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A LIAPUNOV FUNCTION FOR THE INITIAL-BOUNDARY VALUE PROBLEM MODELLING THE MICROWAVE HEATING AND ITS CONSEQUENCES ON THE FORMATION OF HOT-SPOTS

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We prove that if the electricconductivity σ is grater than the adsorpbidity q and the condition of perfect insulation holds on the boundary of the specimen heated, the functional

$$V = \int_{\Omega} \left[\frac{1}{2} \left(\varepsilon |\mathbf{E}|^2 + \mu |\mathbf{H}|^2 \right) + \theta \right] dx \tag{1}$$

is a Liapunov function for the initial boundary value problem modelling the microwave heating. If σ and q are constants the formation of hotspots is impossible.

1. Introduction

Microwave heating is used in many applications, the most popular being certainly the cooking of foods. The mathematical modelling of the process is based on the Maxwell equations coupled with the heat equation. This leads to the following initial-boundary value problem [8], [10], [11]

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$$\varepsilon\frac{\partial \mathbf{E}}{\partial t} = \nabla\times\mathbf{H} - \sigma\mathbf{E} \text{ in } \Omega \tag{2}$$

$$\mu\frac{\partial \mathbf{H}}{\partial t} = -\nabla\times\mathbf{E} \text{ in } \Omega \tag{3}$$

$$\nabla\cdot\mathbf{H} = 0 \text{ in } \Omega \tag{4}$$

$$\mathbf{E}\times\mathbf{n} = 0 \text{ on } \Gamma \tag{5}$$

$$\mathbf{H}\cdot\mathbf{n} = 0 \text{ on } \Gamma \tag{6}$$

$$\mathbf{E}(\mathbf{x},0) = \mathbf{E}_0(\mathbf{x}) \text{ in } \Omega \tag{7}$$

$$\mathbf{H}(\mathbf{x},0) = \mathbf{H}_0(\mathbf{x}) \text{ in } \Omega \tag{8}$$

$$\theta_t = \Delta\theta + q|\mathbf{E}|^2 \text{ in } \Omega \tag{9}$$

$$\frac{\partial\theta}{\partial\mathbf{n}} = 0 \text{ on } \Gamma \tag{10}$$

$$\theta(x,0) = \bar{\theta}(x) \text{ in } \Omega, \tag{11}$$

where Ω is an open and bounded subset of $\mathbf{R}^3$ representing the heated specimen, Γ designates the boundary of Ω and $\mathbf{n}$ the exterior pointing unit nornal vector to Γ. The electric conductivity is denoted by σ and q is the thermal absorptivity. The initial values $\mathbf{E}_0(x)$ and $\mathbf{H}_0(x)$ are supposed to satisfy the condition

$$\mathbf{E}_0(x)\times\mathbf{n} = 0 \text{ on } \Gamma \tag{12}$$

$$\mathbf{H}_0(x)\cdot\mathbf{n} = 0 \text{ on } \Gamma \tag{13}$$

$$\nabla\cdot\mathbf{H}_0(x) = 0 \text{ in } \Omega. \tag{14}$$

The problem is non-linear, since both the electric conductivity σ and the thermal absorptivity q are functions of the temperature θ. This fact makes the problem very complex. In particular the formation of hot-spots, i.e. of small regions of very high temperature is strictly related to the nonlinear character of the problem, see [4], [12], [5]. In this paper we consider a zero-order approximation of the nonlinear problem assuming σ and q to be two positive constants. This simplifies the problem, and has the advantage to put in evidence the role of the condition

$$\sigma > q \tag{15}$$

which is crucial to guarantee that the temperature remains bounded.. This is done in Section 5 via the introduction of a Liapunov function which exists only if the condition (15) is satisfied. The crucial role of the conductivity σ was stressed by N.F. Smyth in [16]. We also cite the related works [13], [14], [15] and [9].

The condition (10) is an ideal condition which one would like to have in order not to disperse the heat generated.

