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ADDENDUM TO THE PAPER: FIXED POINTS FOR NON-EXPANSIVE SET-VALUED MAPPINGS

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The aim of this paper is to improve or simplify some theorems which have been published in the paper [3] in this journal after a stay of the author at the University of Catania. In particular in [3] the main results were that in general the known properties of the set of fixed points for contractive set-valued mappings fail as soon as one replaces "contractive" by "non-expansive". In fact, as we shall prove in this addendum, some of them hold true when the surrounding space is finite-dimensional, or the domain is compact.

Some results of [3] are reproved here with a simpler proof for making this addendum more self-contained.

Like in [3], the general frame is the following. Let E be a Banach space, C a closed convex subset of E (most of time the whole E), and $F : C \rightrightarrows C$ a set-valued non-expansive mapping with closed convex (non-empty) values, it means

$$d_H(F(x), F(y)) \leq \|x - y\|$$

for all x and y in C, where d_H denotes the Hausdorff distance between closed sets : $d_H(A,B) = \max\big(\sup_{x \in A} d(x,B), \sup_{y \in B} d(y,A)\big)$. And the main object in this frame is the set $\mathrm{Fix}(F) = \{x \in C : x \in F(x)\}$ of fixed points of F. A very special case and studied for long is the case of contractions, it is when F is q-lipschitz for some $q < 1$. Of course translations

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in E provide obvious examples of single-valued isometries without fixed points, but this cannot happen with contractions.

1. Non-boundedness of fixed points

If F is a contraction and $F(x)$ is unbounded it is proved in [2] that the set $\mathrm{Fix}(F)$ is itself unbounded. Nevertheless, for non-expansive mappings, it is proved in Theorem 4.1 of [3] that this does not longer hold : in fact a non-expansive mapping is constructed in the Hilbert space ℓ^2 with unbounded values which has exactly one fixed point. We now show that this cannot happen if E is finite-dimensional.

Theorem 1.1. *Let E be a finite-dimensional normed space, $F : E \rightrightarrows E$ a non-expansive set-valued mapping with closed convex unbounded values. Then* $\mathrm{Fix}(F)$ *is either empty or unbounded.*

Proof. One can assume that 0 is a fixed point. For any integer $n \geq 1$, let $q_n = 1 - 2^{-n}$. Then the mapping $F_n : x \mapsto q_n F(x)$ is a q_n-contraction and $0 \in \mathrm{Fix}(F_n)$. It follows from [1] that $\mathrm{Fix}(F_n)$ is connected and from [2] that it is also unbounded. So, for any $R > 0$, $\mathrm{Fix}(F_n)$ intersects the sphere $S(0,R)$ of center 0 and radius R in some point x_n. And since this sphere is compact, the sequence (x_n) has a cluster value x^*. We will prove that $x^* \in \mathrm{Fix}(F)$. Let $\varepsilon > 0$. There is some n such that $\|x_n - x^*\| < \varepsilon$ and $R(1-q_n) < q_n\varepsilon$. Since $x_n \in \mathrm{Fix}(F_n)$ there is $x'_n \in F(x_n)$ such that $x_n = q_n x'_n$, and since $x'_n \in F(x_n)$ there is some $y_n \in F(x^*)$ such that

$$\|x'_n - y_n\| \leq d_H(F(x_n), F(x^*)) \leq \|x_n - x^*\| < \varepsilon\,.$$

So

$$\begin{aligned} d(x^*, F(x^*)) &\leq \|x^* - y_n\| \leq \|x^* - x_n\| + \|x_n - x'_n\| + \|x'_n - y_n\| \\ &\leq 2\varepsilon + \|x_n(1 - \frac{1}{q_n})\| \leq 2\varepsilon + R\frac{1-q_n}{q_n} < 3\varepsilon \end{aligned}$$

and since $F(x^*)$ is closed and ε arbitrary we conclude that $x^* \in \mathrm{Fix}(F)$ hence that $\mathrm{Fix}(F) \cap S(0,R) \neq \emptyset$. Since R is arbitrary large we are done. □

2. Non-uniqueness of fixed points

As above Theorem 4.1 of [3] provides an example of non-expansive mapping with non-singleton values having a unique fixed point. Nevertheless for contractions it is shown in [2] that this phenomenon cannot happen : if F is a set-valued contraction with convex closed non-singleton values the set $\mathrm{Fix}(F)$ has several fixed points (in fact infinitely many fixed points since it is connected). More precisely if F is a q-contraction, $0 \in \mathrm{Fix}(F)$ and $b \in F(0)$ there is some $c \in \mathrm{Fix}(F)$ with $\|c\| \geq \dfrac{1}{1+q}\|b\|$.

