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FURTHER APPLICATIONS OF TWO MINIMAX THEOREMS

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In this paper, we deal with new applications of two minimax theorems of B. Ricceri ([5],[9]). Here is a particular case of one of the results that we obtain: Let $(T,\mathcal{F},\mu)$ be a non-atomic measure space, with $\mu(T) < +\infty$, $(E,\|\cdot\|)$ a real Banach space, $I \subseteq E$ an unbounded set whose closure does not contain 0. Moreover, let p,q,r,s be four numbers such that $0 < s < q \leq p$, $p \geq 1$, $r > 1$. Set $X := \{f \in L^p(T,E) : f(T) \subseteq I\}$. Then, one has

$$\inf_{u\in X} \frac{(\int_T \|u(t)\|^s d\mu)^r}{\int_T \|u(t)\|^q d\mu} = 0 \; .$$

1. Introduction

There is no doubt that the most famous minimax result is the Fan-Sion theorem ([1],[11]) which, for a given function of two variables, requires the quasi-convexity with respect to one variable and the quasi-concavity with respect to the other. Starting from [14], many topological minimax theorems (that is, with assumptions of purely topological nature) were established ([2],[3],[12],[13]). But only in 1992, H. König ([4]) was able to prove a topological minimax which is a formal generalization of the Fan-Sion theorem. In particular, the quasi-convexity assumption is replaced by requiring that the intersections of

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finite families of sublevel sets is connected. Such an assumption, out of a convex setting, is very hard to be satisfied. In other words, while the theoretical value of König's theorem is unquestionable, its applicability, out of the quasi-convex setting, is very hard. The optimal topological assumption replacing quasi-convexity is, of course, requiring that the single sublevel sets are connected. In [5], B. Ricceri observed that, with such a weaker condition, König's theorem is no longer true ([5], Example 1.1). At the same time, he showed that such a condition is able to ensure the minimax equality when the other variable of the considered function runs over a real interval. Later, refining one of the results of [5], Ricceri established the following (Theorem 5.9 of [6]):

Theorem 1.1. *Let X be a topological space, $I \subseteq \mathbb{R}$ a compact interval, and $f : X \times I \to \mathbb{R}$ a function which is lower semicontinuous in X and upper semicontinuous and quasi-concave in I. Moreover, assume that there exists a set $D \subseteq I$ dense in I such that, for each $\lambda \in D$ and $r \in \mathbb{R}$, the set*

$$\{x \in X : f(x,\lambda) < r\}$$

is connected.

Then, one has

$$\sup_{\lambda \in I} \inf_{x \in X} f(x,\lambda) = \inf_{x \in X} \sup_{\lambda \in I} f(x,\lambda)$$

More recently, a kind of variant of Theorem 1.1 has been obtained (Theorem 1.2 of [9]):

Theorem 1.2. *Let X be a topological space, I be a compact real interval and $f : X \times I \to \mathbb{R}$ an upper semicontinuous function which is continuous in X. Assume that:*

(a_2) there exists a set $D \subseteq I$, dense in I, such that, for each $\lambda \in D$ and $r \in \mathbb{R}$, the set

$$\{x \in X : f(x,\lambda) < r\}$$

is connected.

(b_2) for each $x \in X$, the set of all global maxima of the function $f(x,\cdot)$ is connected.

Then, one has

$$\sup_{\lambda \in I} \inf_{x \in X} f(x,\lambda) = \inf_{x \in X} \sup_{\lambda \in I} f(x,\lambda)$$

The aim of the present paper is to establish further applications of Theorems 1.1 and 1.2 besides the ones already provided in [7] and [9].

2. Functionals having the same infimum

Throughout this section, X is a real Banach space, $\varphi : X \to \mathbb{R}$ is a non-zero continuous linear functional and $\psi : X \to \mathbb{R}$ is a Lipschitzian functional whose Lipschitzian constant L is equal to $\|\varphi\|_{X^*}$.

Let us recall the following application of Theorem 1.1 established in [7] ([7], Theorem 3):

Theorem 2.1. *Let $\gamma : [-1,1] \to \mathbb{R}$ be a continuous function which is derivable in $]-1,1[$. Assume that γ' is strictly increasing in $]-1,1[$, with $\gamma'(]-1,1[) = \mathbb{R}$. Denote by η the inverse of the function γ'.*

Then, one has

$$\max\left\{\inf_{x\in X}(\varphi(x)-\psi(x))-\gamma(-1), \inf_{x\in X}(\varphi(x)+\psi(x))-\gamma(1)\right\}$$
$$= \inf_{x\in X}\big(\varphi(x)+\eta(\psi(x))\psi(x)-\gamma(\eta(\psi(x)))\big).$$

We now want to obtain two results applying Theorem 2.1 with two specific choices of the function γ.