This paper is organised as follows. In Section 2 we prove that the initial-boundary value problem for the Maxwell equations with a constant conductivity has a unique solution. The semigroup theory is used together with the Lax-Milgram lemma. The stationary problem formally corresponding to (2)-(8) is considered in Section 3. A remark is made in Section 4 on the influence of condition (10) on the asymptotic behaviour of the solution of heat equation (9).

2. The Maxwell equations with a constant conductivity term

The proof of existence and uniqueness for the initial-boundary problem for the Mawell equations with a constant conductivity term can be found in [7]. To make this paper self-contained we present in this Section a proof which partly differs from [7]. It is convenient to consider first a problem with less conditions with respect to (2)-(8), i.e. the problem

$$\varepsilon\frac{\partial \mathbf{E}}{\partial t} = \nabla\times\mathbf{H} - \sigma\mathbf{E} \ \text{ in } \Omega \tag{16}$$

$$\mu\frac{\partial \mathbf{H}}{\partial t} = -\nabla\times\mathbf{E} \ \text{ in } \Omega \tag{17}$$

$$\mathbf{E}\times\mathbf{n} = 0 \text{ on } \Gamma \tag{18}$$

$$\mathbf{E}(\mathbf{x},0) = \mathbf{E}_0(\mathbf{x}) \ \text{ in } \Omega \tag{19}$$

$$\mathbf{H}(\mathbf{x},0) = \mathbf{H}_0(\mathbf{x}) \ \text{ in } \Omega. \tag{20}$$

We recall the formula of integration by parts

$$\int_\Omega \mathbf{B}\cdot(\nabla\times\mathbf{A})-\mathbf{A}\cdot(\nabla\times\mathbf{B})dx=\int_\Gamma \mathbf{B}\cdot(\mathbf{n}\times\mathbf{A})d\Gamma \tag{21}$$

valid if **A** and **B** belong to $H(curl,\Omega)$, (see [3] page 31). For the definition of this and the other Sobolev spaces used in this paper we refer to the Appendix. Let us define

$$\mathcal{H}=(L^2(\Omega))^3\times(L^3(\Omega))^3. \tag{22}$$

To emphasize the physical meaning of the various components of $\mathcal{H}$ we use the notation

$$\begin{bmatrix}\mathbf{E}\\ \mathbf{H}\end{bmatrix}=\begin{bmatrix}E_1 & E_2 & E_3\\ H_1 & H_2 & H_3\end{bmatrix}\in\mathcal{H}.$$

In $\mathcal{H}$ we define the scalar product

$$\begin{bmatrix}\mathbf{E}\\ \mathbf{H}\end{bmatrix}\cdot\begin{bmatrix}\bar{\mathbf{E}}\\ \bar{\mathbf{H}}\end{bmatrix}=\int_\Omega(\textstyle\sum_{i=1}^3 E_i\bar{E}_i+\sum_{i=1}^3 H_i\bar{H}_i)dx.$$

This makes $\mathcal{H}$ an Hilbert space. Let

$$\mathcal{D}(\mathcal{A})=\left\{H_0(curl,\Omega)\times H(curl,\Omega)\right\}\subset\mathcal{H}$$

be the domain of the operator

$$\mathcal{A}\begin{bmatrix}\mathbf{E}\\ \mathbf{H}\end{bmatrix}=\begin{bmatrix}\nabla\times H-\sigma\mathbf{E}\\ -\nabla\times\mathbf{E}\end{bmatrix}.$$

Lemma 2.1. *The operator $\mathcal{A}$ is dissipative.*

Proof. Using (21) and recalling that $\mathbf{E}\times\mathbf{n}=0$ on Γ we have

$$\left(\mathcal{A}\begin{bmatrix}\mathbf{E}\\ \mathbf{H}\end{bmatrix}\right)\cdot\begin{bmatrix}\mathbf{E}\\ \mathbf{H}\end{bmatrix}=\int_\Omega\left[(\nabla\times\mathbf{H}-\sigma\mathbf{E})\cdot\mathbf{E}-(\nabla\times\mathbf{E})\cdot\mathbf{H}\right]dx=-\sigma\int_\Omega|\mathbf{E}|^2dx\leq 0.$$

This means precisely that $\mathcal{A}$ is dissipative. □

Theorem 1. The problem (16)-(20) has one and only one solution.

