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SUMMATION FORMULAE FOR BASIC HYPERGEOMETRIC FUNCTIONS VIA FRACTIONAL q-CALCULUS

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The object of this paper is to illustrate how the q-fractional calculus approach can be employed to derive a number of summation formulae for the generalized basic hypergeometric functions of one and more variables in terms of the q-gamma functions.

1. Introduction

Recently, we have investigated the applications of q-Leibniz rule of fractional order q-derivatives (see [9] and [10]) and deduced several interesting transformations involving various basic hypergeometric functions of one variable including the basic analogue of Fox's H-function. Earlier, Agarwal [1], Denis [3] and Shukla [7] have also applied the q-Leibniz rule to derive certain interesting transformations for basic hypergeometric functions of one variable.

In the present paper, we have explored the possibility for derivation of some known or new summation formulae for basic hypergeometric functions of one and more variables, using certain fundamental results of q-fractional calculus.

A widely used fundamental result in the q-fractional calculus, is the formula

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cf. Agarwal [1] (see also [2])

$$D^{\beta}_{z,q}(z^{\alpha-1}) = \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\beta)}\, z^{\alpha-\beta-1}, \quad Re(\alpha) > 0, \tag{1}$$

valid for all values of β.

Agarwal [1] defined the q-extension of the Leibniz rule for the fractional q-derivatives for a product of two functions in terms of a series involving fractional q-derivatives of the individual functions in the following manner:

$$D^{\beta}_{z,q}\{U(z)V(z)\} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}\,(q^{-\beta};q)_n}{(q;q)_n}\, D^{\beta-n}_{z,q^n}\{U(zq^n)\}\, D^{n}_{z,q}\{V(z)\}, \tag{2}$$

where $U(z)$ and $V(z)$ are two regular functions such that

$$U(z) = \sum_{r=0}^{\infty} a_r z^r, \quad |z| < R_1; \qquad V(z) = \sum_{r=0}^{\infty} b_r z^r, \quad |z| < R_2,$$

then for the result (2), $|z| < R = \min(R_1, R_2)$.

We shall make use of the following notations and definitions in the sequel: For real or complex a, $0 < |q| < 1$, the q-shifted factorial is defined as:

$$(a;q)_n \equiv (q^a;q)_n = \begin{cases} 1 & ;\text{if} \quad n = 0 \\ (1-q^a)(1-q^{a+1})\cdots(1-q^{a+n-1}) & ;\text{if} \quad n \in N. \end{cases} \tag{3}$$

In terms of the q-gamma function, (3) can be expressed as

$$(a;q)_n = \frac{\Gamma_q(a+n)(1-q)^n}{\Gamma_q(a)}, \quad n > 0, \tag{4}$$

where the q-gamma function (cf. Gasper and Rahman [5]), is given by

$$\Gamma_q(a) = \frac{(q;q)_\infty}{(q^a;q)_\infty(1-q)^{a-1}}. \tag{5}$$

Indeed, it is easy to verify that

$$\lim_{q\to 1^-} \Gamma_q(a) = \Gamma(a) \quad and \quad \lim_{q\to 1^-} \frac{(q^a;q)_n}{(1-q)^n} = (a)_n, \tag{6}$$

where

$$(a)_n = a(a+1)\cdots(a+n-1). \tag{7}$$

The generalized basic hypergeometric series ${}_r\Phi_s(.)$, is defined as:

$$ {}_r\Phi_s\left[\begin{array}{ll} a_1,\cdots,a_r; & \\ & q,z \\ b_1,\cdots,b_s; & \end{array}\right] = \sum_{n=0}^{\infty}\frac{(a_1,\cdots,a_r;q)_n}{(q,b_1,\cdots,b_s;q)_n}z^n\,, \tag{8} $$

where for the convergence of the series (8), we have $0<|q|<1$, for all z if $r\le s$, and $|z|<1$ if $r=s+1$.

