x)$, satisfying $B(x) \cap E = \emptyset$; clearly dist $(x,E) \geq R$; if dist $(x,E) = R$ then $x \in \partial(E_R)$ and $B(x) \subset \Big(\partial(E_R)\Big)_R$; if dist $(x,E) > R$, then $B(x) \subset E'_R$. Therefore

$$(co_R(E))^c \subset \Omega.$$

Then $\Omega = (co_R(E))^c$. □

Remark 3.7. The previous theorem is the analogous, for the R-hulloid, of the property of the convex hull of a body E: $co(E)$ is the intersection of all closed half spaces supporting E.

If E is a compact set, part of the following theorem has been proved in [2, Proposition 2].

Theorem 3.5. Let E be a body, $k \in co_R(E)$, $l = inf_{x \in E'_R}|k-x| = \text{dist}\,(k, E'_R)$. Then l is a minimum and $l \geq R$. Moreover $l = R$ if and only if $k \in \partial co_R(E)$ and there exists $x_0 \in E'_R$ satisfying $B(x_0) \subset E^c$, $\partial B(x_0) \ni k$.

Proof. As $co_R(E) = \cap\{B^c : B^c \supset E\}$, then dist $(E'_R, co_R(E)) \geq R$. Let $x_n \in E'_R$ satisfying $|x_n - k| \to l \geq R$; by possibly passing to a subsequence, one can assume that $x_n \to x_0 \in E'_R$, where $|x_0 - k| = l$. If $|x_0 - k| = R$ then $k \in co_R(E) \cap \partial B(x_0)$. As $l = R$, it cannot be $k \in Int(co_R(E))$. Therefore the claim of the theorem holds. □

Theorem 3.6. Let E be a body, $k \in \partial co_R(E)$. Then there exists $B \subset E^c$ satisfying $k \in \partial B$. Moreover if $\mathfrak{F} = \{B \subset E^c : \partial B \cap co_R(E) \neq \emptyset\}$, then $\mathfrak{F}$ is not empty and if $B \in \mathfrak{F}$ then $\partial B \cap E \neq \emptyset$.

Proof. If $k \in \partial co_R(E)$, by previous theorem there exists $x_0 \in E'_R$ with the property $B(x_0) \subset E^c$, $\partial B(x_0) \ni k$. If dist $(x_0, E) = l > R$ then $k \in B^1 = B(x_0, l) \subset E^c$, this is impossible by Lemma 3.5 and $\mathfrak{F}$ is non empty. Let $B(x) \in \mathfrak{F}$ and, by contradiction, let $\partial B \cap E = \emptyset$; then, $R_1 = \text{dist}\,(x,E) > R$. Thus $B(x,R_1)^c$ is an R-body containing E, then $co_R(E) \subset B(x,R_1)^c$; as $\partial B(x) \subset B(x,R_1)$ so $\partial B(x) \cap co_R(E) = \emptyset$, contradiction with $B(x) \in \mathfrak{F}$. □

Corollary 3.7. Let A be an R-body. Then :

(i) $\Xi(A) := \{x : B(x) \subset A^c\}$ (the set of centers of balls of radius R contained in A^c) is closed;

(ii) $\forall y \in \partial A$, there exists $x_0 \in \Xi(A)$ with the property: $y \in \partial B(x_0)$.

Proof. Let x_0 be an accumulation point of $\Xi(A)$ and $\Xi(A) \ni x_n \to x_0$; let $b \in B(x_0)$, then $\lim_{n\to\infty} |b-x_n| = |b-x_0|$ where $|b-x_0| < R$. Thus for n sufficiently large $|b-x_n| < R$, therefore $b \in B(x_n) \subset A^c, \forall b \in B(x_0)$. Then $B(x_0) \subset A^c$, $x_0 \in \Xi(A)$ and (i) holds.

(ii) follows by Theorem 3.6. □

Lemma 3.8. *Let A be a body; if A^c is union of closed balls of radius R, then A is an R-body.*

Proof. For every $y \in A^c$ there exists $(B(z))^- \subset A^c$, $y \in (B(z))^-$. As A and $(B(z))^-$ are closed and disjoint, there exists $R_1 > R$ so that $B(z,R_1) \subset A^c$. Then there exists a ball $B \subset A^c, B \ni y$. Thus A^c is union of open balls of radius R and A is an R-body. □

Let us notice that there exist R-bodies A such that A^c is not union of closed balls of radius R. As example, let $A = B^c$.

Theorem 3.8. Let A be a body, which is not an R-body. Then there exists $y_0 \in A^c$ such that y_0 belongs to no closed ball of radius R, contained in A^c.

Proof. By contradiction, let us assume that every $y \in A^c$ is contained in a closed ball of radius R contained in A^c, then A^c is union of closed balls of radius R and satisfies the hypothesis of Lemma 3.8, then A is an R-body. Impossible. □

Let $\mathcal{C}^d$ be the metric space of the compact bodies in $\mathbb{E}^d$ with the Hausdorff distance $\delta_H(F,G) := \min\{\varepsilon \geq 0 : F \subset G_\varepsilon, G \subset F_\varepsilon\}$.

