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TENSOR JOIN OF HYPERGRAPHS AND ITS SPECTRA

R. VISHNUPRIYA - R. RAJKUMAR

In this paper, we introduce three operations on hypergraphs by using tensors. We show that these three formulations are equivalent and we commonly call them as the tensor join. We show that any hypergraph can be viewed as a tensor join of hypergraphs. Tensor join enable us to obtain several existing and new classes of operations on hypergraphs. We compute the adjacency, the Laplacian, the normalized Laplacian spectrum of weighted hypergraphs constructed by this tensor join. Also we deduce some results on the spectra of hypergraphs in the literature. As an application, we construct several pairs of the adjacency, the Laplacian, the normalized Laplacian cospectral hypergraphs by using the tensor join.

1. Introduction

In spectral graph theory, the properties of graphs are investigated by the eigenvalues of various associated matrices, such as adjacency matrix, Laplacian matrix, signless Laplacian matrix, normalized Laplacian matrix etc; see, [5]. Likewise, in spectral hypergraph theory, spectra of different connectivity tensors and matrices associated to hypergraphs were studied in the literature; see, [1–4, 16, 17, 21]. Recently, Anirban Banerjee [2] introduced some connectivity matrices namely, the adjacency matrix, the Laplacian matrix and the normalized Laplacian matrix for unweighted hypergraphs. Therein, some of the properties

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of hypergraphs were studied using the spectrum of these associated matrices. Subsequently, Amitesh Sarkar and Anirban Banerjee [22] extend the definiton of the adjacency matrix of a hypergraph introduced in [2] to a weighted hypergraph. In the rest of this paper, we consider the matrix representation of hypergraphs defined in [22].

In the literature, several graph operations were defined and the spectra of graphs constructed by these graph operations were determined; see [5, 7, 14, 15, 18–20] and the references therein. Recently, Gayathri and Rajkumar [8] introduced a graph operation, namely, $\mathcal{M}$-join. Using this operation several new graph operations were defined and various graph operations in the literature were generalized. There in, the spectral properties of these graphs were investigated. In this direction, there are several hypergraph operations were defined in literature; see the survey paper [12]. In [22], several hypergraph operations, such as the weighted join, the generalized corona were introduced and the adjacency spectra of the hypergraphs formed by these operations were determined. Also some families of cospectral hypergraphs with respect to the adjacency matrix were constructed using these operations. The adjacency spectra of the Cartesian product of hypergraphs was obtained in [2].

Motivated by these, in this paper, we introduce some operations on hypergraphs via tensors. We obtain the spectra of the adjacency, the Laplacian, the normalized Laplacian matrices of the hypergraphs constructed by these operations.

The rest of the paper is arranged as follows: In Section 2, we recall some basic notations, definitions and results of graphs/hypergraphs and matrices. In Section 3, we introduce a special type of tensor, namely an indicating tensor corresponding to a finite sequence of mutually disjoint sets. Also, we define several particular cases of this tensor. In Section 4, we introduce three hypergraph operations by using indicating tensors. We show that these three formulations are equivalent and we commonly call them as the tensor join. We show that any hypergraph can be viewed as a tensor join of hypergraphs. Tensor join enable us to obtain several existing and new classes of operations on hypergraphs. In Section 5, we compute the spectrum of the adjacency, the Laplacian and the normalized Laplacian matrices of weighted hypergraphs constructed by the tensor join operations introduced in the previous section. Also we deduce some existing results on spectra of hypergraphs. By using the results proved in this section, we construct infinite families of simultaneously adjacency, Laplacian, normalized Laplacian cospectral hypergraphs by using this tensor join operation.

2. Preliminaries and notations

A *hypergraph* $H(V,E)$ consists of a non-empty set V and a multiset E of subsets of V. The elements of V are called *vertices* and the elements of E are called *hyperedges*, or simply *edges* of H. An edge of cardinality one is called a *loop*. The *rank* and the *co-rank* of a hypergraph H are defined as $r(H) = \underset{e\in E}{max}\{|e|\}$ and $\rho(H) = \underset{e\in E}{min}\{|e|\}$ respectively. A hypergraph is said to be *uniform* if all of it's edges have the same cardinality. If it is m, then the hypergraph is said to be *m-uniform*; otherwise, it is called *non-uniform*. A vertex of a hypergraph is said to be isolated if it does not belong to any edge of that hypergraph. Throughout this paper, we consider only hypergraphs having finite number of vertices.

