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AN APPLICATION OF PHRAGMÉN-LINDELÖF'S THEOREM IN AN ABSTRACT ELLIPTIC EQUATION OF SECOND ORDER

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In this paper, we prove, that Theorem of Phragmén-Lindelöf's can be applied in abstract elliptic equation of second order.

1.

Consider the differential equation

$$Lu \equiv -\frac{d^2u}{dt^2} + B\frac{du}{dt} + Cu \tag{1}$$

defined on $[0,\infty)$, where $u(t)$, the solution of (1), takes values in the complex Hilbert space H, and B and C are self-adjoint linear operators with dense domains in H. We suppose that $C \geq \delta I$, where δ is a constant and I is the identity operator. Suppose that equation (1) is of elliptic type:

$$B^2 \leq \theta^2 C, \tag{2}$$

where $0 \leq \theta < 2$. This means that

$$D(B^2) \subset D(C) \quad \text{and} \quad (B^2x,x) \leq \theta^2(Cx,x), \text{for all} \quad x \in D(C)$$

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hold true. Respectively $(Lu,u) \geq 0$, for u of compact support. Let bounded conditions given by

$$u(0) = u_0, u(\infty) = 0. \tag{3}$$

Equation (1) is solved in $(0,\infty)$. Condition (2) can be written in the form $\circ H$

$$\left\| BC^{-\frac{1}{2}} \right\| \leq \theta; \tag{4}$$

i.e.,

$$||Bx|| \leq \theta ||C^{-\frac{1}{2}}x||, \text{for each} \quad x \in D(C). \tag{4'}$$

By substituting

$$u(t) = v(t) + e^{-t}u_0,$$

equation (1) with conditions (3) is transformed into a nonhomogeneus equation with homogeneous conditions:

$$-\frac{d^2v}{dt^2} + B\frac{dv}{dt} + Cv = f(t) \equiv e^{-t}(u_0 + Bu_0 - Cu_0); v(0) = 0, u_0 \in D(C).$$

Thus, a homogeneous problem with nonhomogeneous conditions is transformed into a nonhomogeneous problem with homogeneous conditions. Define the space H_2 by

$$H_2 = H_2(0,\infty;H) = \left\{ u(t) \in D(C) : \int_0^\infty \left[\left\| \frac{d^2u}{dt^2} \right\|^2 + ||Cu||^2 \right] dt < \infty \right\} \tag{$*$}$$

and the inner product in it by

$$(f,g) = \int_0^\infty \left[(f'',g'') + (f,Cg) \right] dt.$$

Denote by H_2° the adherence of the set of all smooth continuous functions with compact support $u(t) \in D(C)$ for almost all $t \in (0,\infty)$, in the sense of the norm of the space H_2. That the problem

$$Lu = f, \text{for} f \in L_2(0,\infty)$$

is solvable in the space H_2°, it is shown by the following theorem.

Theorem 1.1. *Let L be an elliptic operator of second order satisfying the condition (4). We consider the boundary-value problem*

$$Lu = f \in L_2.$$

The above problem has the unique smooth solution, for each $f \in L_2$; i.e. there exists $u \in H_2^{\circ}$ such that

$$Lu = f.$$

Moreover, the mapping $u \to Lu$, for $u \in H_2$, $Lu \in L_2$ is a homeomorphic correspondence between the space H_2° and the entire space L_2.

Proof. See [7] □

2.

Definition 2.1. The problem with initial conditions (3) is correctly posed if it satisfies the following properties:

1. Solutions are uniquely determined by initial elements;
2. The set D of all initial elements of solutions is dense in the Banach space E ;
3. For every finite segment $[0,T]$, there exists a constant $M = M(T)$ such that every solution satisfies the following inequality

$$||u(t)|| \leq M||u(0)||, \text{for} \quad 0 \leq t \leq T. \tag{5}$$

Definition 2.2. For every $u_0 \in E$ (even if $u_0 \notin D$) the function

$$u(t) = U(t)u_0 \tag{6}$$

is called *generalised solution* of a problem with an initial condition, defined by the initial element u_0. The operator $U(t)$ is called *resolute (generalized) operator* of a problem with an initial condition.

The family of linear operators $\{U(t) : t \geq 0\}$ is a semi-group, see [8].