Proof. We apply the Hille-Yoshida Theorem, see [2], page 101. We already know that $\mathcal{A}$ is dissipative. It remains to show that for every

$$\begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix} \in \mathcal{H}$$

the equation

$$\begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix} \in \mathcal{D}(\mathcal{A}), \qquad (I-\mathcal{A})\begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix}$$

or, in components,

$$(1+\sigma)\mathbf{E} - \nabla \times \mathbf{H} = \mathbf{F}, \quad \mathbf{E} \in H_0(curl, \Omega) \tag{23}$$

$$\mathbf{H} + \nabla \times \mathbf{E} = \mathbf{G}, \quad \mathbf{H} \in H(curl, \Omega) \tag{24}$$

has one and only one solution. Let us take formally the *curl* of (24). We have

$$\nabla \times \mathbf{H} = \nabla \times \mathbf{G} - \nabla \times (\nabla \times \mathbf{E}). \tag{25}$$

Substitute (25) in (23). We obtain

$$(1+\sigma)\mathbf{E} - \nabla \times \mathbf{G} + \nabla \times (\nabla \times \mathbf{E}) = \mathbf{F}. \tag{26}$$

Given the degree of regularity at our disposal (26) is meaningless. However, we can rewrite (26) in weak form. To this end, let us multiply (26) by a test function $\mathbf{W} \in H_0(curl, \Omega)$ and integrate by parts the resulting equation over Ω. Using (21) and recalling (18) we obtain the following variational problem for the determination of $\mathbf{E} \in H_0(curl, \Omega)$

$$\int_\Omega (1+\sigma)\mathbf{E} \cdot \mathbf{W} dx + \int_\Omega (\nabla \times \mathbf{E}) \cdot (\nabla \times \mathbf{W}) dx = \int_\Omega \mathbf{F} \cdot \mathbf{W} dx + \int_\Omega \mathbf{G} \cdot (\nabla \times \mathbf{W}) dx, \tag{27}$$

for all $\mathbf{W} \in \mathcal{H}_0(curl, \Omega)$, (27) is the weak form of equation (26) and could have been taken as starting point. We claim that (27) has one and only one solution. Indeed $H_0(curl, \Omega)$ is an Hilbert space if endowed with the scalar product

$$\int_\Omega (\nabla\times\mathbf{E})\cdot(\nabla\times\mathbf{W})dx+\int_\Omega \mathbf{E}\cdot\mathbf{W}dx,$$

where $\mathbf{E}\in H_0(curl,\Omega)$ and $\mathbf{W}\in H_0(curl,\Omega)$. In $H_0(curl,\Omega)\times H_0(curl,\Omega)$ we define the bilinear form

$$a(\mathbf{E},\mathbf{W})=\int_\Omega (1+\sigma)\mathbf{E}\cdot\mathbf{W}dx+\int_\Omega (\nabla\times\mathbf{E})\cdot(\nabla\times\mathbf{W})dx \tag{28}$$

which is coercive since

$$a(\mathbf{E},\mathbf{E})\geq\int_\Omega |\mathbf{E}|^2dx+\int_\Omega |\nabla\times\mathbf{E}|^2dx.$$

On the other hand, $a(\mathbf{E},\mathbf{W})$ is also bounded as can be easily verified. Moreover, the right hand side of (27) defines a linear and continuous functional in $H_0(curl,\Omega)$. Hence (27) has, by the Lax-Milgram Lemma, one and only one solution in $H_0(curl,\Omega)$. In a similar way we can obtain a variational equation determining $\mathbf{H}$. For, let us take formally the *curl* of (23). We obtain

$$\nabla\times\mathbf{E}=(1+\sigma)^{-1}\nabla\times(\nabla\times\mathbf{H})+(1+\sigma)^{-1}\nabla\times\mathbf{F}. \tag{29}$$