First, we prove the following general result which is inspired to the proof of Theorem 4 of [7]:

Theorem 2.2. *Let $f, g : X \to \mathbb{R}$. Assume that, for each bounded set $C \subset X$, one has $\inf_C f > -\infty$ and $\inf_C g > 0$. Assume also that*

$$\inf_X (f+g) = \inf_X f \; . \tag{2.1}$$

Then, one has

$$\inf_X f = \liminf_{\|x\|\to+\infty} f(x) \; .$$

Proof. Arguing by contradiction, suppose that $\inf_X f < \liminf_{\|x\|\to+\infty} f(x)$. Fix ρ so that

$$\inf_X f < \rho < \liminf_{\|x\|\to+\infty} f(x) \; . \tag{2.2}$$

So, by (2.2), there is $\delta > 0$ such that

$$f(x) > \rho$$

for all $x \in X$ with $\|x\| > \delta$. Now, by (2.1), there is a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty}(f(x_n)+g(x_n)) = \inf_X f \; . \tag{2.3}$$

Since $g > 0$, we have

$$\lim_{n\to\infty} f(x_n) = \inf_X f \ . \tag{2.4}$$

Hence, by (2.2), there is $\nu \in \mathbf{N}$ such that

$$f(x_n) < \rho$$

for all $n \geq \nu$. This implies that

$$\sup_{n\geq\nu} \|x_n\| < \delta \ . \tag{2.5}$$

Consequently, by assumption, $\inf_n f(x_n) > -\infty$ and so, in view of (2.3) and (2.4), we have

$$\lim_{n\to\infty} g(x_n) = 0 \ .$$

By assumption, this implies that the sequence $\{x_n\}$ is not bounded, contradicting (2.5). The proof is complete.

□

Now, we prove

Theorem 2.3. *For each $\alpha \in]0,1[$, we have*

$$\inf_{x\in X}(\varphi(x) + |\psi(x)|) = \inf_{x\in X}\left(\varphi(x) + |\psi(x)| + (1-\alpha)\left(1 + \frac{|\psi(x)|}{\alpha}\right)^{-\frac{\alpha}{1-\alpha}}\right).$$

Proof. Let $\alpha \in]0,1[$ and consider the function $\gamma : [-1,1] \to \mathbb{R}$ defined by:

$$\gamma(\lambda) = -(1-|\lambda|)^{\alpha} - \alpha|\lambda|$$

for all $\lambda \in [-1,1]$. γ is continuous in $[-1,1]$ and derivable in $]-1,1[$, also we have

$$\gamma'(\lambda) = \begin{cases} \alpha\frac{|\lambda|}{\lambda}\left(\frac{1}{(1-|\lambda|)^{1-\alpha}} - 1\right) & \text{if } |\lambda| < 1,\ \lambda \neq 0 \\ 0 & \text{if } \lambda = 0 \end{cases}.$$

So, γ' is strictly increasing and $\gamma'(]-1,1[) = \mathbb{R}$. The inverse of γ' is given by

$$\eta(\mu) = \begin{cases} \frac{|\mu|}{\mu}\left(1 - \left(1 + \frac{|\mu|}{\alpha}\right)^{-\frac{1}{1-\alpha}}\right) & \text{if } \mu \neq 0 \\ 0 & \text{if } \mu = 0 \end{cases},$$

so for each $x \in X \setminus \psi^{-1}(0)$ we have

$$\eta(\psi(x))\psi(x) - \gamma(\eta(\psi(x)))$$
$$= |\psi(x)| \left(1 - \left(1 + \frac{|\psi(x)|}{\alpha}\right)^{-\frac{1}{1-\alpha}}\right) + \left(1 - \left(1 - \left(1 + \frac{|\psi(x)|}{\alpha}\right)^{-\frac{1}{1-\alpha}}\right)\right)^{\alpha} +$$
$$+ \alpha \left(1 - \left(1 + \frac{|\psi(x)|}{\alpha}\right)^{-\frac{1}{1-\alpha}}\right) = \alpha + |\psi(x)| + (1-\alpha)\left(1 + \frac{|\psi(x)|}{\alpha}\right)^{-\frac{\alpha}{1-\alpha}}$$

and this is true also for $\psi(x) = 0$. From [7], we know that

$$\max\left\{\inf_{x\in X}(\varphi(x) + \psi(x)), \inf_{x\in X}(\varphi(x) - \psi(x))\right\} = \inf_{x\in X}(\varphi(x) + |\psi(x)|) \,.$$

Now, we can apply Theorem 2.1, obtaining

$$\inf_{x\in X}(\varphi(x) + |\psi(x)|) + \alpha = \inf_{x\in X}\left(\alpha + \varphi(x) + |\psi(x)| + (1-\alpha)\left(1 + \frac{|\psi(x)|}{\alpha}\right)^{-\frac{\alpha}{1-\alpha}}\right),$$

and hence

$$\inf_{x\in X}(\varphi(x) + |\psi(x)|) = \inf_{x\in X}\left(\varphi(x) + |\psi(x)| + (1-\alpha)\left(1 + \frac{|\psi(x)|}{\alpha}\right)^{-\frac{\alpha}{1-\alpha}}\right).$$