From a bounded sequence in $\mathcal{C}^d$ one can select a convergent subsequence in the Hausdorff metric (see e.g. [13, Theorem 1.8.4]).

Let $\mathcal{R}^d = \{A \subset \mathcal{C}^d : A$ is an R-body $\}$. Let $A \subset \mathbb{E}^d$ be a body, $\varepsilon > 0$. Let $A_\varepsilon^- := \{x \in E^d : dist(A,x) \leq \varepsilon\} = (A_\varepsilon)^-$. $D = B^-$ will be any closed ball of radius R.

Theorem 3.9. Let $A^{(n)}$ be a sequence of compact R-bodies; let us assume that $A^{(n)} \to A \in \mathcal{C}^d$ in the Hausdorff metric. Then, A is an R_ε-body for every $0 < R_\varepsilon < R$.

Proof. By contradiction, let us assume that A it is not an R_ε-body. Then by Theorem 3.8, there exists $y_0 \in A^c$ with the property

$$y_0 \text{ belongs to no closed ball, of radius } R_\varepsilon\text{, subset of } A^c. \tag{5}$$

As dist $(y_0,A) > 0$, then $y_0 \in (A_\sigma)^c$ for suitable $\sigma > 0$. As $A^{(n)} \to A$ in the Hausdorff metric, there exists a sequence $\varepsilon_n \to 0^+$ satisfying $A^{(n)} \subset A_{\varepsilon_n}$, $A \subset A_{\varepsilon_n}$. For n sufficiently large $(A_\sigma)^c \subset (A_{\varepsilon_n})^c$ and $y_0 \in (A_{\varepsilon_n})^c \subset (A^{(n)})^c$. As $A^{(n)} \in \mathcal{R}^d$, then there exist open balls $B(x_n)$ satisfying $y_0 \in B(x_n) \subset (A^{(n)})^c$; then dist $(x_n,A^{(n)}) \geq R$, dist $(x_n,A) \geq R - \varepsilon_n$.

As $|x_n - y_0| < R$, by possibly passing to a subsequence, $x_n \to x_0 \in \mathbb{E}^d$. The point x_0 satisfies: $|x_0 - y_0| \leq R$, dist $(x_0,A) \geq R$. Then $B(x_0) \subset A^c$ and $D := (B(x_0))^-$ is a closed ball of radius R containing y_0. If $y_0 \in B(x_0)$, then $D_\rho = B^-(x_0,\rho)$, with $\rho = max\{|y_0 - x_0|, R_\varepsilon\}$ is a closed ball which provides a contradiction with (5). In case $y_0 \in \partial B(x_0)$ the closed ball enclosed in D, tangent to $\partial B(x_0)$ at y_0, with radius R_ε, provides a contradiction with property (5). □

Remark 3.9. In section 5 it will be shown that in $\mathbb{E}^3$ a limit (in Hausdorff metric) of a sequence of R-bodies may be not an R-body. In Corollary 4.7 it will be proved that, in $\mathbb{E}^2$, a limit of a sequence of R-bodies (in Hausdorff metric) is an R-body too.

Theorem 3.10. Let $\Sigma = \partial B(r) \subset \mathbb{E}^d$ be a sphere of radius $r \geq R$ and let E be a body subset of Σ. Then E is an R-body.

Proof. Σ is a topological space with the topology induced by $\mathbb{E}^d$ and E is closed in that topology. Then $\Sigma \setminus E$ is union of $(d-1)$-dimensional open balls in Σ. Let $D = (B(r))^-$, as $\mathbb{E}^d \setminus \Sigma = B(r) \cup D^c$, then

$$\mathbb{E}^d \setminus E = B(r) \cup D^c \cup (\Sigma \setminus E)$$

is union of the following open balls of radius R:

(i) all open balls of radius R contained in $B(r)$, which fill $B(r)$ since $r \geq R$;

(ii) all open balls B of radius R contained in D^c;

(iii) all open balls B of radius R satisfying the property: $B \cap \Sigma$ is a $(d-1)$-dimensional open ball in $\Sigma \setminus E$.

So E is an R-body. □

With a similar proof, the following fact can be proved.

Theorem 3.11. Let $E \subset \mathbb{E}^d$ be a body, subset of a hyperplane Π. Then E is an R-body.

Remark 3.10. In [10], p.9, a question of Borsuk was stated: 'Are the R-bodies locally contractible?'.

The Borsuk's question has a negative answer: let Π be an hyperplane in E^d. By Theorem 3.11 every body, subset of Π, is an R-body; then there exist not locally contractible bodies subsets of Π.

4. Properties of R-bodies in $\mathbb{E}^2$.

4.1. *R*-hulloid of three points in $\mathbb{E}^2$.

Let R be a fixed positive real number. Let T be a not degenerate triangle in $\mathbb{E}^2$, $V = \{x_1, x_2, x_3\}$ be the set of its vertices, $r(V)$ be the radius of the circle circumscribed to T. By Theorem 3.10, if $r(V) \geq R$, then $co_R(V) = V$.