Substituting (29) in (24) we have

$$\mathbf{H}+(1+\sigma)^{-1}\nabla\times(\nabla\times\mathbf{H})+(1+\sigma)^{-1}\nabla\times\mathbf{F}=\mathbf{G}. \tag{30}$$

We rewrite (30) in weak form. To this end, let us multiply (30) by a test function $\mathbf{V}\in H_0(curl,\Omega)$ and then integrate by parts over Ω. Using (21) and recalling (18) we arrive at the problem: to find $\mathbf{H}\in H(curl,\Omega)$ such that

$$\int_\Omega \mathbf{H}\cdot\mathbf{V}dx+(1+\sigma)^{-1}\int_\Omega (\nabla\times\mathbf{H})\cdot(\nabla\times\mathbf{V})dx+ \tag{31}$$
$$(1+\sigma)^{-1}\int_\Omega \mathbf{F}\cdot(\nabla\times\mathbf{V})dx=\int_\Omega \mathbf{G}\cdot\mathbf{V}dx$$

for all $\mathbf{V}\in H_0(curl,\Omega)$. We can apply again the Lax-Milgram Lemma to (31) and in this way we determine $\mathbf{H}$. This proves the existence and uniqueness for problem (16)- (20) □

To have a theorem of existence and uniqueness for the problem (2)-(8) which, with respect to problem (16)- (20) contains two additional conditions we use the results of [6] page 356, where it is proved that the additional conditions (4), (6) are automatically verified if we assume (12), (13) and (14). Hereafter we always assume that the data satisfy the conditions (12), (13) and (14.

3. The stationary problem

Together with the initial-boundary value problem for the Maxwell equations with a constant conductivity term (2)-(8) it appears natural to consider as corresponding stationary problem the following: to find

$$\mathbf{E}(x) \in H_0(curl, \Omega), \quad \mathbf{H}(x) \in H(curl, \Omega) \cap H_0(div, \Omega),$$

such that

$$\nabla \times \mathbf{H} - \sigma \mathbf{E} = 0 \quad \text{in } \Omega \tag{32}$$

$$\nabla \times \mathbf{E} = 0 \quad \text{in } \Omega \tag{33}$$

$$\nabla \cdot \mathbf{H} = 0 \quad \text{in } \Omega \tag{34}$$

$$\mathbf{E} \times \mathbf{n} = 0 \quad \text{on } \Gamma \tag{35}$$

$$\mathbf{H} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma. \tag{36}$$

Theorem 2. Assume Ω to be simply connected and $\sigma > 0$. Then $\mathbf{E}(x) = 0$, $\mathbf{H}(x) = 0$ is the only solution of (32)-(36).

Proof. Let $\mathbf{E}(x)$, $\mathbf{H}(x)$ be a solution of (32)-(36). Multiply (32) by $\mathbf{E}$ and integrate over Ω. We have

$$\int_\Omega \mathbf{E} \cdot (\nabla \times \mathbf{H}) dx = \sigma \int_\Omega |\mathbf{E}|^2 dx. \tag{37}$$

By (21) and in view of (35) we have, by (33),

$$\int_\Omega \mathbf{E} \cdot (\nabla \times \mathbf{H}) dx = \int_\Omega (\nabla \times \mathbf{E}) \cdot \mathbf{H} dx = 0. \tag{38}$$

Hence by (37)

$$\sigma \int_\Omega |\mathbf{E}|^2 dx = 0. \tag{39}$$

Since $\sigma > 0$ we have $\mathbf{E}(x) = 0$. Recalling (32) we conclude that

$$\nabla \times \mathbf{H} = 0. \tag{40}$$

Since Ω is simply connected there exists a scalar potential $\psi(x)$ such that

$$\mathbf{H} = \nabla\psi. \tag{41}$$

But (34) **H** is also solenoidal, therefore

$$\Delta\psi = 0 \quad \text{in } \Omega. \tag{42}$$

By (36) and (41) we have

$$\frac{\partial\psi}{\partial\mathbf{n}} = 0 \quad \text{on } \Gamma. \tag{43}$$

All the constants are solutions of (42) (43) and we conclude, by (41), that $\mathbf{H}(x) = 0$. □

Remark 1. If we drop one of the hypoteses of Theorem 3.1 the situation is more complex and the solution of (32)-(36) need not to be unique.