Theorem 2.1. *Let E be a finite-dimensional normed space, F be a set-valued non-expansive mapping with closed convex values, $a \in \mathrm{Fix}(F)$ and $b \in F(a)$. Then there exists some $c \in \mathrm{Fix}(F)$ such that $\|c-a\| \geq \frac{1}{2}\|b-a\|$.*

Proof. We can and do assume that $a = 0$ and $b \in F(0)$. For any integer $n \geq 1$, let $q_n = 1 - 2^{-n}$ and define F_n by $F_n(x) = q_n F(x)$. Then F_n is a contraction, $0 \in \mathrm{Fix}(F_n)$ and $q_n b \in F_n(0)$. By the result of [2] we recalled above, there is a fixed point x'_n of F_n satisfying $\|x'_n\| \geq \frac{\|q_n b\|}{1+q_n}$. Since $\mathrm{Fix}(F_n)$ is connected and contains 0 the set $\{\|z\| : z \in \mathrm{Fix}(F_n)\}$ is an intervall containing 0 and $\|x'_n\|$; hence we can choose $x_n \in \mathrm{Fix}(F_n)$ such that $\|x_n\| = \frac{q_n}{1+q_n}\|b\|$, in particular $\|x_n\| \leq \frac{1}{2}\|b\|$.

By compactness of the ball $B(0, \frac{\|b\|}{2})$, there exists a cluster value x^* for the sequence (x_n). Then for any $\varepsilon > 0$ one can find $n \geq 1$ such that $\|x^* - x_n\| < \varepsilon$ and $2^{-n}\|b\| < \varepsilon$. Since $\tilde{x}_n = \frac{x_n}{q_n}$ belongs to $F(x_n)$ there exists $y_n \in F(0)$ such that $\|y_n - \tilde{x}_n\| \leq d_H(F(x^*), F(x_n)) \leq \|x^* - x_n\|$ and we get

$$\begin{aligned} d(x^*, F(x^*)) &\leq \|x^* - y_n\| \leq \|x^* - x_n\| + \|x_n - \tilde{x}_n\| + \|\tilde{x}_n - y_n\| \\ &\leq \varepsilon + \|\tilde{x}_n\|(1-q_n) + \varepsilon \leq 2\varepsilon + \|b\|\frac{1-q_n}{1+q_n} \\ &\leq 2\varepsilon + 2^{-n}\|b\| < 3\varepsilon \end{aligned}$$

Again this proves that $x^* \in \mathrm{Fix}(F)$ and that $\|x^*\| = \frac{\|b\|}{2}$.

In fact theorem 1.1 appears as a corollary of this one since we can choose $\|b\|$ arbitrary large with $b \in F(0)$, so get $\|x^*\|$ arbitrary large. □

3. Non-connectedness

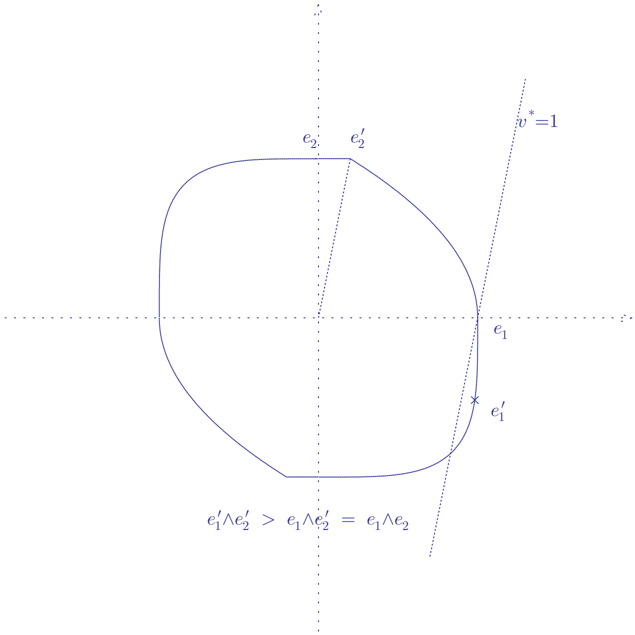
The aim of this section is to give a simpler proof of theorems 6.3 and 7.5 from [3], concerning 2-dimensional normed spaces. The method of proof is essentially the same as in [3] and again is split into two cases following the properties of the norm. Let $(E, \|.\|)$ be a 2-dimensional normed space and B its unit ball.