□

We point out the following particular case of Theorem 2.3 (for $\alpha = \frac{1}{2}$):

$$\inf_{x\in X}(\varphi(x) + |\psi(x)|) = \inf_{x\in X}\left(\varphi(x) + |\psi(x)| + \frac{1}{4|\psi(x)| + 2}\right).$$

Remark 2.1. Notice that Theorem 2.3 is no longer true, in general, if $L > \|\varphi\|_{X^*}$. Indeed, fix $\lambda > 1$ and take $\psi(x) = \lambda\|\varphi\|_{X^*}\|x\|$. So, we have

$$\varphi(x) + \psi(x) \geq (\lambda - 1)\|\varphi\|_{X^*}\|x\|$$

for all $x \in X$. This implies that

$$\inf_{x\in X}(\varphi(x) + \psi(x)) = 0$$

and

$$\lim_{\|x\|\to+\infty}(\varphi(x) + \psi(x)) = +\infty \,.$$

Of course, for each bounded set $C \subset X$ and $\alpha \in]0,1[$, we have

$$\inf_{x\in C}(1-\alpha)\left(1+\frac{\psi(x)}{\alpha}\right)^{-\frac{\alpha}{1-\alpha}} > 0$$

and hence, in view of Theorem 2.2, we have

$$\inf_{x\in X}(\varphi(x)+\psi(x)) < \inf_{x\in X}\left(\varphi(x)+\psi(x)+(1-\alpha)\left(1+\frac{\psi(x)}{\alpha}\right)^{-\frac{\alpha}{1-\alpha}}\right).$$

The other result is as follows:

Theorem 2.4. *We have*

$$\inf_{x\in X}(\varphi(x)+|\psi(x)|) = \inf_{x\in X}\left(\varphi(x)+\sqrt{\psi(x)^2+2|\psi(x)|}+\arcsin\frac{1}{1+|\psi(x)|}\right)-1.$$

Proof. Consider the function γ defined by

$$\gamma(\lambda) = \arcsin|\lambda| - |\lambda|$$

for all $\lambda \in [-1,1]$. Following the exact same steps as before, we compute

$$\gamma'(\lambda) = \begin{cases} \frac{|\lambda|}{\lambda}\left(\frac{1}{\sqrt{1-\lambda^2}}-1\right) & \text{if } |\lambda|<1,\ \lambda\neq 0 \\ 0 & \text{if } \lambda = 0 \end{cases}$$

and

$$\eta(\mu) = \begin{cases} \frac{|\mu|}{\mu}\sqrt{1-\frac{1}{(1+|\mu|)^2}} & \text{if } \mu\neq 0 \\ 0 & \text{if } \mu = 0 \end{cases}$$

hence

$$\begin{aligned} &\eta(\psi(x))\psi(x)-\gamma(\eta(\psi(x))) \\ &= |\psi(x)|\sqrt{1-\frac{1}{(1+|\psi(x)|)^2}}-\arcsin\sqrt{1-\frac{1}{(1+|\psi(x)|)^2}}+\sqrt{1-\frac{1}{(1+|\psi(x)|)^2}} \\ &\qquad = \sqrt{\psi(x)^2+2|\psi(x)|}+\arcsin\frac{1}{1+|\psi(x)|}-\frac{\pi}{2} \end{aligned}$$

so if we apply Theorem 2.1, since $\gamma(1)=\gamma(-1)=\frac{\pi}{2}-1$, we obtain

$$\begin{aligned} &\inf_{x\in X}(\varphi(x)+|\psi(x)|)-\frac{\pi}{2}+1 \\ &\qquad = \inf_{x\in X}\left(\varphi(x)+\sqrt{\psi(x)^2+2|\psi(x)|}+\arcsin\frac{1}{1+|\psi(x)|}-\frac{\pi}{2}\right) \end{aligned}$$

hence

$$\inf_{x\in X}(\varphi(x)+|\psi(x)|)=\inf_{x\in X}\left(\varphi(x)+\sqrt{\psi(x)^2+2|\psi(x)|}+\arcsin\frac{1}{1+|\psi(x)|}\right)-1.$$

□

Remark 2.2. Notice that also Theorem 2.4, in general, is no longer true when $L>\|\varphi\|_{X^*}$. To see this, it is enough to consider $X=\mathbb{R}$, with $\varphi(x)=x$ and $\psi(x)=2|x|\ \forall x\in\mathbb{R}$. A few easy computations show that

$$0=\inf_{x\in\mathbb{R}}(x+2|x|)<\inf_{x\in\mathbb{R}}\left(x+\sqrt{4x^2+4|x|}+\arcsin\frac{1}{1+2|x|}-1\right)=\frac{\pi}{2}-1$$

3. Infimum of certain functionals on L^p

Throughout this section, $(T,\mathcal{F},\mu)$ is a measure space, with $\mu(T)<+\infty$, E is a real Banach space and $p\geq 1$.