Theorem 2.3. *The solution of a problem (1) with conditions (3) and (4) is bounded. Moreover*

$$||u(t)|| = ||U(t)u_0||_H \leq ||u_0||. \tag{7}$$

Proof. See [8] □

Considering Definition 1.1 we conclude that the problem (1) with initial conditions (3) is correctly posed. The semi-group $\{U(t) : t \geq 0\}$ of resolute operators of the correctly posed problem (1) with initial condition (3) has the following properties:

1. $U(t)U(s) = U(s)U(t)$ for $t, s \geq 0$.
2. $U(0) = I$
3. The semi-group $U(t)$ is strongly continuous.
4. The semi-group $U(t)$ is bounded in $[0, \infty)$ (since $||U(t)|| \leq 1$ for every $t \in [0, \infty)$).

Therefore, by the Hille-Yosida-Phillips Theorem the equation (1) can be written in the following form

$$\frac{du}{dt} = Au, \tag{1'}$$

where A is the generator of the semi-group $U(t)$, for $t \geq 0$, of resolute operators of correctly problem (1) with initial conditions (3).

Theorem 2.4. *If $\theta < \sqrt{2}$, then the solution $v \in H_2^\circ$ of the equation*

$$-\frac{d^2v}{dt^2} + B\frac{dv}{dt} + Cv = f, for \quad f \in L_2, \tag{8}$$

can be obtained by the method of successive approximations for every $f \in L_2$ (starting from the solution of the equation $L_0 v = f$). The solution is given by

$$v = (I + L_0^{-1}BD)^{-1}L_0^{-1}f = \sum_{n=0}^{\infty}(-1)^n(L_0^{-1}BD)^n L_0^{-1}f,$$

where D is the differential operator $Dv = v^{'}$.

Proof. See [8] □

3.

The classical Phragmén-Lindelöf principle can be formulated as follows. Let $u(x,t)$ denote a harmonic function in the half-strip $0 \leq x \leq l, t \geq 0$, vanishing on the semi-infinite lines which form the boundary; i.e.,

$$u(0,t) = u(l,t) = 0, \text{for} \quad t \geq 0.$$

If such a function is bounded in the half-strip, then it decreases exponentially. Our goal is to investigate if this result could be extended for $t \geq 0$ an abstract function for which satisfies equation

$$Lu = 0,$$

where L is an elliptic operator of second order with coefficients independent of the variable t, and the function u satisfies the Dirichlet zero-boundary conditions $u(0) = 0$, for $t > 0$. The set of solutions of an elliptic equation of the kind considered above forms a linear space of vector-valued functions. This linear space is *translation invariant* with respect to t, since the coefficients of the differential operator in question, as well as the boundary conditions, are independent of t. The elliptic character of the operator guarantees the smoothness of solutions in the interior of their domain of definition. This property is in fact the *principle of interior compactness*. In [4] (Theorem 1.1) it is proved that elements of a translation invariant interior compact subspace satisfy the Phragmén-Lindelöf principle; i.e. if for an element u of such a space the integral $\int_0^\infty ||u(t)||e^{\alpha t}dt$ is finite, then already $\int_0^\infty ||u(t)||e^{\alpha t}dt$ is also finite, where β is a constant greater than α, depending on α but not on the function u. An interesting feature of this theorem is that it relates the behavior of functions at infinity to their behavior over a finite range. A number of results follow from this theorem, such as the asymptotic expansion for each element into Fourier series over exponentials; the belonging space, the unique continuation property and an abstract generalization of a theorem of Weinstein. This results are equivalent to the discreteness of the spectrum of the translation operator. Let S be a linear space of functions $u(t)$ for the positive reals, $t > 0$, whose functional values belong to a Banach space E. Assume that S is translation invariant i.e., that if $u(t)$ lies in S, so does $u(t+\eta)$ for η positive, and that elements of S belong to L_2 over every subinterval of the positive real axis. Introduce the notation

$$||u||_a^b = \left\{ \int_a^b \left(\left\| \frac{d^2u}{dt^2} \right\|^2 + ||Cu||^2 \right)^2 dt \right\}^{\frac{1}{2}}, 0 < a < b < \infty,$$

where $||u(t)||$ stands for the Banach norm of $u(t)$ defined in H_2 by relation (*).

Definition 3.1. A space S is called *interior compact* if the unit sphere with respect to the norm $||u||_a^b$ is precompact with respect to the norm $||u||_{a'}^{b'}$ whenever $a < a' < b' < b$.