Proposition 4.1. Let $\{x_1, x_2, x_3\}$ be the vertices of a triangle T inscribed in a circumference C of radius r. Three possible cases may occur:

i) ([6, pag 16]) if T is acute-angled then the three circumferences of radius r, each one through two vertices of T, different from C, meet in the orthocenter y of T;

ii) if T is obtuse-angled in x_3 then the two circumferences of radius r through the vertices $\{x_1, x_3\}$ and $\{x_2, x_3\}$, respectively, different from C, meet C in x_3 and in a point exterior to T;

iii) if T is right-angled at x_3 then the two circumferences of radius r through the vertices $\{x_1, x_3\}$ and $\{x_2, x_3\}$, different from C, are tangent at x_3.

Proof. i) it is related to the Johnson's Theorem [5]; ii) and iii) follows by construction. □

Theorem 4.2. Let $V = \{x_1, x_2, x_3\}$ be the set of the vertices of a triangle T with circumradius $r = r(V)$. If $r(V) < R$, then

$$co_R(V) = V \cup \tilde{T},$$

where $\tilde{T} \subset T$ is the curvilinear triangle bordered by three arcs of circumferences of radius R; each one through two vertices of T and relative to the circle not containing the remaining vertex of T. If T is a right-angled or obtuse-angled then the vertex of the greatest angle of T is also a vertex of $\tilde{T}$, that is the end point of two consecutive arcs of $\partial\tilde{T}$.

Proof. Let $B(q_i, r)$,$B(c_i, R)$ be the open circles, not containing x_i, with boundary through the two vertices of T different from $x_i, i = 1, 2, 3$. In the case i) of Proposition 4.1, the orthocenter y of T is in the interior of T and $y \in \cap_{i=1,2,3}\partial B(q_i, r)$. As $R > r$: $T \cap B(c_i, R) \subsetneq T \cap B(q_i, r)$, then dist $(y, B(c_i, R)) > 0$, $(i = 1, 2, 3)$. Thus

$$\tilde{T} :\equiv T \cap \left(\bigcup_{j=1}^{3} B(c_j, R)\right)^C \tag{6}$$

is a curvilinear triangle with $y \in Int(\tilde{T})$; moreover $\partial\tilde{T}$ is union of of three arcs of the circumferences $\partial B(c_i, R)$ $(i = 1, 2, 3)$.

If T is obtuse-angled at x_3, case ii) of Proposition 4.1), the two circumferences $\partial B(c_i,R)$ containing x_3 and another vertex of T cross each other in x_3 and in a point exterior to T. If T is right-angled at x_3 the two circumferences $\partial B(q_i,r)$ meet and are tangent to each other in x_3, then again the circumferences $\partial B(c_i,R)$ cross each other in x_3 and in a point exterior to T. In both cases dist $(x_3, B(c_3,R)) > 0$ and $\tilde{T}$, given by (6), is a curvilinear triangle with a vertex at x_3. □

4.2. Two dimensional R-bodies, equivalent definitions

Definition 4.3. Let a_1,a_2 be two points in $\mathbb{E}^2$, with $0 < |a_1 - a_2| < 2R$. Let $B(x_1), B(x_2)$ the two open circles with the boundaries through a_1,a_2. Let us define

$$H(a_1,a_2) = B(x_1) \cup B(x_2),$$

and let $\mathfrak{h}(a_1,a_2)$ be the intersection of all closed balls of radius R containing a_1,a_2.

Definition 4.4. Let A be a planar body. A satisfies the property $\mathfrak{Q}_R$ if :

$$\forall a_1,a_2,a_3 \in A \quad \text{the } R\text{-hulloid of the set } \{a_1,a_2,a_3\} \text{ is a subset of } A.$$

When x,y are points on a circumference ∂B, let us denote with $arc_{\partial B}(x,y)$ the shorter arc on ∂B from x to y.

Lemma 4.1. *Let A be a planar body. If A satisfies the property $\mathfrak{Q}_R$, then*

$$\{a_1,a_2\} \subset A, 0 < |a_1 - a_2| < 2R : \mathfrak{h}(a_1,a_2) \setminus \{a_1,a_2\} \subset A^c \Rightarrow H(a_1,a_2) \subset A^c. \tag{7}$$

Proof. Let $H(a_1,a_2) = B(x_1) \cup B(x_2)$. Let us assume, by contradiction, that there exist $a_3 \in A \cap (B(x_1) \setminus \mathfrak{h}(a_1,a_2))$. Let $T = co(\{a_1,a_2,a_3\})$, then $r(T) < R$. By Theorem 4.2 there exist $y_1,y_2 \in arc_{\partial B(x_2)}(a_1,a_2)$ satisfying

$$arc_{\partial B(x_2)}(y_1,y_2) \subset co_R(\{a_1,a_2,a_3\}) \subset A.$$

As

$$arc_{\partial B(x_2)}(y_1,y_2) \subset \mathfrak{h}(a_1,a_2) \setminus \{a_1,a_2\} \subset A^c,$$

this is impossible. The proof is similar if $a_3 \in B(x_2)$. □

Theorem 4.5. Let A be a planar body. A is an R-body if and only if A satisfies the property $\mathfrak{Q}_R$.