We denote by $L^p(T,E)$ the space of all equivalence classes of strongly μ-measurable functions u with $\int_T\|u(t)\|^p d\mu<+\infty$, equipped with the norm

$$\|u\|_{L^p(T,E)}=\left(\int_T\|u(t)\|^p d\mu\right)^{\frac{1}{p}}.$$

We will write $L^p(T)$ instead of $L^p(T,\mathbb{R})$.

A set $D\subseteq L^p(T,E)$ is said to be decomposable if, for every $u,v\in D$ and every $A\in\mathcal{F}$, the function

$$t\longmapsto\chi_A(t)u(t)+(1-\chi_A(t))v(t)$$

is an element of D, where χ_A is the characteristic function of A.

A function $f:T\times E\to\mathbb{R}$ is said to be a Carathéodory function if it is measurable in T and continuous in E.

In [9], as an application of Theorem 1.2, the following result has been obtained (Theorem 2.4 of [9]):

Theorem 3.1. *Let $X\subseteq L^p(T,E)$ a decomposable set, $[a,b]$ a compact real interval, and $\gamma:[a,b]\to\mathbb{R}$ a convex (resp. concave) and continuous function.*

Moreover, let $\varphi, \psi, \omega : T \times E \to \mathbb{R}$ *be three Carathéodory functions such that, for some* $M \in L^1(T)$, $k \in \mathbb{R}$, *one has*

$$\max\{|\varphi(t,x)|, |\psi(t,x)|, |\omega(t,x)|\} \leq M(t) + k\|x\|^p$$

for all $(t,x) \in T \times E$ *and*

$$\gamma(a)\int_T \psi(t,u(t))d\mu + a\int_T \omega(t,u(t))d\mu \neq \gamma(b)\int_T \psi(t,u(t))d\mu + b\int_T \omega(t,u(t))d\mu$$

for all $u \in X$ *such that* $\int_T \psi(t,u(t))d\mu > 0$ *(resp.* $\int_T \psi(t,u(t))d\mu < 0$*).*

Then, one has

$$\sup_{\lambda\in[a,b]}\inf_{u\in X}\left(\int_T \varphi(t,u(t))d\mu + \gamma(\lambda)\int_T \psi(t,u(t))d\mu + \lambda\int_T \omega(t,u(t))d\mu\right) =$$
$$= \inf_{u\in X}\sup_{\lambda\in[a,b]}\left(\int_T \varphi(t,u(t))d\mu + \gamma(\lambda)\int_T \psi(t,u(t))d\mu + \lambda\int_T \omega(t,u(t))d\mu\right).$$

Let $I \subseteq E$ be a non-empty set. We denote by $\mathcal{A}_I$ the class of all pairs of continuous functions $\omega, \psi : E \to \mathbb{R}$, with $\omega(x) \geq 0$ and $\psi(x) > 0$ for all $x \in I$, such that

$$\sup_{x\in E}\frac{|\omega(x)| + |\psi(x)|}{1 + \|x\|^p} < +\infty$$

and

$$\sup_{x\in I}\frac{\omega(x)}{\psi(x)} < +\infty\ .$$

Moreover, we denote by $\mathcal{B}_I$ the family of all decomposable subsets X of $L^p(T,E)$ such that $u(T) \subseteq I$ for all $u \in X$, and containing each constant function taking its value in I.

Remark 3.1. Of course, if $(\omega, \psi) \in \mathcal{A}_I$ and $X \in \mathcal{B}_I$, we have

$$\inf_{x\in I}\frac{\omega(x)}{\psi(x)} \leq \frac{\int_T \omega(u(t))d\mu}{\int_T \psi(u(t))d\mu} \leq \sup_{x\in I}\frac{\omega(x)}{\psi(x)}$$

for all $u \in X$.

In this setting, applying Theorem 3.1, we get the following two results.