By precompact we mean that every bounded sequence with respect to the norm $||u||_a^b$ contains a Cauchy sequence with respect to the norm $||u||_{a'}^{b'}$. The completion of an interior compact space S under L_2 convergence in every finite subinterval of the real axis is still interior compact. Therefore we shall assume in what follows that S is already closed in this topology.

Theorem 3.2. *([4])(Abstract Phragmén-Lindelöf Principle). Let S be a translation invariant interior compact space; then there exists a positive number α such that for all elements of S, for which $\int_0^\infty \|u(t)\|^2 dt$ converges, the integral $\int_0^\infty \|u(t)\|^2 e^{at} dt$ also converges.*

Theorem 3.3. *([4], page 364, Theorem 1.1) The Phragmén-Lindelöf principle is still valid if the space is interior compact with respect to a single pair of intervals.*

4.

In this paper we shall show (Theorem 4.3) that Theorem 3.2 is applicable to the space of solutions of an elliptic equation of second order for $t > 0$. First we provide two Lemmas whose proofs are given in [6].

Lemma 4.1. *Let K be closed bounded set in R^n , which is a subset of an open set G. Then, the distance d between K and G is positive.*

Lemma 4.2. *If K is closed bounded set in R^n, which is a subset of an open set G, then there exists a function Ψ (from C_0^∞) such that:*

1. $\Psi(x) = 1$ *, for* $x \in K$ *;*
2. *support of function* Ψ *lies in G;*
3. $0 \le \Psi(x) \le 1$ *for all* x *.*

This variant for C^∞ is a special case of Urison's Lemma.

We will prove that the solution of equation (1) with conditions (2) and (3) satisfies the principle of interior compactness in the context of Definition 3.1.

Theorem 4.3. *(Principle of interior compactness). Let $G = (a,b)$ be an arbitrary interval and $G' = (a', b')$ subinterval its with no common boundary points; i.e., $G' \subset G$. Then, the set of solutions of equation (1) with conditions (2) and (3) satisfies the condition $\|u\|_G \le 1$ of precompactness the norm $\|u\|_{G'}$.*

Proof. Let $\varphi(t) \in C_0^\infty(G)$, $\varphi(t) = 1$ in the neighborhood of G' , where such function φ exists, based on lemma 4.2. Then

$$\tilde{u}(t) = (\varphi(t)u(t))/_{G'}.$$

We begin with the inequality

$$\int_a^b \|L\varphi u\|^2 dt \geq M_1 \int\limits_a^b \left(\|\frac{d^2(\varphi u)}{dt^2}\|^2 + \|C\varphi u\|^2 \right) dt,$$

or, having in mind the assumptions

$$\int\limits_a^b \|L\varphi u\|^2 dt \geq M_1 \int\limits_{a'}^{b'} \left(\|\frac{d^2 u}{dt^2}\|^2 + \|Cu\|^2 \right) dt, \tag{9}$$

Further

$$\begin{aligned} L\varphi u &= -\frac{d^2(\varphi u)}{dt^2} + B\frac{d(\varphi u)}{dt} + C\varphi u \\ &= -\varphi \frac{d^2 u}{dt^2} - 2\varphi' \frac{du}{dt} - u\varphi'' + \varphi B \frac{du}{dt} + \varphi' Bu + \varphi Cu \\ &= \varphi \left(\frac{d^2 u}{dt^2} + B\frac{du}{dt} + Cu \right) - 2\varphi' \frac{du}{dt} - u\varphi'' + \varphi' Bu. \end{aligned}$$

Since the function u is a solution of equation (1), i.e., $Lu = 0$, we have

$$L\varphi u = -2\varphi' \frac{du}{dt} - u\varphi'' + \varphi' Bu.$$

Having in mind (9), we obtain

$$\int_{a'}^{b'} \left(\left\| \frac{d^2 u}{dt^2} \right\|^2 + ||Cu||^2 \right)^2 dt \leq \int_a^b \left\| -2\varphi' \frac{du}{dt} - \varphi'' u + \varphi' Bu \right\|^2 dt \leq$$

$$\left\{ 2\int_a^b |\varphi'(t)|^2 \left\| \frac{du}{dt} \right\|^2 dt \right\}^{\frac{1}{2}} + \left\{ \int_a^b |\varphi'(t)|^2 \|Bu\|^2 dt \right\}^{\frac{1}{2}} + \left\{ \int_a^b |\varphi''(t)|^2 \|u\|^2 dt \right\}^{\frac{1}{2}}.$$