2. Main results

In this section, we shall establish some known summation formulae associated with the basic hypergeometric functions and some new summation formulae for the basic Appell function $\Phi^{(1)}(.)$, the basic hypergeometric function $\Phi_D^{(3)}(.)$ and the basic Lauricell function $\Phi_D^{(n)}(.)$ as the applications of the q-Leibniz rule for the fractional order q-derivatives of a product of two basic functions.
The results are obtained by assigning particular values to the functions $U(z)$ and $V(z)$ in the q-Leibniz rule (2). For example
(i) If we put $U(z)=z^{c-a-1}$, $V(z)=z^{-b}$ and $\beta=-a$ in the q-Leibniz rule (2), and then making use of result (1), we obtain

$$ \frac{\Gamma_q(c-a-b)}{\Gamma_q(c-b)} = \sum_{n\ge 0}\frac{(-1)^n q^{n(n-1)/2+n(c-a)}(q^a;q)_n\Gamma_q(c-a)\Gamma_q(1-b)}{(q;q)_n\Gamma_q(c+n)\Gamma_q(1-b-n)}. \tag{9} $$

On further simplifications, equation (9) yields to a summation formula for the basic hypergeometric function ${}_2\Phi_1(.)$, namely

$$ \frac{\Gamma_q(c-a-b)\,\Gamma_q(c)}{\Gamma_q(c-b)\,\Gamma_q(c-a)} = {}_2\Phi_1\left[\begin{array}{ll} q^a,q^b; & \\ & q,q^{c-a-b} \\ q^c; & \end{array}\right]. \tag{10} $$

Which is the well known q-Gauss summation theorem.
(ii) Again, if we put $U(z)=z^{c+n-1}$, $V(z)=z^{-b}$ and $\beta=n$ in the relation (2), we obtain the terminating q-Gauss summation formula, namely

$$ \frac{(q^{c-b};q)_n}{(q^c;q)_n} = {}_2\Phi_1\left[\begin{array}{ll} q^{-n},q^b; & \\ & q,q^{n+c-b} \\ q^c; & \end{array}\right]. \tag{11} $$

(iii) If we take $U(z)=z^{c+n-1}$, $V(z)=z^{-(a+n)}$ and $\beta=n$ in the q-analogue of Leibniz rule, we obtain the following terminating summation formula, namely

$$ \frac{(-1)^n q^{n(c-a)-n(n+1)/2}(q^{1+a-c};q)_n}{(q^c;q)_n} = {}_2\Phi_1\left[\begin{array}{ll} q^{-n},q^{a+n}; & \\ & q,q^{c-a} \\ q^c; & \end{array}\right]. \tag{12} $$

(vi) Again, if we put $U(z) = z^{a+n-1}$, $V(z) = z^{b+n-1}$ and $\beta = n$ in the relation (2), we obtain the another summation formula, namely

$$\frac{(q^{a+b-1};q)_{2n}}{(q^a;q)_n(q^{a+b-1};q)_n} = {}_2\Phi_1\left[\begin{array}{ll} q^{-n}, q^{1-b-n}; & \\ & q, q^{a+b+2n-1} \\ q^a; & \end{array}\right]. \tag{13}$$

(v) Further, on putting $U(z) = z^{c-a-1}$, $V(z) = z^{-(b_1+b_2)}$ and $\beta = -a$ in the relation (2), and on making use of result (1), it yields to

$$\begin{aligned}\frac{\Gamma_q(c-a-b_1-b_2)}{\Gamma_q(c-b_1-b_2)} &= \\ = \sum_{n\geq 0}\frac{(-1)^n q^{n(n-1)/2+n(c-a)}(q^a;q)_n\Gamma_q(c-a)\Gamma_q(1-b_1-b_2)}{(q;q)_n\Gamma_q(c+n)\Gamma_q(1-b_1-b_2-n)}&.\end{aligned} \tag{14}$$

On further simplification the above equation (14) reduces to

$$\frac{\Gamma_q(c-a-b_1-b_2)\ \Gamma_q(c)}{\Gamma_q(c-b_1-b_2)\ \Gamma_q(c-a)} = \sum_{n\geq 0}\frac{q^{n(c-a)}\ (q^a;q)_n\ (q^{b_1+b_2};q)_n}{(q;q)_n(q^c;q)_n\ q^{n(b_1+b_2)}}. \tag{15}$$

On using the q-binomial theorem cf. Gasper and Rahman [5], namely

$$(ab;q)_n = \sum_{k=0}^{n}\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}b^k(a;q)_k(b;q)_{n-k}, \tag{16}$$

in the equation (15), it yields to

$$\begin{aligned}\frac{\Gamma_q(c-a-b_1-b_2)\ \Gamma_q(c)}{\Gamma_q(c-b_1-b_2)\ \Gamma_q(c-a)} &= \sum_{n\geq k_1}\frac{q^{n(c-a)}\ (q^a;q)_n}{(q;q)_n(q^c;q)_n\ q^{n(b_1+b_2)}} \\ &\sum_{k_1\geq 0}\frac{(q;q)_n}{(q;q)_{k_1}(q;q)_{n-k_1}}(q^{b_2})^{k_1}\ (q^{b_1};q)_{k_1}\ (q^{b_2};q)_{n-k_1},\end{aligned} \tag{17}$$