Let $\mathcal{P}^*(A)$ denote the set of all non-empty subsets of a set A. A hypergraph $H(V,E)$ is said to be *complete* if $E = \mathcal{P}^*(V)$. We denote the *complete hypergraph* on n vertices with no loops as K_n. For, $0 \leq r \leq n$, the *complete r-uniform hypergraph* on n vertices, denoted by K_n^r, is the hypergraph whose edge set is the set of all possible r-subsets of V.

For a nonempty subset S of positive integers, a *S-hypergraph* on V is a hypergraph with vertex set V and edge set $E = \bigcup_{s\in S} E_s$, where E_s is a non-empty set of s-subsets of V. The *complement of a S-hypergraph* $H(V,E)$, denoted by $H^c(V,E^c)$ is the S-hypergraph on V whose edge set consists of the subsets of V with cardinality in S which do not lie in E [9]. The *degree of a vertex* v in a hypergraph H, denoted by $d(v)$, is the number of edges containing v in H.

Definition 2.1. ([22]) Let $H(V,E,W)$ be a hypergraph with vertex set $V = \{1,2,\ldots,n\}$, edge set E and a weight function $W : E \to \mathbb{R}_{\geq 0}$ defined by $W(e) = w_e$ for all $e \in E$. The *adjacency matrix* $A(H)$ of $H(V,E,W)$ is the $n \times n$ symmetric matrix in which

$$(i,j)\text{-th entry of } A(H) = \begin{cases} \sum\limits_{e\in E; i,j\in e} \frac{w_e}{|e|-1} & \text{if } i \neq j,\ i \text{ and } j \text{ are adjacent;} \\ 0 & \text{otherwise.} \end{cases}$$

If we take $w_e = 1$, then $A(H)$ becomes the adjacency matrix of the unweighted hypergraph $H(V,E)$ defined in [2]. The *valency* of a vertex i of H, denoted by $d(i)$ is defined as $d(i) = \sum\limits_{e\in E; i\in e} w_e$. The *Laplacian matrix* $L(H)$ of $H(V,E,W)$ is defined by $L(H) = D(H) - A(H)$, where $D(H)$ is the diagonal matrix whose entries are the valencies $d(i)$ of the vertices i of H. If the hypergraph $H(V,E,W)$ has no isolated vertices, then its *normalized Laplacian matrix* $\mathcal{L}(H)$ is defined as $\mathcal{L}(H) = D(H)^{-1/2}L(H)D(H)^{-1/2}$.

A weighted/unweighted hypergraph is said to be *r-regular* if valency/ degree of each of its vertices is r.

For a matrix M, we use the notation $P_M(x)$ to denote its characteristic polynomial and $\sigma(M)$ to denote its multiset of eigenvalues (spectrum). The spectrum of $A(H)$, $L(H)$ and $\mathcal{L}(H)$ are said to be the *A-spectrum*, the *L-spectrum* and the *$\mathcal{L}$-spectrum* of the hypergraph H, respectively. Two hypergraphs are said to be *$A-$cospectral (resp. $L-$cospectral, $\mathcal{L}-$cospectral)* if they have the same A-spectrum (resp. L-spectrum, $\mathcal{L}$-spectrum). The largest eigenvalue of $A(H)$ is said to be the *Perron adjacency eigenvalue of H*, whereas its other eigenvalues are said to be the *non-Perron adjacency eigenvalues of H*.

Let $A_1, A_2, \ldots, A_m$ be square matrices of order n with entries from $\mathbb{C}$. Then $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{C}$ are said to be *co-eigenvalues of $A_1, A_2, \ldots, A_m$*, if there exists a vector $X \in \mathbb{C}^n$ such that $A_i X = \lambda_i X$ for $i = 1, 2, \ldots, m$ [6].

Let I_n denote the identity matrix of size $n \times n$ and $J_{n\times m}$ denote the matrix of size $n \times m$ whose all the entries are 1. In particular, we denote $J_{n\times n}$ simply as J_n. The Kronecker product of two matrices A and B is denoted by $A \otimes B$.