4. Remark on the non-homogeneous heat equation with Neumann boundary condition

When in problem (2)-(8) the electric field $\mathbf{E}(x,t)$ is known via the results of Section 2 it remains to consider the problem

$$\theta_t = \Delta\theta + q|\mathbf{E}|^2 \quad \text{in } \Omega \tag{44}$$

$$\theta(x,0) = \bar{\theta}(x) \tag{45}$$

$$\frac{\partial\theta}{\partial\mathbf{n}} = 0 \quad \text{on } \Gamma. \tag{46}$$

The solution of problem (44)-(46) is well-known. In this Section we would like to put in evidence a consequence of the Neumann boundary condition (46) expressing the thermal insulation of the body. The problem (44)-(46) can be split in the following two problems

$$\frac{\partial\theta_1}{\partial t} = \Delta\theta_1 \quad \text{in } \Omega \tag{47}$$

$$\theta_1(x,0) = \bar{\theta}(x) \tag{48}$$

$$\frac{\partial\theta_1}{\partial\mathbf{n}} = 0 \quad \text{on } \Gamma \tag{49}$$

and

$$\frac{\partial \theta_2}{\partial t} = \Delta\theta_2 + q|\mathbf{E}|^2 \quad \text{in } \Omega \tag{50}$$

$$\theta_2(x,0) = 0 \tag{51}$$

$$\frac{\partial \theta_2}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma. \tag{52}$$

For the solution of (44)-(46) we have $\theta(x,t) = \theta_1(x,t) + \theta_2(x,t)$. We are interested in the behaviour for $t \to \infty$ of the solution of (47)-(49). To this end we construct the solution of (47)-(49) as a Fourier's development in series of the normalized eigensolutions $X_j(x)$ and eigenvalues λ_j of the Laplace operator with Neumann boundary condition i.e.

$$-\Delta X_j(x) = \lambda_j X_j(x) \quad \text{in } \Omega, \quad j = 0,1,2,.... \tag{53}$$

$$\frac{\partial X_j}{\partial \mathbf{n}} \quad \text{on } \Gamma \tag{54}$$

$$\int_\Omega X_i(x) X_j(x) dx = \delta_{ij}. \tag{55}$$

The existence of a complete orthonormal system of eigensolutions for problem (53)-(55) is a consequence of the spectral theory for self-adjoint operator. It is immediately verified that $\lambda_0 = 0$ is an eigenvalue having as correspondent eigensolutions all the constants. On the other hand, by normalization we immediately found the value of the normalized eigenvector X_0 to be

$$X_0 = \frac{1}{\sqrt{|\Omega|}}. \tag{56}$$

All other eigenvalues λ_j with j greater than 0 are positive as can be seen if we multiply (53) by $X_j(x)$ and integrate by parts over Ω. Taking into account (54) we have

$$\int_\Omega |\nabla X_j(x)|^2 dx = \lambda_j \int_\Omega |X_j(x)|^2 dx \quad j = 1,2,... \tag{57}$$

This implies $\lambda_j > 0$ if $j > 0$. If we write the Fourier development of $\theta_1(x,t)$ in series of eigensolutions of (53), (54) we find, taking into account (56),

$$\theta_1(x,t) = \frac{1}{|\Omega|}\int_\Omega \bar{\theta}(\xi)d\xi + \sum_{j=1}^{\infty}\left[\int_\Omega \bar{\theta}(\xi)X_j(\xi)d\xi\right]X_j(x)e^{-\lambda_j t}. \tag{58}$$

In (58) all the terms, with the exception of the first one, tend uniformly to zero as $t \to \infty$ since $\lambda_j > 0$ for $j > 0$. Thus

$$\lim_{t\to\infty} \theta_1(x,t) = \frac{1}{|\Omega|}\int_\Omega \bar{\theta}(\xi)d\xi. \tag{59}$$

This is physically natural since by the condition (49) heat cannot escape from Ω. Thus the total heat present in Ω for $t = 0$ remains unchanged. The interest of the present Remark in relation to the original problem (2)-(8) lies in the fact that as $t \to \infty$ the electric field $\mathbf{E}$ vanishes. Thus (59) holds not only for $\theta_1(x,t)$ but also for the entire solution (44)-(46). This is what is proved in the next Section using a Liapunov function.