Lemma 3.1. *Either there exist two linear functionals u^* and v^* on E such that $\|u^* + tv^*\| = 1$ for all $t \in [-1, 1]$, or there exists a basis (e_1, e_2) of E satisfying $\|e_1\| = \|e_2\| = \|e_1^*\| = \|e_2^*\| = 1$ and $\|e_2 + te_1\| > 1$ for all real $t \neq 0$.*

Proof. Let (a_1, a_2) any fixed basis of E and define for $x = x_1 a_1 + x_2 a_2$ and $y = y_1 a_1 + y_2 a_2$: $x \wedge y = x_1 y_2 - x_2 y_1$. The mapping $(x, y) \mapsto x \wedge y$ is continuous and attains its supremum $\mu > 0$ on the compact set $B \times B$ at some (e_1, e_2).

It is clear that $\|e_1\| = \|e_2\| = 1$ hence that $\|e_1^*\| \geq \langle e_1^*, e_1 \rangle = 1$ and $\|e_2^*\| \geq 1$ too.

Moreover since $e_1 \wedge (e_2 + \lambda e_1) = \mu$ one cannot have $\|e_2 + \lambda e_1\| < 1$. Hence $\|e_2^*\| = 1$, and similarly $\|e_1^*\| = 1$.



If we had $\|e_2'\| = 1$ for $e_2' = e_2 + te_1$ with some $t \neq 0$ then $w^* = \frac{1}{\mu}(e_1^* - te_2^*)$ would satisfy $\langle w^*, x \rangle = \frac{1}{\mu} x \wedge e_2^*$ for all $x \in E$. In particular $\langle w^*, e_1 \rangle = \frac{1}{\mu} e_1 \wedge (e_2 + te_1) = \frac{1}{\mu} e_1 \wedge e_2 = 1$, so $\|w^*\| \geq 1$. And $\frac{1}{\mu} x \wedge (e_2 + te_1) \leq 1$ for all $x \in B$ since $(e_2 + te_1) \in B$, so $\|w^*\| = \sup_{x \in B} \langle w^*, x \rangle \leq 1$. It follows that in this case

$$1 = \langle se_1^* + (1-s)w^*, e_1 \rangle \leq \|se_1^* + (1-s)w^*\| \leq \max(\|e_1^*\|, \|w^*\|) = 1$$

for all $s \in [0, 1]$. And it is enough to take $u^* = \frac{e_1^* + w^*}{2}$ and $v^* = \frac{e_1^* - w^*}{2}$ for getting $\|u^* + tv^*\| \leq 1$ for all $t \in [-1, 1]$. $\square$

Lemma 3.2. *Let E be a normed space, u^* be a non-zero linear functional and, for $t \in \mathbb{R}$, P_t be the halfspace $\{z \in E : \langle u^*, z \rangle \geq t\}$.*

Then, for $s, t \in \mathbb{R}$, $d_H(P_s, P_t) = \frac{|s-t|}{\|u^*\|}$.

Proof. This follows immediately from the definition of the norm in E^*. □

Lemma 3.3 (cf. [3], Theorem 6.3)**.** *Let E be a 2-dimensional normed space and E^* be its dual. If there are two non-zero linear functionals u^* and v^* in E^* such that $\|u^* + tv^*\| = 1$ for all $t \in [-1,1]$ (equivalently for $t \in \{-1,0,1\}$), then there exists a non-expansive set-valued mapping $F : E \rightrightarrows E$ with closed convex values such that* $\mathrm{Fix}(F)$ *is non connected.*

Proof. The function $h : E \to \mathbb{R}$ defined by $h(x) = \langle u^*, x\rangle + \sin^2(\langle v^*, x\rangle)$ is $\mathcal{C}^1$ and satisfies $h'(x) = u^* + \sin(2\langle v^*, x\rangle)v^*$, hence $\|h'(x)\| = 1$. Thus h is 1-lipschitz, an so is the set-valued function

$$F : x \mapsto \{z \in E : \langle u^*, z\rangle \geq h(x)\}$$

Moreover

$$x \in \mathrm{Fix}(F) \iff \langle u^*, x\rangle \geq \langle u^*, x\rangle + \sin^2(\langle v^*, x\rangle) \iff \sin(\langle v^*, x\rangle) = 0$$

so $\mathrm{Fix}(F)$ is the countable union $\bigcup_{k\in\mathbb{Z}}\{x : \langle v^*, x\rangle = k\pi\}$ of pairwise disjoint lines, which is not connected. □

Lemma 3.4 (cf. [3], Lemma 7.2)**.** *Let h be a continuous positive function from $\mathbb{R}^+$ to $\mathbb{R}$. Then there exists a convex positive decreasing 1-lipschitz function φ on $\mathbb{R}^+$ such that $0 < \varphi(t) \leq h(t)$ for all $t \in \mathbb{R}^+$.*