Proof. Let A be an R-body then $co_R(\{a_1,a_2,a_3\}) \subset co_R(A) = A$ and $\mathfrak{Q}_R$ holds for A.

On the other hand let assume the property $\mathfrak{Q}_R$ holds for a body A. Let us prove that A is an R-body, by showing:

$$\text{if} \quad y_0 \in A^c \quad \text{then} \quad \exists B \ni y_0, B \subset A^c. \tag{8}$$

Let $y_0 \in A^c$, then there exists $\delta > 0$ such that dist $(y_0, A) = \delta$. If $\delta \geq R$, then $B(y_0, R) \subset B(y_0, \delta)$ and (8) holds. Let $\delta < R$. By definition of δ, there exists $a_1 \in A \cap \partial B(y_0, \delta)$ and $B(y_0, \delta) \subset A^c$. There are two cases:

i) there exists a point $a_2 \neq a_1$, $a_2 \in A \cap \partial B(y_0, \delta)$;

ii) $A \cap \partial B(y_0, \delta) = \{a_1\}$.

In the case i), $\mathfrak{h}(a_1, a_2) \setminus \{a_1, a_2\} \subset B(y_0, \delta) \subset A^c$. Let $H(a_1, a_2) = B(x_1) \cup B(x_2)$; by Lemma 4.1 the following inclusion holds:

$$H(a_1, a_2) \subset A^c. \tag{9}$$

As $y_0 \in B(x_1)$ or $y_0 \in B(x_2)$ and both balls $B(x_1), i = 1,2$ have empty intersection with A, then y_0 satisfies (8).

In the case ii) on $\partial B(y_0, \delta)$ let a_* be the symmetric point of a_1 with respect to the center y_0. For $t > 2$ let $a(t) = a_1 + (t-1)(a_* - a_1)$. Let $t_R > 2$ be such that $|a_1 - a(t_R)| = 2R$. The set function $t \to \mathfrak{h}(a_1, a(t)) \setminus \{a_1\}$, for $2 \leq t < t_R$, is strictly increasing with respect to the inclusion. If for all $2 \leq t < t_R$ the set $\mathfrak{h}(a_1, a(t)) \setminus \{a_1\} \subset A^c$ then $\lim_{t \to t_R -} \mathfrak{h}(a_1, a(t))$ is a closed ball $D \ni y_0$ of radius R, $A^c \supset Int(D) \ni y_0$ and (8) holds. Otherwise, there exists $2 < \tau < t_R$ satisfying $\mathfrak{h}(a_1, a(\tau)) \setminus \{a_1\} \cap A \neq \emptyset$. Let

$$\bar{t} = Inf\{t \in [2, t_R] : \Big(\mathfrak{h}(a_1, a(t)) \setminus \{a_1\}\Big) \cap A \neq \emptyset\}$$

and let

$$2 \leq t \leq t_R \to F(t) := \Big(\mathfrak{h}(a_1, a(t)) \setminus \{a_1\}\Big) \cap (B(y_0, \delta))^c. \tag{10}$$

By construction $\{F(t)\}$ is a continuous family of bodies, strictly monotone with respect to the inclusion, with dist $(F(t), A) > 0$ for $t < \bar{t}$. Then $F(\bar{t}) \cap A \neq \emptyset$, $Int(F(\bar{t})) \subset A^c$ and dist $(a_1, F(\bar{t})) > 0$. Therefore there exists $a_2 \in \partial F(\bar{t}) \cap \partial A$ of minimum distance from a_1. This implies that $arc_{\partial F(\bar{t})}(a_1, a_2)$ has no interior points of the body A. Then, $\mathfrak{h}(a_1, a_2) \setminus \{a_1, a_2\} \subset A^c$; by arguing as in case i), the inclusion (9) holds and y_0 satisfies (8). □

Theorem 4.6. Let $A \subset \mathbb{E}^2$ be a body. If A is a ρ-body for every positive $\rho < R$ then A is an R-body.

Proof. If A is ρ-body the property $\mathfrak{Q}_\rho$ holds for $\rho < R$. Let us show that it holds for $\rho = R$. Let $a_1, a_2, a_3 \in A$, with $r(\{a_1, a_2, a_3\}) \geq R$, then $co_R(\{a_1, a_2, a_3\}) = \{a_1, a_2, a_3\} \subset A$. In case $r(\{a_1, a_2, a_3\}) < R$ let $\rho > r(\{a_1, a_2, a_3\})$; by Theorem 4.2, with ρ instead of R and a_1, a_2, a_3 in place of x_1, x_2, x_3, it follows

$$co_\rho(\{a_1, a_2, a_3\}) = \{a_1, a_2, a_3\} \bigcup \tilde{T}_\rho.$$

$\tilde{T}_\rho$ a curvilinear triangle subset of A, bounded by arcs of radius ρ. As A is closed and $\tilde{T}_\rho \to \tilde{T}$, then $\tilde{T} \subset A$. Therefore $\mathfrak{Q}_R$ holds too and previous theorem proves that A is an R-body. □

From Theorem 4.6 and Theorem 3.9 it follows

Corollary 4.7. A limit of a sequence of planar R-bodies (in Hausdorff metric) is an R-body too.