5. Existence of a Liapunov function and its consequences

The following theorem is the main result of this paper and explains the crucial role of the condition (60) below.

Theorem 3. If Ω is simply connected and σ and q are positive constants such that

$$\sigma > q \tag{60}$$

the functional

$$V = \int_\Omega \left[\frac{1}{2}\left(\varepsilon|\mathbf{E}|^2 + \mu|\mathbf{H}|^2\right) + \theta\right] dx \tag{61}$$

is a strict Liapunov function for the initial-boundary value problem

$$\varepsilon\frac{\partial \mathbf{E}}{\partial t} = \nabla\times\mathbf{H} - \sigma\mathbf{E} \ \text{ in } \Omega \tag{62}$$

$$\mu\frac{\partial \mathbf{H}}{\partial t} = -\nabla\times\mathbf{E} \ \text{ in } \Omega \tag{63}$$

$$\nabla\cdot\mathbf{H} = 0 \text{ in } \Omega \tag{64}$$

$$\mathbf{E}\times\mathbf{n} = 0 \text{ on } \Gamma \tag{65}$$

$$\mathbf{H}\cdot\mathbf{n} = 0 \ \text{ on } \Gamma \tag{66}$$

$$\mathbf{E}(\mathbf{x},0) = \mathbf{E}_0(\mathbf{x}) \ \text{ in } \Omega \tag{67}$$

$$\mathbf{H}(\mathbf{x},0) = \mathbf{H}_0(\mathbf{x}) \ \text{ in } \Omega \tag{68}$$

$$\theta_t = \Delta\theta + q|\mathbf{E}|^2 \ \text{ in } \Omega \tag{69}$$

$$\frac{\partial\theta}{\partial\mathbf{n}} = 0 \ \text{ on } \Gamma \tag{70}$$

$$\theta(x,0) = \bar{\theta}(x) \ \text{ in } \Omega \tag{71}$$

$$\lim_{t\to\infty}\theta(x,t) = \frac{1}{|\Omega|}\int_\Omega \bar{\theta}(x)dx. \tag{72}$$

Moreover,

$$(\mathbf{E},\mathbf{H},\theta) = \Big(0,0,\frac{1}{|\Omega|}\int_\Omega \bar{\theta}(x)dx\Big) \tag{73}$$

is the only stationary solution for (62)-(72). In addition (73) is globally asymptotically stable.

Proof. The functional(61) is defined in the phase space of the initial conditions $\{\mathbf{E}_0(x),\mathbf{H}_0(x),\bar{\theta}(x)\}$. By the results of Sections 2 and 3 the problem (62)-(72) has one and only one solution

$$(\mathbf{E}(x,t),\mathbf{H}(x,t),\theta(x,t)) \tag{74}$$

globally defined in $[0,\infty)$. Let us compute (61) along (74), in this way the functional (61) becomes a function of the time t of which we can compute the first derivative. We find

$$\frac{dV}{dt} = \int_\Omega \Big(\mathbf{E}\cdot\varepsilon\mathbf{E}_t + \mathbf{H}\cdot\mu\mathbf{H}_t + \theta_t\Big)dx. \tag{75}$$