Let G_1 and G_2 be graphs on m and n vertices, respectively. Let π be a binary relation, that is $\pi \subseteq V(G_1) \times V(G_2)$. Then the *$\pi-$graph of G_1 and G_2*, is the graph whose vertex set is $V(G_1) \cup V(G_2)$ and edge set is $E(G_1) \cup E(G_2) \cup \pi$ [11]. An equivalent formulation of this definition is given as follows [8]: Write the binary relation π as a $0-1$ matrix $N = (n_{ij})$ of size $m \times n$ in which $n_{ij} = 1$ if and only if the i-th vertex of G_1 and the j-th vertex of G_2 are related with respect to π, so the $\pi-$graph of G_1 and G_2 is the graph obtained by taking one copy of G_1 and G_2, and joining the i-th vertex of G_1 to the j-th vertex of G_2 if and only if $n_{ij} = 1$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. This graph is denoted by $G_1 \vee_N G_2$ and is called the *N-join of G and H*. This definition is extended as follows.

Definition 2.2. ([8]) Let $\mathcal{H}_k$ be a sequence of k graphs $H_1, H_2, \ldots, H_k$ with $|V(H_i)| = n_i$ for $i = 1, 2, \ldots, k$ and let $\mathcal{M} = (M_{12}, M_{13}, \ldots, M_{1k}, M_{23}, M_{24}, \ldots, M_{2k}, \ldots, M_{(k-1)k})$, where M_{ij} is a $0-1$ matrix of size $n_i \times n_j$. Then *the $\mathcal{M}$-join of the graphs in $\mathcal{H}_k$*, denoted by $\bigvee_{\mathcal{M}} \mathcal{H}_k$, is the graph $\bigcup\limits_{\substack{i,j=1,\\ i<j}}^{k} \left(H_i \vee_{M_{ij}} H_j\right)$.

The following results are used in the subsequent sections.

Theorem 2.3. *([13, pp. 483]) Let A and B be two matrices of sizes $m \times n$ and $n \times m$ respectively. Then for any invertible $m \times m$ matrix X, $|X + AB| = |X| \times |I_n + BX^{-1}A|$.*

Theorem 2.4. *([10, Corollary 2]) Let a real matrix A be partitioned as*

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \dots & A_{kk} \end{bmatrix}.$$

For $i,j = 1,2,\dots,k$, if A_{ij} are symmetric matrices of order n such that they commutes with each other. Then $\sigma(A) = \sum_{h=1}^{n} \sigma(E_h)$, where the summation denotes the union of the multisets and

$$E_h = \begin{bmatrix} a_{11}^{(h)} & a_{12}^{(h)} & \cdots & a_{1k}^{(h)} \\ a_{21}^{(h)} & a_{22}^{(h)} & \cdots & a_{2k}^{(h)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1}^{(h)} & a_{k2}^{(h)} & \cdots & a_{kk}^{(h)} \end{bmatrix},$$

with $a_{ij}^{(h)}$ is an eigenvalue of A_{ij} corresponding to the same eigenvector X for each $i,j = 1,2,\dots,k$; $h = 1,2,\dots,n$.

3. Indicating tensors

Let $\mathcal{R}(a_1,a_2,\dots,a_m)$ denote the range set of the sequence $(a_i)_{i=1}^{m}$. Let

$$\mathcal{R}^{\blacktriangledown}(a_1,a_2,\dots,a_m) = \begin{cases} \mathcal{R}(a_1,a_2,\dots,a_m)\setminus\{\blacktriangledown\} & \text{if } \blacktriangledown \in \{a_1,a_2,\dots,a_m\}; \\ \mathcal{R}(a_1,a_2,\dots,a_m) & \text{otherwise.} \end{cases}$$

For $n \in \mathbb{N}$, let $[n] := \{1,2,\dots,n\}$. We denote $\mathcal{P}^*([n])\setminus \bigcup_{y\in[n]} \{y\}$ simply by $\widehat{[n]}$.