Remark 4.2. With arguments similar to the proof of Theorem 4.5, it can also be proved that for a planar body A the property $\mathfrak{Q}_R$ is equivalent to the property (7).

4.3. Connected and disconnected R-bodies in $\mathbb{E}^2$

Theorem 4.8. Let E be a connected body in $\mathbb{E}^2$, contained in an open ball B of radius R; then $co_R(E)$ is connected.

Proof. As E is connected, by Proposition 2.6, E admits R-hull A of *reach* $\geq R$; then, by Remark 3.2, $A = co_R(E)$. By Proposition 2.4 the set A is connected. □

In the previous theorem the assumption that E is contained in an open ball of radius R is needed as the following example shows.

Example 1. In E^2 let $\Sigma_0 := \partial B(0, R_0)$, with

$$\frac{R}{\sqrt{3}} < R_o < R.$$

Let $k_i \in \Sigma_0, i = 1, 2, 3$ be the vertices of an equilateral triangle T and let $\partial B(o_j, R)$ the circumference, through the two points $k_i, i \neq j$, with $k_j \notin B(o_j, R)$. Let $D := (B(0, 4R))^-$ and

$$E := D \cap \left(B(0, R_0) \bigcup_{j=1}^{3} B(o_j, R) \right)^c.$$

Then E is a planar connected body with disconnected R-hulloid.

Proof. It is obvious that E is connected since it is homotopic to a ring. E^c is an open set since E^c is the union of D^c and open balls. As $R_0 < R$ and $\forall i \neq j, k_i \in \partial B(o_j), k_j \notin B(o_j)$ the set E^c does not contain the set of the vertices k_i. Let

$$\tilde{T} :\equiv \left(\bigcup_{j=1}^{3} B(o_j,R) \cup B(0,R_0) \right) \setminus \left(\bigcup_{j=1}^{3} B(o_j,R) \right).$$

$\tilde{T}$ is a curvilinear triangle and it is a closed connected set disjoint from E; moreover any point of $\tilde{T}$ can not lie in an open circle of radius R avoiding all the vertices k_i of the equilateral triangle T. Then, by Lemma 3.5, $E \cup \tilde{T} \subset co_R(E)$; as the complementary of $E \cup \tilde{T}$ is $D^c \cup_j B(o_j,R)$, union of open balls of radius R, then $E \cup \tilde{T}$ is an R-body, that is

$$co_R(E) = E \cup \tilde{T}$$

which is a disconnected R-body. □

The previous example can be modified to get a simply connected set E_* such that $co_R(E_*)$ is disconnected. Let us consider $E_* = E \cap W^c$, where W is a small strip from $\partial B(o_1,R)$ to $\partial D(4,R)$. Clearly $co_R(E_*) = co_R(E)$ is disconnected and E_* is a simply connected set.

5. R-hulloid of the vertices of a simplex in $\mathbb{R}^d$

Definition 5.1. Let $d \geq 2$, $1 \leq n \leq d$. Let $\{v_1,\ldots,v_{n+1}\} \subset \mathbb{R}^d$ be a family of affinely indipendent points and let $V = \{v_1,\ldots,v_{n+1}\} \subset \mathbb{R}^d$. An n-simplex is the set $T = co(V)$.

Let $T = co(V)$; the $(d-1)$-simplexes $T_i = co(V \setminus \{v_i\}), (i = 1,\ldots,d+1)$ are called the facets of T. If V lies on a sphere, centered in $Lin(T)$, and its points are equidistant, then T will be called a regular simplex.

It is well known the following fact: let V the set of the vertices of a d-simplex T in $\mathbb{E}^d$. There exists a unique open ball $B(V)$ such that all the vertices in V belong to $\partial B(V)$, called the circumball to $co(V)$. Let us notice that $D(V) = (B(V))^-$ does not coincide (in general) with the closed ball of minimum radius containing V, as an obtuse isosceles triangle shows.

Definition 5.2. Let $1 < n \leq d$; if T is a n-simplex, the circumcenter $c(T)$ and the circumradius $r(T)$ are the center and the radius respectively, of the unique open ball $B(c(T),r(T))$, called circumball of T, such that: i) $c(T) \in Lin(T)$; ii) $\partial B(c(T),r(T)) \supset V$.

Let us denote

$$r(V) :\equiv r(co(V)), c(V) :\equiv c(co(V)), B(V) :\equiv B(c(V), r(V)).$$

From Theorem 3.10 it follows that

Corollary 5.3. If $r(V) \geq R$ then

$$co_R(V) = V. \tag{11}$$

Definition 5.4. Let $R > 0$. The R-hulloid of V will be called full if its interior is not empty.