Theorem 3.2. *Let* $(\omega, \psi) \in \mathcal{A}_I$*,* $X \in \mathcal{B}_I$ *and let* $r > 1$*. Set*

$$a := \left(\frac{1}{r}\inf_{x\in I}\frac{\omega(x)}{\psi(x)}\right)^{\frac{1}{r-1}}$$

and

$$b := \left(\frac{1}{r}\sup_{x\in I}\frac{\omega(x)}{\psi(x)}\right)^{\frac{1}{r-1}} .$$

Then, one has

$$\inf_{u\in X}\frac{(\int_T \omega(u(t))d\mu)^r}{\int_T \psi(u(t))d\mu} = \left(\mu(T)\frac{r^{\frac{r}{r-1}}}{r-1}\sup_{\lambda\in[a,b]}\inf_{x\in I}\big(\lambda\omega(x)-\lambda^r\psi(x)\big)\right)^{r-1} \tag{3.1}$$

Proof. By Remark 3.1, we have

$$\left\{\left(\frac{\int_T \omega(u(t))d\mu}{r\int_T \psi(u(t))d\mu}\right)^{\frac{1}{r-1}} : u\in X\right\} \subseteq [a,b] .$$

Since X contains each constant function taking its value in I, we clearly have

$$\inf_{u\in X}\left(\int_T \big(\lambda\omega(u(t))-\lambda^r\psi(u(t))\big)d\mu\right) = \mu(T)\inf_{x\in I}\big(\lambda\omega(x)-\lambda^r\psi(x)\big)$$

for all $\lambda\in[a,b]$, and hence

$$\sup_{\lambda\in[a,b]}\inf_{u\in X}\left(\int_T \big(\lambda\omega(u(t))-\lambda^r\psi(u(t))\big)d\mu\right) = \mu(T)\sup_{\lambda\in[a,b]}\inf_{x\in I}\big(\lambda\omega(x)-\lambda^r\psi(x)\big). \tag{3.2}$$

Now, since $\int_T \psi(u(t))d\mu > 0$ for all $u\in X$, we can apply Theorem 3.1, with $\gamma(\lambda) = -\lambda^r$ and $\varphi = 0$, obtaining

$$\sup_{\lambda\in[a,b]}\inf_{u\in X}\left(\int_T \big(\lambda\omega(u(t))-\lambda^r\psi(u(t))\big)d\mu\right)$$
$$= \inf_{u\in X}\sup_{\lambda\in[a,b]}\left(\lambda\int_T \omega(u(t))d\mu - \lambda^r\int_T \psi(u(t))d\mu\right). \tag{3.3}$$

Fix $u\in X$. The function $\lambda\mapsto\lambda\int_T \omega(u(t))d\mu-\lambda^r\int_T\psi(u(t))d\mu$ is concave in $[0,+\infty[$ and its derivative vanishes at the point $\left(\frac{\int_T \omega(u(t))d\mu}{r\int_T\psi(u(t))d\mu}\right)^{\frac{1}{r-1}}$ which lies in $[a,b]$. Consequently, we have

$$\inf_{u\in X}\sup_{\lambda\in[a,b]}\left(\lambda\int_T \omega(u(t))d\mu - \lambda^r\int_T \psi(u(t))d\mu\right)$$
$$= \inf_{u\in X}\left(\left(\frac{\int_T \omega(u(t))d\mu}{r\int_T\psi(u(t))d\mu}\right)^{\frac{1}{r-1}}\int_T \omega(u(t))d\mu - \left(\frac{\int_T \omega(u(t))d\mu}{r\int_T\psi(u(t))d\mu}\right)^{\frac{r}{r-1}}\int_T\psi(u(t))d\mu\right)$$
$$= \inf_{u\in X}\frac{r-1}{r^{\frac{r}{r-1}}}\left(\frac{(\int_T \omega(u(t))d\mu)^r}{\int_T\psi(u(t))d\mu}\right)^{\frac{1}{r-1}} .$$

Therefore, in view of (3.2) and (3.3), we have

$$\inf_{u\in X}\frac{r-1}{r^{\frac{r}{r-1}}}\left(\frac{(\int_T \omega(u(t))d\mu)^r}{\int_T \psi(u(t))d\mu}\right)^{\frac{1}{r-1}} = \mu(T)\sup_{\lambda\in[a,b]}\inf_{x\in I}\left(\lambda\,\omega(x)-\lambda^r\psi(x)\right)$$

which is equivalent to (3.1).

□

It is worth noticing the following corollary of Theorem 3.2:

Theorem 3.3. *Let* $(\omega,\psi)\in\mathcal{A}_I$*,* $X\in\mathcal{B}_I$ *and let* $r>1$. *Assume that*

$$\inf_{x\in I}(\omega(x)-\lambda\,\psi(x)) = -\infty \tag{3.4}$$

for all $\lambda>0$.

Then, one has

$$\inf_{u\in X}\frac{(\int_T \omega(u(t))d\mu)^r}{\int_T \psi(u(t))d\mu} = 0\ .$$

Proof. Writing

$$\omega(x)-\lambda\,\psi(x) = \psi(x)\left(\frac{\omega(x)}{\psi(x)}-\lambda\right)\ ,$$

from (3.4), we infer that $\inf_{x\in I}\frac{\omega(x)}{\psi(x)}=0$. So, (3.1) holds with $a=0$ and hence, by (3.4) again, the right-hand side of (3.1) is 0, as claimed. □

In turn, a particular case of Theorem 3.3 is as follows

Proposition 3.1. *Let* I *be an unbounded set whose closure does not contain* 0*, and let* q,r,s *be three positive numbers such that* $s<q\le p$ *and* $r>1$.