Proof. One can assume that $h(0) \leq 1$. For $\alpha \geq 0$ denote $H_\alpha = \inf_{t\leq\alpha} h(t)$ and define the affine function ℓ_α on $\mathbb{R}^+$ by

$$\ell_\alpha(t) = H_\alpha\big(1 - \frac{t}{\alpha}\big),$$

then the convex function $\varphi = \sup_{\alpha\geq 1} \ell_\alpha$. It is readily checked that $H_\alpha > 0$, that ℓ_α is decreasing, 1-lipschitz and that $\ell_\alpha \leq h$.

For $t \geq 0$ and $\alpha \geq \max(1,t)$, we have $0 < \frac{1}{2}H_{2\alpha} \leq \ell_{2\alpha}(t) \leq \varphi(t)$ hence $\varphi > 0$. So φ is decreasing positive and 1-lipschitz. □

Lemma 3.5 (cf [3], Lemma 7.5)**.** *Assume that (e_1, e_2) is a basis of the normed space E such that $\|e_1\| = \|e_2\| = \|e_1^*\| = \|e_2^*\| = 1$ and $\|e_2 + se_1\| > 1$ for all $s \neq 0$. Then there exists a non-expansive set-valued mapping $F : E \rightrightarrows E$ with closed convex values such that* $\mathrm{Fix}(F)$ *is not connected.*

Proof. For $t \neq 0$ and $s = t^{-1}$ we have $\|e_1 + te_2\| = |t|.\|e_2 + se_1\| > |t|$. So by previous lemma we can find some positive decreasing and 1-lipschitz function φ such that for all $t \in \mathbb{R}^+$:

$$0 < \varphi(t) \leq h(t) := \min\big(\|e_1 + te_2\|, \|e_1 - te_2\|\big) - t\,.$$

Define $\gamma(t)=\varphi(|t|)+|t|$, $a=-e_1$, $b=e_1$, $D=\{a,b\}\cup\mathbb{R}.e_2$, then the function $g:D\to\mathbb{R}^+$ by $g(a)=g(b)=0$ and $g(te_2)=\gamma(t)$.

Check that g is 1-lipschitz on D. The only non-obvious inequality is the following : since φ is decreasing if $t_1=|t|\le|s|=s_1$

$$\begin{aligned}|g(se_2)-g(te_2)|&=|\gamma(s)-\gamma(t)|=s_1-t_1+\varphi(s_1)-\varphi(t_1)\\&\le s_1-t_1+\varphi(s_1)-\varphi(s_1)=s_1-t_1\\&\le|s-t|=\|se_2-te_2\|\end{aligned}$$

Thus g can be extended into some 1-lipschitz function $\tilde{g}:E\to\mathbb{R}^+$, and we define F on E by

$$F(x)=\{z\in E:\langle e_2^*,z\rangle\ge\tilde{g}(x)\}$$

which is non-expansive following lemma 3.2 since $\tilde{g}$ is 1-lipschitz. It is clear that a and b belong to $\mathrm{Fix}(F)$. It is enough now to check that $\mathrm{Fix}(F)\cap\mathbb{R}.e_2=\emptyset$. But if $x=te_2\in\mathrm{Fix}(F)$ we have

$$t=\langle e_2^*,x\rangle\ge\tilde{g}(x)=g(x)=\gamma(t)=|t|+\varphi(|t|)>|t|,$$

a contradiction. And this shows that $\mathrm{Fix}(F)$ is not connected. □

Corollary 3.6. *There exist on the 2-dimensional euclidean space $\mathbb{R}^2$ a linear functional f of norm 1 and a 1-lipschitz function $\tilde{g}$ such that the sublevels of $f+\tilde{g}$ are not all connected.*

Proof. In lemma 3.5, we can take (e_1,e_2) an orthonormal basis and $\varphi(t)=\sqrt{1+t^2}-t$. Choosing $f=-e_2^*$, and $\tilde{g}$ and F as above we get

$$L_0=\{x:(f+\tilde{g})(x)\le0\}=\{x:\langle e_2^*,x\rangle\ge\tilde{g}(x)\}=\mathrm{Fix}(F)$$

which is not connected. □

Theorem 3.7 (cf. [3], Corollary 7.7)**.** *Let E be a normed space of dimension at least 2. Then there exists a non-expansive set-valued mapping $F:E\rightrightarrows E$ with closed convex values such that* $\mathrm{Fix}(F)$ *is not connected.*

Proof. Take two non-proportional continuous linear functionals φ and ψ on E, define $\Phi:E\to\mathbb{R}^2$ by $\Phi(x)=\big(\varphi(x),\psi(x)\big)$ and equip $\mathbb{R}^2$ with the quotient norm : $|||u|||=\inf_{x\in\Phi^{-1}(u)}\|x\|$.