If $d = 2$, let V be the set of the vertices of a triangle with circumradius less than R; by Theorem 4.2, $co_R(V)$ is full.

5.1. Examples of R-hulloid of the vertices of a simplex in $\mathbb{E}^d$

Convex sets on $\mathcal{S}^{d-1}$ have been studied in [14]. Here properties of regular simplexes on $\mathcal{S}^{d-1}$ are recalled and used. If S is a regular simplex, centroid and circumcenter coincide.

Lemma 5.1. *Let $d > 1, R_0 > 0, \Sigma_0 := \partial B(0, R_0)$ in $\mathbb{E}^d$. Let $W = \{k_1, \ldots, k_{d+1}\} \subset \Sigma_0$ be the set of the vertices of a regular d-simplex S on Σ_0. Then*

$$< k_i, k_j >= -R_0^2/d, \quad i \neq j \tag{12}$$

and

$$|k_i - k_j| = \sqrt{2\frac{d+1}{d}} R_0. \tag{13}$$

Let $W_i = W \setminus \{k_i\}$ and let $\Sigma_i \subset \Sigma_0$ be the $(d-1)$-dimensional sphere through the points of W_i. Then Σ_i has center $-k_i/d$; moreover $\forall p \in \Sigma_0$

$$\textit{the spherical distance on } \Sigma_0 \textit{ from } \; p \textit{ to } \; W \; \textit{ is less or equal to } \; R_0 \arccos 1/d. \tag{14}$$

Proof. As the centroid of S is 0, then

$$\sum_{i=1}^{d+1} k_i = 0, \quad |k_i|^2 = R_0^2, \quad < k_i, k_j >= R_0^2 \cos\phi \quad (i, j = 1, \ldots, d+1), i \neq j$$

and

$$0 =< k_j, \sum_{i=1}^{d+1} k_i >= (R_0)^2 + d\,(R_0)^2 \cos\phi \quad (j = 1, \ldots, d+1).$$

Therefore $\cos\phi = -\frac{1}{d}$; so (12) and (13) hold.

As $S_i = co(W_i)$ is an equilateral $(d-1)$-simplex, the centroid of S_i will be $\frac{1}{d}\sum_{j\neq i} k_j = -k_i/d$ and coincides with the center of Σ_i. Let $\tilde{F}_j$ the spherical $(d-1)$-dimensional ball on Σ_0 of center $-k_j$ bounded by Σ_j. Then $\tilde{F}_j$ has spherical radius

$$R_0 \arccos \frac{\langle -k_i, k_j\rangle}{R_0^2} = R_0 \arccos 1/d.$$

As $\cup_{j=1}^{d+1} \tilde{F}_j = \Sigma_0$ the thesis follows. □

Theorem 5.5. Let $d > 2$ and let S be the regular simplex introduced in Lemma 5.1; let $R = \frac{d}{2}R_0$. Then the set W of its vertices is not an R-body and $co_R(W) = W \cup \{0\}$ is not full.

Proof. Let $B(o_j, \rho_j)$ with the property that

$$\partial B(o_j,\rho_j) \supset \{0, k_1, \ldots, k_{j-1}, k_{j+1}, \ldots, k_{d+1}\}.$$

Clearly $o_j = -\lambda k_j, (\lambda > 0)$. As $|o_j - 0|^2 = |o_j - k_i|^2, i \neq j$ then

$$(\lambda R_0)^2 = (\lambda R_0)^2 + (R_0)^2 + 2\lambda (R_0)^2 \cos\phi,$$

therefore $\lambda = \frac{d}{2}$, $o_j = -\frac{d}{2}k_j$ and $\rho_j = |o_j - 0| = \frac{dR_0}{2} = R$.

From (13) it follows

$$|o_i - o_j| = 2R\sqrt{\frac{1}{2} + \frac{1}{2d}}, \quad j \neq i. \tag{15}$$

Claim $\mathcal{Q}$: *Let* $R - R_0 < |z| \leq R$, $Q_z :\equiv B(0,R_0) \cap B(z,R)$. *Then* $\partial Q_z \cap \Sigma_0$ *is a spherical* $(d-1)$ *dimensional ball on* Σ_0 *of radius* r. *If* $|z| < R$ *then*

$$r > R_0 \arccos 1/d.$$

Proof: let $v = z/|z|$, the family of $Q_{\lambda v}$ is ordered by inclusion for $R - R_0 < \lambda \leq R$, with minimum set for $\lambda = R$; for $\lambda = R$ the spherical $(d-1)$ dimensional ball $\partial Q_{R_0 z/|z|}$ has radius $R_0 \arccos 1/d$.

If $R - R_0 < |z| < R$, then from Claim $\mathcal{Q}$ and (14), any open ball $B(z,R)$, which contains the point 0 contains at least one of the vertices $k_i, i = 1, \ldots, d+1$. As $0 \notin W$ the set W is not an R-body. Moreover since

$$(W \cup \{0\})^c = \bigcup_{j=1}^{d+1} B(o_j, R) \bigcup (co(W))^c,$$

then $W \cup \{0\}$ is an R-body containing W; then $W \cup \{0\}$ is the R-hulloid of W and it has empty interior. □

Theorem 5.6. In $\mathbb{E}^3$ there exist sequences of R-bodies with limit, in the Hausdorff metric, a body that is not an R-body.