Then, for each $X\in\mathcal{B}_I$*, one has*

$$\inf_{u\in X}\frac{(\int_T \|u(t)\|^s d\mu)^r}{\int_T \|u(t)\|^q d\mu} = 0\ .$$

Proof. It is enough to notice that the pair $(\|\cdot\|^s,\|\cdot\|^q)$ belongs to $\mathcal{A}_I$ and that (3.4) is satisfied. □

Theorem 3.4. *Let* $(\omega,\psi)\in\mathcal{A}_I$*, with* $\inf_{x\in I}\frac{\omega(x)}{\psi(x)}>0$*, and let* $X\in\mathcal{B}_I$. *Set*

$$a:=\log\left(\inf_{x\in I}\frac{\omega(x)}{\psi(x)}\right)$$

and

$$b:=\log\left(\sup_{x\in I}\frac{\omega(x)}{\psi(x)}\right).$$

Then, one has

$$\inf_{u\in X}\left(\int_T \omega(u(t))d\mu\left(\log\left(\frac{\int_T \omega(u(t))d\mu}{\int_T \psi(u(t))d\mu}\right)-1\right)\right)=\mu(T)\sup_{\lambda\in[a,b]}\inf_{x\in I}\left(\lambda\,\omega(x)-e^{\lambda}\psi(x)\right). \tag{3.5}$$

Proof. By Remark 3.1, we have

$$\left\{\log\left(\frac{\int_T \omega(u(t))d\mu}{\int_T \psi(u(t))d\mu}\right) : u\in X\right\}\subseteq[a,b].$$

As we have seen in the proof of Theorem 3.2, we have

$$\sup_{\lambda\in[a,b]}\inf_{u\in X}\left(\int_T\left(\lambda\,\omega(u(t))-e^{\lambda}\psi(u(t))\right)d\mu\right)=\mu(T)\sup_{\lambda\in[a,b]}\inf_{x\in I}\left(\lambda\,\omega(x)-e^{\lambda}\psi(x)\right). \tag{3.6}$$

Then, since $\int_T \psi(u(t))d\mu>0$ for all $u\in X$, we can apply Theorem 3.1 with $\gamma(\lambda)=-e^{\lambda}$ and $\varphi=0$, obtaining

$$\begin{aligned}&\sup_{\lambda\in[a,b]}\inf_{u\in X}\left(\int_T\left(\lambda\,\omega(u(t))-e^{\lambda}\psi(u(t))\right)d\mu\right)\\&\qquad=\inf_{u\in X}\sup_{\lambda\in[a,b]}\left(\lambda\int_T\omega(u(t))d\mu-e^{\lambda}\int_T\psi(u(t))d\mu\right).\end{aligned} \tag{3.7}$$

Fix $u\in X$. The derivative of the concave function $\lambda\mapsto\lambda\int_T\omega(u(t))d\mu-e^{\lambda}\int_T\psi(u(t))d\mu$ vanishes at the point $\log\left(\frac{\int_T\omega(u(t))d\mu}{\int_T\psi(u(t))d\mu}\right)$ which lies in $[a,b]$. So, we have

$$\begin{aligned}&\inf_{u\in X}\sup_{\lambda\in[a,b]}\left(\lambda\int_T\omega(u(t))d\mu-e^{\lambda}\int_T\psi(u(t))d\mu\right)\\&=\inf_{u\in X}\left(\log\left(\frac{\int_T\omega(u(t))d\mu}{\int_T\psi(u(t))d\mu}\right)\int_T\omega(u(t))d\mu-e^{\log\left(\frac{\int_T\omega(u(t))d\mu}{\int_T\psi(u(t))d\mu}\right)}\int_T\psi(u(t))d\mu\right)\\&\qquad=\inf_{u\in X}\left(\int_T\omega(u(t))d\mu\left(\log\left(\frac{\int_T\omega(u(t))d\mu}{\int_T\psi(u(t))d\mu}\right)-1\right)\right).\end{aligned}$$

Therefore, in view of (3.6) and (3.7), we obtain (3.5). □

By taking ψ,ω that satisfy the conditions of either Theorem 3.2 or Theorem 3.4, we can compute the infimum of a variety of functionals of the type figuring in the left-hand sides of (3.1) and (3.5).

We remark that Theorem 3.1 is just one of the possible ways of applying Theorem 1.2 to integral functionals in L^p-spaces. Another way is the following.