Definition 3.1. For $i = 1,2,\dots,k$, let A_i be mutually disjoint sets having n_i elements. Let $\mathcal{A}$ be the sequence $(A_i)_{i=1}^{k}$. Then *an indicating tensor corresponding to $\mathcal{A}$*, denoted by $T[\mathcal{A}] := (T[\mathcal{A}]_{p_1p_2\dots p_N})$, is a $0-1$ tensor of order $N := n_1+n_2+\dots+n_k$ and dimension $(\underbrace{n_1+1,\dots,n_1+1}_{n_1 \text{ times}},\underbrace{n_2+1,\dots,n_2+1}_{n_2 \text{ times}},$ $\dots,\underbrace{n_k+1,\dots,n_k+1}_{n_k \text{ times}})$, where $p_1,p_2,\dots,p_{n_1} \in A_1 \cup \{\blacktriangledown\}$, $p_{n_1+n_2+\dots+n_i+1},\dots,$ $p_{n_1+n_2+\dots+n_{i+1}} \in A_{i+1} \cup \{\blacktriangledown\}$ for $i = 1,2,\dots,k-1$; $\blacktriangledown$ is an arbitrary symbol that is not an element of any A_i, $i = 1,2,\dots,k-1$; and is satisfying the following:

(i) If there exists $p_1,p_2,\dots,p_N$ such that $\mathcal{R}^{\blacktriangledown}(p_1,p_2,\dots,p_N) \subseteq A_i$ for some $i \in [k]$, then $T[\mathcal{A}]_{p_1p_2\dots p_N} = 0$.

(ii) If there exists $p_1,p_2,\dots,p_N$ such that $T[\mathcal{A}]_{p_1p_2\dots p_N} = 1$, then $T[\mathcal{A}]_{p'_1p'_2\dots p'_N} = 1$ whenever $\mathcal{R}^{\blacktriangledown}(p'_1,p'_2,\dots,p'_N) = \mathcal{R}^{\blacktriangledown}(p_1,p_2,\dots,p_N)$.

Notice that if $p_1 = p_2 = \cdots = p_N = \blacktriangledown$, then we have $\mathcal{R}^{\blacktriangledown}(p_1, p_2, \ldots, p_N) = \Phi \subseteq A_i$ and so $T[\mathcal{A}]_{p_1 p_2 \ldots p_N} = 0$.

Example 3.2. Let $A_1 = \{1\}$, $A_2 = \{2,3\}$ and $A_3 = \{4,5,6\}$. Let $\mathcal{A} = (A_i)_{i=1}^3$. Then an indicating tensor $T[\mathcal{A}]$ of order 6 and dimension $(2,3,3,4,4,4)$ whose entries are given by,

$$T[\mathcal{A}]_{i_1 i_2 \ldots i_6} = \begin{cases} 1 & \text{if } \mathcal{R}^{\blacktriangledown}(i_1, i_2, \ldots, i_6) = \{1,2,4,5,6\} \text{ or } \{1,3\}; \\ 0 & \text{otherwise.} \end{cases}$$

More explicitly, the entries $T[\mathcal{A}]_{122456}$, $T[\mathcal{A}]_{122465}$, $T[\mathcal{A}]_{122546}$, $T[\mathcal{A}]_{122564}$, $T[\mathcal{A}]_{122645}$, $T[\mathcal{A}]_{122654}$, $T[\mathcal{A}]_{12\blacktriangledown 456}$, $T[\mathcal{A}]_{12\blacktriangledown 465}$, $T[\mathcal{A}]_{12\blacktriangledown 546}$, $T[\mathcal{A}]_{12\blacktriangledown 564}$, $T[\mathcal{A}]_{12\blacktriangledown 645}$, $T[\mathcal{A}]_{12\blacktriangledown 654}$, $T[\mathcal{A}]_{1\blacktriangledown 2456}$, $T[\mathcal{A}]_{1\blacktriangledown 2465}$, $T[\mathcal{A}]_{1\blacktriangledown 2546}$, $T[\mathcal{A}]_{1\blacktriangledown 2564}$, $T[\mathcal{A}]_{1\blacktriangledown 2645}$, $T[\mathcal{A}]_{1\blacktriangledown 2654}$, $T[\mathcal{A}]_{133\blacktriangledown\blacktriangledown\blacktriangledown}$, $T[\mathcal{A}]_{1\blacktriangledown 3\blacktriangledown\blacktriangledown\blacktriangledown}$, $T[\mathcal{A}]_{13\blacktriangledown\blacktriangledown\blacktriangledown\blacktriangledown}$ take the value 1 and the remaining entries are zero.

Definition 3.3. We call an indicating tensor obtained by taking A_i instead of $A_i \cup \{\blacktriangledown\}$ for $i = 1,2,\ldots,k$ in Definition 3.1 as *an indicating tensor of type-2 corresponding to* $\mathcal{A}$ and is denoted by $T^*[\mathcal{A}]$.