By lemmas 3.1, 3.3 and 3.5 we can construct a a non-expansive set-valued mapping $G:\mathbb{R}^2\rightrightarrows\mathbb{R}^2$ with closed convex values such that $\mathrm{Fix}(G)$ is non connected. It is enough now to define

$$F(x)=\Phi^{-1}\big(G\circ\Phi(x)\big)$$

for $x\in E$ for getting the desired set-valued mapping. □

4. Connectedness

We shall consider in this section the case where H is a convex compact subset of a Banach space and F is a non-expansive set-valued mapping $F : H \rightrightarrows H$ with closed convex values. In particular, if E is a finite-dimensional normed space, F is a non-expansive set-valued mapping $F : H \rightrightarrows H$ with closed convex values and $R = \sup_{x\in E} \sup_{y\in F(x)} \|y\| < +\infty$, one can take for H the ball of radius R and get that $\mathrm{Fix}(F) = \mathrm{Fix}(F_{|H})$.

Theorem 4.1. *If H is a convex compact subset of a Banach space and F is a non-expansive set-valued mapping $F : H \rightrightarrows H$ with closed convex values, then* $\mathrm{Fix}(F)$ *is necessarily non-empty compact and connected.*

Proof. For any integer $n \geq 1$ let $q_n = 1 - 2^{-n}$ and $F_n : H \rightrightarrows H$ be the set-valued mapping

$$x \mapsto q_n F(x) + 2^{-n} H$$

which is well defined since $F_n(x) \subset H$. Clearly F_n is q_n-contractive. Thus $\mathrm{Fix}(F_n)$ is non-empty, closed hence compact, and connected following [1]. The set $\mathcal{K}(H)$ of non-empty compact subsets of H equipped with Vietoris topology is compact.

Lemma 4.2. *The set* $\mathrm{Fix}(F)$ *is the limit in $\mathcal{K}(H)$ of the sequence* $(\mathrm{Fix}(F_n))$.

Proof. Let K be any cluster value of the sequence $\big(\mathrm{Fix}(F_n)\big)$. Notice first that $\mathrm{Fix}(F) \subset \mathrm{Fix}(F_n)$: indeed if $x \in \mathrm{Fix}(F)$ then

$$x = q_n x + 2^{-n} x \in q_n F(x) + 2^{-n} H = F_n(x),$$

so $\mathrm{Fix}(F) \subset \mathrm{Fix}(F_n)$ and $\mathrm{Fix}(F) \subset K$. Conversely, suppose that $x^* \in K$. Then x^* is a cluster value of some sequence (x_n) with $x_n \in \mathrm{Fix}(F_n)$. Then for $\varepsilon > 0$, one can find n such that $\|x^* - x_n\| < \varepsilon$ and $2^{-n}M < \varepsilon$ where $M = \sup_{x\in H} \|x\|$.

So there exist $z_n \in F(x_n)$ and $u_n \in H$ such that $x_n = q_n z_n + 2^{-n} u_n$. And since F is non-expansive, there is $z'_n \in F(x^*)$ such that

$$\|z_n - z'_n\| \leq d_H\big(F(x^*), F(x_n)\big) \leq \|x^* - x_n\|.$$

Thus

$$\|x_n - z_n\| = \|2^{-n} u_n - (1 - q_n) z_n\| \leq 2^{-n}\|u_n\| + 2^{-n}\|z_n\| \leq 2^{1-n} M$$

and

$$\begin{aligned} d(x^*, F(x^*)) &\leq \|x^* - z'_n\| \leq \|x^* - x_n\| + \|x_n - z_n\| + \|z_n - z'_n\| \\ &\leq 2\varepsilon + 2^{1-n} M < 4\varepsilon \end{aligned}$$

and since ε is arbitrary we conclude that $x^* \in \mathrm{Fix}(F)$. Hence $K \subset \mathrm{Fix}(F)$. It follows that $\mathrm{Fix}(F)$ is the only cluster value in $\mathcal{K}(H)$ of the sequence $\big(\mathrm{Fix}(F_n)\big)$, thus that this sequence converges to $\mathrm{Fix}(F)$. □

And since the set of connected compact subsets of H is closed in $\mathcal{K}(H)$, we conclude that $\mathrm{Fix}(F)$ is connected. So the proof of theorem 4.1 is complete. □

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