Theorem 3.5. *Let $X \subseteq L^p(T,E)$ be a decomposable set, $[a,b]$ a compact real interval and $\gamma, \delta \in C^0([a,b]) \cap C^1(]a,b[)$ two functions such that $\gamma'(\lambda) \neq 0$ for all $\lambda \in [a,b]$ and $\frac{\delta'}{\gamma'}$ is strictly monotone in $]a,b[$. Moreover, let $\varphi, \psi, \omega : T \times E \to \mathbb{R}$ be three Carathéodory functions such that, for some $M \in L^1(T)$, $k \in \mathbb{R}$, one has*

$$\max\{|\varphi(t,x)|, |\psi(t,x)|, |\omega(t,x)|\} \leq M(t) + k\|x\|^p$$

for all $(t,x) \in T \times E$ and

$$\gamma(a)\int_T \psi(t,u(t))d\mu + \delta(a)\int_T \omega(t,u(t))d\mu \neq \gamma(b)\int_T \psi(t,u(t))d\mu + \delta(b)\int_T \omega(t,u(t))$$

in each of the two following cases:

(i) $\frac{\delta'}{\gamma'}$ *is strictly increasing, $u \in X$ and $\gamma'(\lambda)\int_T \omega(t,u(t))d\mu > 0$ for all $\lambda \in]a,b[$;*

(ii) $\frac{\delta'}{\gamma'}$ *is strictly decreasing, $u \in X$ and $\gamma'(\lambda)\int_T \omega(t,u(t))d\mu < 0$ for all $\lambda \in]a,b[$.*

Then, one has

$$\sup_{\lambda \in [a,b]} \inf_{u \in X} \left(\int_T \varphi(t,u(t))d\mu + \gamma(\lambda)\int_T \psi(t,u(t))d\mu + \delta(\lambda)\int_T \omega(t,u(t))d\mu \right)$$
$$= \inf_{u \in X} \sup_{\lambda \in [a,b]} \left(\int_T \varphi(t,u(t))d\mu + \gamma(\lambda)\int_T \psi(t,u(t))d\mu + \delta(\lambda)\int_T \omega(t,u(t))d\mu \right).$$

Proof. Consider the function $f : X \times [a,b] \to \mathbb{R}$ defined by

$$f(u,\lambda) = \int_T \varphi(t,u(t))d\mu + \gamma(\lambda)\int_T \psi(t,u(t))d\mu + \delta(\lambda)\int_T \omega(t,u(t))d\mu$$

for all $(u,\lambda) \in X \times [a,b]$. Fix $u \in X$. Assume that $\int_T \omega(t,u(t))d\mu \neq 0$. We check that $f(u,\cdot)$ has a unique global maximum in $[a,b]$. Indeed, if $f'_\lambda(u,\cdot) \neq 0$ for all $\lambda \in]a,b[$, then $f(u,\cdot)$ is strictly monotone and so it reaches its maximum only either at a or at b. Otherwise, since $\frac{\delta'}{\gamma'}$ is strictly monotone, $f'_\lambda(u,\cdot)$ vanishes only at the point $\tilde{\lambda} \in]a,b[$ such that $\frac{\delta'(\tilde{\lambda})}{\gamma'(\tilde{\lambda})} = -\frac{\int_T \psi(t,u(t))d\mu}{\int_T \omega(t,u(t))d\mu}$. If $\tilde{\lambda}$ is a global maximum of $f(u,\cdot)$, then it is the only one in view of Rolle's theorem. So, suppose that $\tilde{\lambda}$ is not a global maximum of $f(u,\cdot)$. Now, assume that $\frac{\delta'}{\gamma'}$ is strictly increasing. If $\gamma'(\lambda)\int_T \omega(t,u(t))d\mu < 0$ for all $\lambda \in]a,b[$, we would have $f'_\lambda(u,\lambda) < 0$ for all $\lambda \in]\tilde{\lambda},b]$ and $f'_\lambda(u,\lambda) > 0$ for all $\lambda \in [a,\tilde{\lambda}[$ and hence $\tilde{\lambda}$ would be a global maximum of $f(u,\cdot)$. Consequently, we have $\gamma'(\lambda)\int_T \omega(t,u(t))d\mu > 0$ for all $\lambda \in]a,b[$ and hence, by assumption, $f(u,a) \neq f(u,b)$. Therefore, $f(u,\cdot)$ reaches its maximum only at a or at b. With similar arguments, we get the same conclusion when $\frac{\delta'}{\gamma'}$ is strictly decreasing. Now, suppose that $\int_T \omega(t,u(t))d\mu = 0$. In this case, $f(u,\cdot)$ is either constant or strictly monotone. We then infer that, for every $u \in X$, the set of all global maxima of $f(u,\cdot)$ is connected. On the other hand, it is clear that the function f is continuous in $X \times [a,b]$. Furthermore, by Théorème 7 of [10], $f(\cdot,\lambda)$ is inf-connected for all $\lambda \in [a,b]$. Now, we can apply Theorem 1.2, and the proof is complete. □

Now, exactly as we did for Theorem 3.1, we want to apply Theorem 3.5 to a specific case, obtaining results on the infimum of certain integral functionals. This gives us the following result.