Proof. Let us use the notations of Lemma 5.1 in the special case $d = 3$.

Let $k_i, i = 1,\dots,4$ the vertices of a regular simplex in $\mathbb{E}^3$ on the sphere $\Sigma_0 := \partial B(0,R_0)$, $R_0 = \frac{2R}{3}$.

For any fixed $i = 1,\dots 4$ the vertices $k_j, j \neq i$ belong to the boundary of the ball $B(o_i,R)$, with $o_i = -\frac{3}{2}k_i$.

From (12) it follows that

$$< o_j, k_i >= \frac{2}{9}R^2, \quad i \neq j, \quad i,j = 1,\dots,4.$$

Let $\varepsilon \to 0^+$ and let $x_i^{(n)} = k_i + \varepsilon_n \frac{k_i}{|k_i|}, i = 1,\dots,4$. The points $x_i^{(n)}$ are the vertices of a regular simplex $T^{(n)}$ in $\mathbb{E}^3$. For $i \neq j$ let $R_n = |o_i - x_j^{(n)}|$, then

$$R_n^2 = R^2 + \varepsilon_n^2 + 2 < k_i - o_j, k_i/|k_i| > \varepsilon_n = R^2 + \varepsilon_n^2 + \frac{2}{3}R\varepsilon_n > R^2.$$

For all $n \in \mathbb{N}$ let

$$W^{(n)} := \{x_1^{(n)},\dots,x_4^{(n)}\} = T^{(n)} \cap (\cup_{i=1}^4 B(o_i,R_n))^c.$$

As the complementary of the union of open balls of radius greater than R is an R-body and $T^{(n)}$ is convex then $V^{(n)}$ is an R-body too. The limit of $W^{(n)}$ is $W = \{x_1,\dots,x_4\}$ which is not an R-body as proved in Theorem 5.5. □

Theorem 5.7. Let $d \geq 3$; in $\mathbb{E}^d$ there exist connected bodies in a ball of radius $\sqrt{2}R$ with disconnected R-hulloid.

Proof. Let us consider the regular simplex S in $\mathbb{E}^d$, defined in Theorem 5.5, with vertices on $\Sigma_0 := \partial B(0,R_0)$, $R_0 := \frac{2R}{d}$.

The $(d-2)$ spherical surface $L_{i,j} :\equiv \partial B(o_i,R) \cap \partial B(o_j,R)$, $i \neq j$, has center at $\frac{o_i+o_j}{2}$ and contains 0. Then, by (15), $L_{i,j}$ has radius

$$|(o_i+o_j)/2| = \sqrt{R^2 - R^2(\frac{1}{2}+\frac{1}{2d})} = R\sqrt{\frac{1}{2}-\frac{1}{2d}}.$$

Then, the maximum distance of $L_{i,j}$ from 0 is

$$2R\sqrt{\frac{1}{2}-\frac{1}{2d}} < \sqrt{2}R.$$

Let $D := (B(0,\sqrt{2}R))^-$ and let

$$E := D \cap \left(\bigcup_{j=1}^{d+1} B(o_j,R) \cup \{0\} \right)^c. \tag{16}$$

Claim 1: *E is connected.*

First let us consider the $(d-1)$ spherical balls $U_i = B(o_i) \cap \partial B(0,\sqrt{2}R)$ centered at $c_i = \sqrt{2}o_i$. As $0 \in \partial B(o_i,R)$, then by elementary geometric arguments, the spherical radius of U_i is $\frac{\pi}{4}\sqrt{2}R$. By (15), the spherical distance between o_i and o_j on $\partial B(0,R)$ is

$$2R \arcsin \sqrt{\frac{1}{2} + \frac{1}{2d}} > \frac{\pi}{2}R.$$

Then, the spherical distance between c_i and c_j is greater than $\frac{\pi}{2}\sqrt{2}R$. Since the $(d-1)$ spherical balls U_i have radius $\frac{\pi}{4}\sqrt{2}R$, they are disjoints and

$$\mathcal{E} = \partial B(0,\sqrt{2}R) \setminus \cup_{i=1}^{d+1} S_i$$

is a connected subset of ∂E. Let us consider now $x \in E$, then $x \notin B(o_i,R)$; since $0 \in \partial B(o_i,R)$, then $\lambda x \notin B(o_i)$ for $\lambda \geq 1$. Therefore the segment connecting x to $\sqrt{2}\frac{x}{|x|}R \in \mathcal{E}$ is a subset of E. Claim 1 follows.

Claim 2: *E^c is an open set.*

As

$$E^c = D^c \cup \left(\bigcup_{j=1}^{d+1} B(o_j,R) \cup \{0\} \right),$$

it is enough to show that $\{0\} \subset Int(E^c)$. This follows from the fact that $\{0\}$ is in the interior of the simplex S, and $Int(S) \subset E^c$.