which on further simplifications, reduces to a summation formula for the basic Appell function $\Phi^{(1)}(.)$ as under:

$$\frac{\Gamma_q(c-a-b_1-b_2)\ \Gamma_q(c)}{\Gamma_q(c-b_1-b_2)\ \Gamma_q(c-a)} = \Phi^{(1)}\left[q^a, q^{b_1}, q^{b_2}; q^c; q, q^{c-a-b_1}, q^{c-a-b_1-b_2}\right], \tag{18}$$

where the basic Appell function $\Phi^{(1)}(.)$ defined as:

$$\Phi^{(1)}\left[a,b,b';c;q;x,y\right] = \sum_{m,n\geq 0}\frac{(a;q)_{m+n}(b;q)_m(b';q)_n}{(c;q)_{m+n}(q;q)_m(q;q)_n}x^m y^m, \tag{19}$$

and $|x| < 1, |y| < 1, 0 < |q| < 1$.
(vi) Again, if we take $U(z) = z^{c+n-1}$, $V(z) = z^{-(b_1+b_2)}$ and $\beta = n$ in the equation (2), it yields to the terminating summation formula for the basic Appell function $\Phi^{(1)}(.)$, namely

$$\frac{(q^{c-b_1-b_2};q)_n}{(q^c;q)_n} = \Phi^{(1)}\left[q^{-n},q^{b_1},q^{b_2};q^c;q,q^{n+c-b_1},q^{n+c-b_1-b_2}\right]. \tag{20}$$

(vii) Similarly, if we take $U(z) = z^{c-a-1}$, $V(z) = z^{-(b_1+b_2+b_3)}$ and $\beta = -a$ in the relation (2), and on making use of result (1), it yields to

$$\frac{\Gamma_q(c-a-b_1-b_2-b_3)}{\Gamma_q(c-b_1-b_2-b_3)} =$$

$$= \sum_{n\geq 0} \frac{(-1)^n q^{n(n-1)/2+n(c-a)}(q^a;q)_n \Gamma_q(c-a)\Gamma_q(1-b_1-b_2-b_3)}{(q;q)_n \Gamma_q(c+n)\Gamma_q(1-b_1-b_2-b_3-n)}. \tag{21}$$

On further simplification, the above equation (21) reduces to

$$\frac{\Gamma_q(c-a-b_1-b_2-b_3)\,\Gamma_q(c)}{\Gamma_q(c-b_1-b_2-b_3)\,\Gamma_q(c-a)} = \sum_{n\geq 0} \frac{q^{n(c-a)}\,(q^a;q)_n\,(q^{b_1+b_2+b_3};q)_n}{(q;q)_n (q^c;q)_n\, q^{n(b_1+b_2+b_3)}}, \tag{22}$$

which on making use of q-identities, reduces to a summation formula for the basic hypergeometric function of three variable, namely

$$\frac{\Gamma_q(c-a-b_1-b_2-b_3)\,\Gamma_q(c)}{\Gamma_q(c-b_1-b_2-b_3)\,\Gamma_q(c-a)} =$$

$$= \Phi_D^{(3)}\left[q^a,q^{b_1},q^{b_2},q^{b_3};q^c;q,q^{c-a-b_1},q^{c-a-b_1-b_2},q^{c-a-b_1-b_2-b_3}\right], \tag{23}$$

where the basic hypergeometric function $\Phi_D^{(3)}(.)$ defined as:

$$\Phi_D^{(3)}[a,b,b',b'';c;q;x,y,z] =$$
$$= \sum_{m,n,p\geq 0} \frac{(a;q)_{m+n+p}(b;q)_m(b';q)_n(b'';q)_p}{(c;q)_{m+n+p}(q;q)_m(q;q)_n(q;q)_p} x^m y^m z^p. \tag{24}$$

(viii) Again, if we put $U(z) = z^{c+n-1}$, $V(z) = z^{-(b_1+b_2+b_3)}$ and $\beta = n$ in the equation (2), it yields to a reduction formula as under:

$$\frac{(q^{c-b_1-b_2-b_3};q)_n}{(q^c;q)_n} =$$

$$= \Phi_D^{(3)}\left[q^{-n}, q^{b_1}, q^{b_2}, q^{b_3}; q^c; q, q^{n+c-b_1}, q^{n+c-b_1-b_2}, q^{n+c-b_1-b_2-b_3}\right]. \tag{25}$$