Example 3.4. Let $A_1 = \{1\}$, $A_2 = \{2,3\}$ and $A_3 = \{4\}$. Let $\mathcal{A} = (A_i)_{i=1}^3$. Then an indicating tensor $T^*[\mathcal{A}]$ of type-2 of order 4 and dimension $(1,2,2,1)$ whose entries are given by,

$$T^*[\mathcal{A}]_{i_1 i_2 i_3 i_4} = \begin{cases} 1 & \text{if } \mathcal{R}(i_1, i_2, i_3, i_4) = \{1,2,4\} \text{ or } \{1,3,4\} \text{ or } \{1,2,3,4\}; \\ 0 & \text{otherwise.} \end{cases}$$

More explicitly, $T^*[\mathcal{A}]_{1224} = T^*[\mathcal{A}]_{1334} = T^*[\mathcal{A}]_{1234} = T^*[\mathcal{A}]_{1324} = 1$ and the remaining entries are zero.

For an indicating tensor $T[\mathcal{A}]$ and an indicating tensor $T^*[\mathcal{A}]$ of type-2, we define the following notations.

(i) $E(T[\mathcal{A}]) := \{\mathcal{R}^{\blacktriangledown}(p_1, p_2, \ldots, p_N) \mid T[\mathcal{A}]_{p_1 p_2 \ldots p_N} = 1\}$.

(ii) $E(T^*[\mathcal{A}]) := \{\mathcal{R}(p_1, p_2, \ldots, p_N) \mid T^*[\mathcal{A}]_{p_1 p_2 \ldots p_N} = 1\}$.

(iii) For each $p \in A_i,\ q \in A_j\ (1 \leq i \leq j \leq k)$, $c \in [N]$,
$E^c_{p,q}(T[\mathcal{A}]) := \{S \in E(T[\mathcal{A}]) \mid \{p,q\} \subseteq S,\ |S| = c\}$.

In the following we introduce some special classes of indicating tensors.

(1) For each $m \in \{1,2,\ldots,N\}$, let $T[\mathcal{A};m]$ denote an indicating tensor corresponding to $\mathcal{A}$ in which $T[\mathcal{A};m]_{p_1 p_2 \ldots p_N} = 0$ whenever $|\mathcal{R}^{\blacktriangledown}(p_1, p_2, \ldots, p_N)| \neq m$.

(2) For a non empty subset B of $\{k, k+1, \ldots, N\}$, let ${}_BT[\mathcal{A}]$ denote the indicating tensor corresponding to $\mathcal{A}$ in which

$$
{}_BT[\mathcal{A}]_{p_1p_2\ldots p_N} = \begin{cases} 1 & \text{if } |\mathcal{R}^{\blacktriangledown}(p_1,p_2,\ldots,p_N)| \in B \text{ and} \\ & \mathcal{R}^{\blacktriangledown}(p_1,p_2,\ldots,p_N) \cap A_i \neq \Phi \text{ for all } i \in [k]; \\ 0 & \text{otherwise.} \end{cases}
$$

(3) Let $J[\mathcal{A}]$ denote the indicating tensor corresponding to $\mathcal{A}$ in which

$$
J[\mathcal{A}]_{p_1p_2\ldots p_N} = \begin{cases} 0 & \text{if } \mathcal{R}^{\blacktriangledown}(p_1,p_2,\ldots,p_N) \subseteq A_i \text{ for some } i \in [k]; \\ 1 & \text{otherwise.} \end{cases}
$$

(4) For $i = 1, 2, \ldots, k$, let $A_i = \{u_{i_1}, u_{i_2}, \ldots, u_{i_n}\}$. For each $r \in [n]$, let ${}_rT[\mathcal{A}]$ denote the indicating tensor corresponding to $\mathcal{A}$ with

$$
{}_rT[\mathcal{A}]_{p_1p_2\ldots p_{nk}} = \begin{cases} 1 & \text{if } \mathcal{R}^{\blacktriangledown}(p_1,p_2,\ldots,p_{nk}) = \bigcup\limits_{i=1}^{k} \{u_{i_{l_1}}, u_{i_{l_2}}, \ldots, u_{i_{l_r}}\} \\ & \text{for some } \{l_1, l_2, \ldots, l_r\} \subseteq [n]; \\ 0 & \text{otherwise.} \end{cases}
$$