Claim 3: *The set of the vertices of S is contained in E .*

For each i the vertex $k_i \in \partial B(o_j,R), j \neq i$ and $k_i \notin B(o_i,R)^-$.

E is a closed set from Claim 2; from Claim 3 and (16) it follows that E is not an R-body, since any open ball of radius R, containing $0 \in E^c$, cannot be contained in E^c.

Claim 4: *The point* 0 *has a positive distance from E.*

Let us consider for $i = 1,\dots,d+1$ the simplexes

$$S_i = co(\{0,k_1,\dots,k_{i-1},k_{i+1},k_{d+1}\}).$$

Then $S = \cup_i S_i$. Let $0 < \varepsilon <$ dist $(0,S_i)$, where S_i are the facets of S; as $B(0,\varepsilon) \subset \cup_i B(0,\varepsilon) \cap S_i$, then

$$\text{dist } (0,E) \geq \varepsilon.$$

Let us consider now the body $E \cup \{0\}$. Since

$$(E \cup \{0\})^c = D^c \cup \left(\bigcup_{j=1}^{d+1} B(c_j, R) \right),$$

then $E \cup \{0\}$ is by definition an R-body and is the minimal R-body containing E. Then $co_R(E) = E \cup \{0\}$ which is a not connected set, since is the union of two closed disjoint sets. □

6. R-bodies and other classes of bodies

In Remark 3.3 it is noticed that the class of R-bodies contains the class of bodies which have reach greater or equal than R.

The following class has been introduced in [7]: the class $\mathcal{K}_2^{1/R}$ of bodies A satisfying the following property:

$$\forall x \in A^c \text{ there exists a closed ball } D(R) \ni x : D(R) \cap Int(A) = \emptyset. \tag{17}$$

Theorem 6.1. The following strict inclusion holds:

$$R\text{-bodies} \subsetneq \mathcal{K}_2^{1/R}. \tag{18}$$

Moreover let $A \in \mathcal{K}_2^{1/R}$ and $A = (Int(A))^-$, then:
i) if $d = 2$, then A is an R-body;
ii) if $d > 2$, then A can be not an R-body.

Proof. The inclusion (18) is obvious: since if A is an R-body and $x \in A^c$, then $x \in B(R)$ and $B(R) \cap A = \emptyset$; therefore $\partial B(R) \cap Int(A) = \emptyset$. Then, if $x \in D(R) = \partial B(R) \cup B(R)$ thus $D(R) \cap Int(A) = \emptyset$. The inclusion is strict: let $E = D(0,r) \cap B(0,R)^c \cup \partial B(0,r_1)$, with $r_1 < R < r$. Then E is not an R-body as if $x \in B(0,R) \setminus \partial B(0,r_1)$ there is no ball $B \subset E^c$ containing x; on the other hand $E \in \mathcal{K}_2^{1/R}$.

Let $d = 2$ and $A \in \mathcal{K}_2^{1/R}$, $A = (Int(A))^-$. By contradiction, if A is not an R-body, then, by Theorem 4.5, there exist $a_1, a_2, a_3 \in A$ such that there exists $z \in co_R(\{a_1,a_2,a_3\}) \cap A^c$. Since $z \neq a_i, i = 1,2,3$, then $co_R(\{a_1,a_2,a_3\})$ strictly contains $\{a_1,a_2,a_3\}$; by Corollary 5.3 with $V = \{a_1,a_2,a_3\}$, it follows that $r(V) < R$. Thus by Theorem 4.2, it follows that

$$co_R(V) = V \cup \tilde{T}.$$

$\tilde{T}$ is a curvilinear triangle with $(int(\tilde{T}))^- = \tilde{T}$. Since $z \in \tilde{T} \cap A^c$ and A^c is open, then there exists $\tilde{z} \in Int(\tilde{T}) \cap A^c$. As

$$\tilde{z} \in Int(\tilde{T}) \subset int(co_R(V),$$

every ball $D(R) \ni \tilde{z}$ contains at least one of the vertices a_i in its interior, let a_1. Then $D(R)$ contains a neighborhood U of $a_1 \in A$. Since $A = (Int(A))^-$, a_1 can not be an isolated point of A, and in U there are points of $int(A)$. Therefore property (17) does not hold for $\tilde{z} \in A^c$ and $A \notin \mathcal{K}_2^{1/R}$, contradiction.

In case ii), let us consider the set E defined by (16) of Theorem 5.7. E is not an R-body but $E \cup \{0\}$ is it. Then any point of E^c, different from 0 satisfies property (17); moreover

$$Int(E) = Int(D) \cap_{j=1}^{d+1} D(o_j, R)^c \cap \{0\}^c,$$

then 0 satisfies property (17) too, since the closed ball $D(o_1, R)$ does not intersect $Int(E)$. Then $E \in \mathcal{K}_2^{1/R}$ and E is not an R-body. Moreover it easy to see that $E = (Int(E))^-$. □

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