Based on ellipticity $(4')$ and by putting

$$M_2 = \max\{\max|\varphi(t)|, \max|\varphi'(t)|, \max|\varphi''(t)|\},$$

we obtain

$$\int_{a'}^{b'} \left(\left\| \frac{d^2 u}{dt^2} \right\|^2 + ||Cu||^2 \right) dt \leq M_3 \int_a^b \left(\left\| \frac{du}{dt} \right\|^2 + ||C^{\frac{1}{2}} u||^2 \right) dt. \tag{10}$$

Based on assumptions $||u||_G \leq 1$,

$$\int_a^b \left(\left\| \frac{du}{dt} \right\|^2 + ||C^{\frac{1}{2}} u||^2 \right) dt \leq 1,$$

and having in mind (10), we have

$$\int_{a'}^{b'} \left(\left\| \frac{d^2u}{dt^2} \right\|^2 + ||Cu||^2 \right) dt \leq M_4. \tag{11}$$

But, set (9) is compact in the metric

$$\int_{a'}^{b'} \left(\left\| \frac{du}{dt} \right\|^2 + ||C^{\frac{1}{2}} u||^2 \right) dt.$$

In other words, we need to show that a set $\{u(t)\}$ such that

$$\int_a^b \left(\left\| \frac{du}{dt} \right\|^2 + ||C^{\frac{1}{2}} u||^2 \right) dt \leq 1,$$

is compact in the metric

$$\int_{a'}^{b'} \left(\left\| \frac{du}{dt} \right\|^2 + ||C^{\frac{1}{2}} u||^2 \right) dt, \tag{12}$$

where $(a', b') \subset (a, b)$.

We may assume $(a', b') \subset (-\pi, \pi)$. In the space $L_2(-\pi, \pi; H)$ we obtain the system

$$e_{n,m} = e^{int} e_m,$$

where

$$Ce_m = \lambda_m e_m; \lambda_m \to \infty, \text{because} \quad 0 < C^{-1} \in \sigma_\infty.$$

The system $e_{n,m}$ is totally orthonormed:

$$u(t) = \sum_{n,m} a_{n,m} e^{int} e_m; a_{n,m} = \frac{1}{2\pi i} \int_{-\pi}^{\pi} (u(t), e_m) e^{int} dt.$$

So, it is proved that from (12) it follows that

$$\int_{a'}^{b'} \left(\left\| \frac{d^2u}{dt^2} \right\|^2 + ||Cu||^2 \right) dt \leq M.$$

Then,

$$u'' = \sum_{n,m} n^2 a_{n,m} e^{int} e_m ; Cu = \sum_{n,m} \lambda_m a_{n,m} e^{int} e_m,$$

or

$$\int_{-\pi}^{\pi} \left(\left\| \frac{d^2u}{dt^2} \right\|^2 + ||Cu||^2 \right) dt = \sum_{n,m} (n^4 + \lambda_m^2) a_{n,m}^2 \leq M. \tag{13}$$

We should prove that set (13) is compact in metric

$$\sum_{n,m} (n^2 + \lambda_m) a_{n,m}^2.$$

We evaluate "the tail" $\sum_{n,m} (n^4 + \lambda_m^2) a_{n,m}^2$ uniformly of $a_{n,m}$. Indeed,

$$N \geq \sum_{n,m>N} (n^4 + \lambda_m^2) a_{n,m}^2 \geq \sum_{n,m>N} (N^2 n^2 + \lambda_N \lambda_n) a_{n,m}^2 \geq$$

$$\min(N, \lambda_N) \sum_{n,m>N} (n^2 + \lambda_m) a_{n,m}^2 \Rightarrow \sum_{n,m>N} (n^2 + \lambda_m) a_{n,m}^2 \leq \frac{M}{\min(N, \lambda_N)} < \varepsilon.$$

This is the compactness, since

$$\sum_{n,m \leq N} (n^2 + \lambda_m) a_{n,m}^2 \leq 1$$

is compact, and further based on the Hausdorf theorem on existence of finite ε- lattices. □

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