Substituting in (75) in place of $\varepsilon\mathbf{E}_t$, $\mu\mathbf{H}_t$ and θ_t respectively (62), (63) and (69) we find

$$\frac{dV}{dt} = \int_\Omega [\mathbf{E}\cdot(\nabla\times\mathbf{H} - \sigma\mathbf{E}) + \mathbf{H}\cdot(-\nabla\times\mathbf{E})]dx + \int_\Omega (\Delta\theta + q|\mathbf{E}|^2)dx \tag{76}$$

which can be rewritten as

$$\frac{dV}{dt} = \int_\Omega (\mathbf{E}\cdot\nabla\times\mathbf{H} - \mathbf{H}\cdot\nabla\times\mathbf{E})dx + \int_\Omega (q-\sigma)|\mathbf{E}|^2 dx + \int_\Omega \Delta\theta dx. \tag{77}$$

The first integral (77) vanishes by (21) in view of (65). The last integral in (77) disappears by the condition (70), which is seen here to be essential in the present treatment. Thus we arrive at

$$\frac{dV}{dt} = (q-\sigma)\int_\Omega |\mathbf{E}|^2 dx. \tag{78}$$

Hence, by (60),

$$\frac{dV}{dt} < 0. \tag{79}$$

This permits to say that V is a strict Liapunov function. We have $\frac{dV}{dt} = 0$ if and only if

$$\int_\Omega |\mathbf{E}|^2 dx = 0 \tag{80}$$

i.e.

$$\mathbf{E} = 0. \tag{81}$$

Thus we can repeat, from a different perspective, the considerations of Section 3. More precisely, from (62) and (81) we obtain

$$\nabla \times \mathbf{H} = 0. \tag{82}$$

Since Ω is simply connected there exists a scalar potential ψ such that

$$\mathbf{H}(x) = \nabla\psi(x) \tag{83}$$

and by (64)

$$\Delta\psi = 0 \quad \text{ in } \Omega. \tag{84}$$

But by (66)

$$\frac{\partial\psi}{\partial\mathbf{n}} = 0 \quad \text{ on } \Gamma. \tag{85}$$

Hence ψ is a constant. By (83) we conclude that $\mathbf{H}(x) = 0$. The considerations of Section 3 imply that

$$\mathbf{E}(x) = 0, \quad \mathbf{H}(x) = 0, \quad \theta(x) = \frac{1}{|\Omega|}\int_\Omega \bar{\theta}(x)dx \tag{86}$$

is a stationary solution of problem (62)-(72). By the theory of Liapunov function in infinite dimensions, see [6], [1], the stationary solution (86) is globally asymptotically stable. □

The physical meaning of the functional

$$V = \int_\Omega \frac{1}{2}\left(\varepsilon|\mathbf{E}|^2 + \mu|\mathbf{H}|^2\right)dx + \int_\Omega \theta dx \tag{87}$$

is clear. The first integral in (87) is the electro-magnetic energy presents at a certain time in Ω, the second integral in (87) is the thermal energy which exists at the same instant in Ω. If in problem (62)-(72) $q < \sigma$ the total energy is constantly decreasing with time and the contrary happens if $q > \sigma$.

If in the definition of "hot-spot" we require the temperature to diverge in a finite time, then the present model cannot predict the formation of hot-spots. However, the functional (87) is a Liapunov function even in the full non linear problem, i.e. when σ and q depend on the temperature, see [4].

Appendix. We collect here the definitions of the functions space used in this paper.

$$\mathcal{H}(curl,\Omega) = \{\mathbf{A} \in L^2(\Omega,\mathbf{R}^3), \quad \nabla\times\mathbf{A} \in L^2(\Omega,\mathbf{R}^3)\} \tag{88}$$

$$\mathcal{H}_0(curl,\Omega) = \{\mathbf{A} \in \mathcal{H}(curl,\Omega), \quad \mathbf{A}\times\mathbf{n} \text{ on } \Gamma\} \tag{89}$$

$$\mathcal{H}(div,\Omega) = \{\mathbf{A} \in L^2(\Omega,\mathbf{R}^3), \quad \nabla\cdot\mathbf{A} \in L^2(\Omega,\mathbf{R}^3)\} \tag{90}$$

$$\mathcal{H}_0(div,\Omega) = \{\mathbf{A} \in \mathcal{H}(div,\Omega), \quad \mathbf{A}\cdot\mathbf{n} \text{ on } \Gamma\}. \tag{91}$$

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