(5) Let $I[\mathcal{A}] := {}_1T[\mathcal{A}]$ and we call this as the *identity indicating tensor corresponding to $\mathcal{A}$.*

(6) Let $H(V(H), E(H))$ be a hypergraph with $V(H) = \{1, 2, \ldots, n\}$. Let $1 < k \leq \rho(H)$ and let $(G_i(U_i, E_i))_{i=1}^{k}$ be a sequence of hypergraphs with $U_i = \{u_{i1}, u_{i2}, \ldots, u_{in}\}$. Let $\mathcal{A} = (U_i)_{i=1}^{k}$. Let $N_H[\mathcal{A}]$ denote the indicating tensor corresponding to $\mathcal{A}$ with

$$
N_H[\mathcal{A}]_{p_1p_2\ldots p_{nk}} = \begin{cases} 1 & \text{if } \mathcal{R}^{\blacktriangledown}(p_1,p_2,\ldots,p_{nk}) = \bigcup\limits_{i=1}^{k} \{u_{i_{l_1}}, u_{i_{l_2}}, \ldots, u_{i_{l_{s_i}}}\} \\ & \text{where } s_i \geq 1,\ D_i = \{l_1, l_2, \ldots, l_{s_i}\} \subseteq V(H) \\ & \text{such that the set of all } D_i \text{ forms a partition} \\ & \text{of } e \text{ for some } e \in E(H). \\ 0 & \text{otherwise.} \end{cases}
$$

(7) For $i = 1, 2, \ldots, k$, let $|A_i| = n$. We denote the indicating tensor $J[\mathcal{A}] - {}_rT[\mathcal{A}]$ by $\mathfrak{I}_r[\mathcal{A}]$. When $r = 1$, we denote it simply by $\mathfrak{I}[\mathcal{A}]$.

(8) We denote the indicating tensor $N_H[\mathcal{A}] + {}_rT[\mathcal{A}]$ by ${}_{H_r}N[\mathcal{A}]$. When $r = 1$, we denote it simply by ${}_HN[\mathcal{A}]$.

4. Tensor join of hypergraphs

In the rest of the paper, whenever we consider a sequence of weighted/ unweighted hypergraphs $(G_i)_{i=1}^k$, without loss of generality, we assume that the vertex sets of G_is are mutually disjoint for $i = 1, 2, \ldots, k$.

Definition 4.1. Let $\mathcal{G} = (G_i(V_i, E_i))_{i=1}^k$ be a sequence of k hypergraphs. Let $\mathcal{V} = (V_i)_{i=1}^k$. Consider an indicating tensor $T[\mathcal{V}]$. Then the $T[\mathcal{V}]$*-join of hypergraphs in* $\mathcal{G}$, denoted by $\bigvee\limits_{T[\mathcal{V}]} \mathcal{G}$, is the hypergraph constructed as follows:

- Take one copy of $G_i,\ i = 1, 2, \ldots, k$;
- For each $D \subseteq \bigcup\limits_{i=1}^{k} V_i$, join the vertices in D as an edge in $\bigvee\limits_{T[\mathcal{V}]} \mathcal{G}$ if and only if $D \in E(T[\mathcal{V}])$.

If $\mathcal{G} = (G_1, G_2)$, then we denote the $T[\mathcal{V}]$-join of hypergraphs in $\mathcal{G}$ by $G_1 \bigvee\limits_{T[\mathcal{V}]} G_2$.

Example 4.2. Consider the hypergraphs $G_1(V_1, E_1)$, $G_2(V_2, E_2)$ and $G_3(V_3, E_3)$ as shown in Figures 1(a), 1(b) and 1(c) respectively. Let $\mathcal{G} = (G_i)_{i=1}^3$ and $\mathcal{V} =$