(ix) If we take $U(z) = z^{c-a-1}$, $V(z) = z^{-(b_1+\cdots+b_m)}$ and $\beta = -a$ in the relation (2), it yields to

$$D_{z,q}^{-a}\left\{z^{c-a-(b_1+\cdots+b_m)-1}\right\} = \sum_{n\geq 0} \frac{(-1)^n q^{n(n+1)/2}\,(q^a;q)_n}{(q;q)_n}\, D_{z,q}^{-a-n}\left\{(zq^n)^{c-a-1}\right\}$$
$$D_{z,q}^{n}\left\{z^{-(b_1+\cdots+b_m)}\right\}. \tag{26}$$

On using the fractional q-derivetive formula (1), equation (26) reduces to

$$\frac{\Gamma_q(c-a-b_1-\cdots-b_m)}{\Gamma_q(c-b_1-\cdots-b_m)} = \sum_{n\geq 0} \frac{(-1)^n q^{n(n-1)/2}\,(q^a;q)_n}{(q;q)_n}\, q^{n(c-a)}$$
$$\frac{\Gamma_q(c-a)\,\Gamma_q(1-b_1-\cdots-b_m)}{\Gamma_q(c+n)\,\Gamma_q(1-b_1-\cdots-b_m-n)}, \tag{27}$$

which on further simplifications, yields to

$$\frac{\Gamma_q(c-a-b_1-\cdots-b_m)\,\Gamma_q(c)}{\Gamma_q(c-b_1-\cdots-b_m)\,\Gamma_q(c-a)} = \sum_{n\geq 0} \frac{q^{n(c-a)}\,(q^a;q)_n\,(q^{b_1+\cdots+b_m};q)_n}{(q;q)_n(q^c;q)_n\, q^{n(b_1+\cdots+b_m)}}. \tag{28}$$

By making use of the q-multinomial theorem cf. Gasper and Rahman [5], namely

$$(a_1a_2\cdots a_{m+1};q)_n =$$
$$= \sum_{\substack{k_1,k_2,\cdots,k_m\geq 0\\ n\geq k_1+k_2+\cdots+k_m}} \frac{(q;q)_n\, a_2^{k_1} a_3^{k_1+k_2}\cdots a_{m+1}^{k_1+k_2+\cdots+k_m}}{(q;q)_{k_1}(q;q)_{k_2}\cdots(q;q)_{k_m}(q;q)_{n-(k_1+k_2+\cdots+k_m)}} \tag{29}$$
$$(a_1;q)_{k_1}(a_2;q)_{k_2}\cdots(a_m;q)_{k_m}(a_{m+1};q)_{n-(k_1+k_2+\cdots+k_m)},$$

where $m = 1,2,\cdots$, $n = 0,1,\cdots$, the equation (28) reduces to

$$\frac{\Gamma_q(c-a-b_1-\cdots-b_m)\,\Gamma_q(c)}{\Gamma_q(c-b_1-\cdots-b_m)\,\Gamma_q(c-a)} = \sum_{n\geq k_1+\cdots+k_{m-1}} \frac{q^{n(c-a)}\,(q^a;q)_n}{(q;q)_n(q^c;q)_n\, q^{n(b_1+\cdots+b_m)}}$$
$$\sum_{k_1,\cdots,k_{m-1}\geq 0} \frac{(q;q)_n\,(q^{b_2})^{k_1}\,(q^{b_3})^{k_1+k_2}\cdots(q^{b_m})^{k_1+\cdots+k_{m-1}}}{(q;q)_{k_1}\cdots(q;q)_{k_{m-1}}(q;q)_{n-(k_1+\cdots+k_{m-1})}}$$
$$(q^{b_1};q)_{k_1}\cdots(q^{b_{m-1}};q)_{k_{m-1}}(q^{b_m};q)_{n-(k_1+\cdots+k_{m-1})}. \tag{30}$$