Theorem 3.6. *Let $I \subseteq E$ be a non-empty set, $X \in \mathcal{B}_I$ and $\omega, \psi : \mathbb{R} \to \mathbb{R}$ two continuous functions such that $\omega(x) > 0$ for all $x \in I$ and*

$$\sup_{x\in E} \frac{\omega(x) + |\psi(x)|}{1 + \|x\|^p} < +\infty \,.$$

Then, we have

$$\inf_{u\in X} \sqrt{\left(\int_T \psi(u(t))d\mu\right)^2 + \left(\int_T \omega(u(t))d\mu\right)^2}$$
$$= \mu(T) \sup_{\lambda\in[-\frac{\pi}{2},\frac{\pi}{2}]} \inf_{x\in I}(\psi(x)\sin\lambda + \omega(x)\cos\lambda) \quad (3.8)$$

Proof. We are going to apply Theorem 3.5 taking $[a,b] = [-\frac{\pi}{2},\frac{\pi}{2}]$, $\gamma(\lambda) = \sin\lambda$ and $\delta(\lambda) = \cos\lambda$. Since $\frac{\delta'}{\gamma'}$ is strictly decreasing and $\gamma'(\lambda)\int_T \omega(u(t))d\mu > 0$ for all $\lambda \in]a,b[$, $u \in X$, no other condition has to be satisfied. Consequently, we have

$$\inf_{u\in X} \sup_{\lambda\in[-\frac{\pi}{2},\frac{\pi}{2}]} \left(\int_T \psi(u(t))d\mu \sin\lambda + \int_T \omega(u(t))d\mu \cos\lambda\right)$$
$$= \sup_{\lambda\in[-\frac{\pi}{2},\frac{\pi}{2}]} \inf_{u\in X} \left(\int_T \psi(u(t))d\mu \sin\lambda + \int_T \omega(u(t))d\mu \cos\lambda\right). \quad (3.9)$$

On the other hand, since $X \in \mathcal{B}_I$, we have

$$\sup_{\lambda\in-\frac{\pi}{2},\frac{\pi}{2}]} \inf_{u\in X} \left(\int_T \psi(u(t))d\mu \sin\lambda + \int_T \omega(u(t))d\mu \cos\lambda\right)$$
$$= \mu(T) \sup_{\lambda\in[-\frac{\pi}{2},\frac{\pi}{2}]} \inf_{x\in I}(\psi(x)\sin\lambda + \omega(x)\cos\lambda). \quad (3.10)$$

Fix $u \in X$. An easy checking shows that the function $\lambda \mapsto \int_T \psi(u(t))d\mu \sin\lambda + \int_T \omega(u(t))d\mu \cos\lambda$ reaches its maximum at the point $\arctan\left(\frac{\int_T \psi(u(t))d\mu}{\int_T \omega(u(t))d\mu}\right)$. So, we have

$$\begin{aligned}
&\sup_{\lambda\in[-\frac{\pi}{2},\frac{\pi}{2}]}\left(\int_T \psi(u(t))d\mu \sin\lambda + \int_T \omega(u(t))d\mu \cos\lambda\right)\\
&\qquad = \sin\left(\arctan\left(\frac{\int_T \psi(u(t))d\mu}{\int_T \omega(u(t))d\mu}\right)\right)\int_T \psi(u(t))d\mu +\\
&\qquad\quad \cos\left(\arctan\left(\frac{\int_T \psi(u(t))d\mu}{\int_T \omega(u(t))d\mu}\right)\right)\int_T \omega(u(t))d\mu\\
&= \frac{\int_T \psi(u(t))d\mu}{\int_T \omega(u(t))d\mu\sqrt{1+\left(\frac{\int_T \psi(u(t))d\mu}{\int_T \omega(u(t))d\mu}\right)^2}}\int_T \psi(u(t))d\mu + \frac{1}{\sqrt{1+\left(\frac{\int_T \psi(u(t))d\mu}{\int_T \omega(u(t))d\mu}\right)^2}}\int_T \omega(u(t))d\mu\\
&\qquad = \sqrt{\left(\int_T \psi(u(t))d\mu\right)^2 + \left(\int_T \omega(u(t))d\mu\right)^2}.
\end{aligned}$$

Now (3.8) follows directly from (3.9) and (3.10). □

Remark 3.2. Let X, ω and ψ be as in Theorem 3.6. Consider the set

$$K = \left\{\left(\int_T \omega(u(t))d\mu, \int_T \psi(u(t))d\mu\right) : u \in X\right\} \subseteq \mathbb{R}^2 .$$

Theorem 3.6 gives us the exact value of the distance of 0 from K. Since, by the Lyapunov convexity theorem, K is convex, this information is very useful in applying Theorem 1 of [8] to $\overline{K}$.

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