On further simplifications, the above result (30) yields to a summation formula for the basic Lauricella function $\Phi_D^{(n)}(.)$, namely

$$\frac{\Gamma_q(c-a-b_1-\cdots-b_m)\,\Gamma_q(c)}{\Gamma_q(c-b_1-\cdots-b_m)\,\Gamma_q(c-a)} =$$

$$= \Phi_D^{(m)}\left[q^a, q^{b_1}, \cdots, q^{b_m}; q^c; q, q^{c-a-b_1}, q^{c-a-b_1-b_2}, \cdots, q^{c-a-b_1-\cdots-b_m}\right], \quad (31)$$

where the basic Lauricell function $\Phi_D^{(n)}(.)$ defined as:

$$\Phi_D^{(n)}[a, b_1, \cdots, b_n; c; q; x_1, \cdots, x_n] =$$

$$= \sum_{m_1,\cdots,m_n \geq 0} \frac{(a;q)_{m_1+\cdots+m_n}}{(c;q)_{m_1+\cdots+m_n}} \prod_{j=1}^{n} \left\{ \frac{(b_j;q)_{m_j}\, x_j^{m_j}}{(q;q)_{m_j}} \right\}, \quad (32)$$

and for convergence $|x_1| < 1, \cdots, |x_n| < 1, 0 < |q| < 1$.
(x) Finally, if we take $U(z) = z^{c+n-1}$, $V(z) = z^{-(b_1+\cdots+b_m)}$ and $\beta = n$ in the relation (2), it yields to a terminating summation formula for the basic Lauricella function $\Phi_D^{(n)}(.)$, namely

$$\frac{(q^{c-b_1-\cdots-b_m};q)_n}{(q^c;q)_n} =$$

$$= \Phi_D^{(m)}\left[q^{-n}, q^{b_1}, \cdots, q^{b_m}; q^c; q, q^{n+c-b_1}, q^{n+c-b_1-b_2}, \cdots, q^{n+c-b_1-\cdots-b_m}\right]. \quad (33)$$

The reduction formulae, established in this section of the present paper are an q-identities, if the parameters are so restricted that each of the functions involved exists.

3. Special cases

In this section, we discuss some of the importent special cases of the main results established in the previous section:
(i) In view of the limit formula (6), one can note that the results (12) and (13) are the respective q-extensions of the known results reported in [6, p. 69], namely

$$\frac{(-1)^n(1+a-c)_n}{(c)_n} = {}_2F_1\left[\begin{array}{c} -n, a+n; \\ \quad\quad 1 \\ c; \end{array}\right], \quad (34)$$

and

$$\frac{(a+b-1)_{2n}}{(a)_n\,(a+b-1)_n} = {}_2F_1\left[\begin{array}{c} -n, 1-b-n; \\ \quad\quad 1 \\ a; \end{array}\right], \quad (35)$$

whereas the result (31) is the q-analogue of the known result due to [4], namely

$$F_D^{(m)}\left[a,b_1,\cdots,b_m;c;1\right]=\frac{\Gamma(c-a-b_1-\cdots-b_m)\,\Gamma(c)}{\Gamma(c-b_1-\cdots-b_m)\,\Gamma(c-a)}. \tag{36}$$

(ii) Further, if we put the arbitrary parameter $m=1,2$ and 3 in the result (31), it reduces to the respective results (10), (18) and (23).
We conclude with the remark that, the q-Leibniz rule introduced by Agarwal [1] is certainly an important tool/ technique for deriving numerous transformations and summation formulae involving various basic hypergeometric functions of one and more variables. The results thus derived in this paper are general in character and likely to find certain applications in the theory of basic hypergeometric functions.

REFERENCES

[1] R. P. Agarwal, *Fractional q-derivatives and q-integrals and certain hypergeometric transformations*, Ganita **27** (1976), 25–32.

[2] W. A. Al-Salam, *Some fractional q-integrals and q-derivatives*, Proc. Edin. Math. Soc. **15** (1966), 135–140.

[3] R. Y. Denis, *On certain special transformations of poly-basic hypergeometric functions*, The Math. Student, **51** (1983), no.1-4, 121–125.

[4] M. K. Gaira - H. S. Dhami, *New results for H-function and G-function by the application of fractional calculus*, IMA Preprint Series no. 1987, University of Minnesota, August (2004).

[5] G. Gasper - M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, Cambridge, 1990.

[6] E. D. Rainville, *Special Functions*, Chelsea Publishing Company, Bronx, New York, 1960.

[7] H. L. Shukla, *Certain results involving basic hypergeometric functions and fractional q-derivative*, The Math. Student, **61** (1992), no. 1-4, 107–112.

[8] L. J. Slater, *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge, London and New York, 1966.

[9] R. K. Yadav - S. D. Purohit, *Fractional q-derivatives and certain basic hypergeometric transformations*, South East Asian J. Math. and Math. Sc., **2**(2) (2004), 37–46.

[10] R. K. Yadav - S. D. Purohit, *On fractional q-derivatives and transformations of the generalized basic hypergeometric functions*, J. Indian Acad. Math., **28**(2) (2006), 321–326.

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