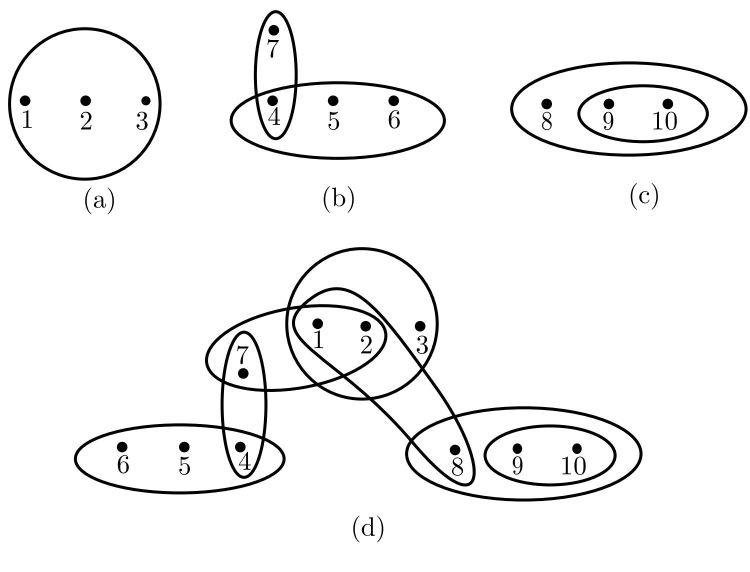

Figure 1: The hypergraphs (a) $G_1(V_1, E_1)$, (b) $G_2(V_2, E_2)$, (c) $G_3(V_3, E_3)$ and (d) $\bigvee\limits_{T[\mathcal{V}]} \mathcal{G}$

$(V_i)_{i=1}^3$. Consider the indicating tensor $T[\mathcal{V}]$ of order 10 and dimension $(4,4,4,5,5,5,5,4,4,4)$ with

$$T[\mathcal{V}]_{i_1 i_2 \dots i_{10}} = \begin{cases} 1 & \text{if } \mathcal{R}^{\blacktriangledown}(i_1,i_2,\dots,i_{10}) = \{1,2,7\} \text{ or } \{1,2,8\}; \\ 0 & \text{otherwise.} \end{cases}$$

Notice that, $E(T[\mathcal{V}]) = \{\{1,2,7\},\{1,2,8\}\}$. Then the hypergraph $\bigvee\limits_{T[\mathcal{V}]} \mathcal{G}$ is as shown in Figure 1(d).

Definition 4.3. Let $\mathcal{G} = (G_i(V_i,E_i))_{i=1}^k$ be a sequence of k hypergraphs. For each $S \in \widehat{[k]}$, let $\mathcal{V}_S = (V_i)_{i \in S}$. Let $\mathcal{T}^* = \{T^*[\mathcal{V}_S] \mid S \in \widehat{[k]}\}$ be a set of indicating tensors of type-2. Then the $\mathcal{T}^*$*-join of hypergraphs in* $\mathcal{G}$, denoted by $\bigvee\limits_{\mathcal{T}^*} \mathcal{G}$, is the hypergraph obtained by taking a copy of each G_i and for each $D \subseteq \bigcup\limits_{i=1}^k V_i$, join the set of vertices in D by an edge in $\bigvee\limits_{\mathcal{T}^*} \mathcal{G}$ if and only if $D \in E(T^*[\mathcal{V}_S])$ for some $S \in \widehat{[k]}$.

Definition 4.4. Let H be a hypergraph with $V(H) = [k]$. Let $\mathcal{G} = (G_i(V_i,E_i))_{i=1}^k$ be a sequence of hypergraphs with $|V_i| = n_i$ for $i = 1,2,\dots,k$. For each $e \in E(H)$, let $\mathcal{V}_e = (V_i)_{i \in e}$, $N_e := \sum\limits_{i \in e} n_i$ and $\mathcal{G}_e = \{G_i \mid i \in e\}$. Let $\mathcal{T} = \{T[\mathcal{V}_e] \mid e \in E(H)\}$, where for each $e \in E(H)$, $T[\mathcal{V}_e]$ is a non-zero indicating tensor with

$$T[\mathcal{V}_e]_{p_1 p_2 \dots p_{N_e}} = 0 \text{ if } \mathcal{R}^{\blacktriangledown}(p_1,p_2,\dots,p_{N_e}) \cap V_i = \Phi \text{ for some } i \in e.$$

Then construct the hypergraph by taking a copy of each G_i and doing the $T[\mathcal{V}_e]$-join of hypergraphs in $\mathcal{G}_e$ for each edge $e \in E(H)$. We denote this hypergraph by $\mathcal{G}(H,\mathcal{T})$ and call it as the $(H,\mathcal{T})$*-join of hypergraphs in* $\mathcal{G}$.

Notice that, $V(\mathcal{G}(H,\mathcal{T})) = \bigcup\limits_{i=1}^k V_i$ and $E(\mathcal{G}(H,\mathcal{T})) = \bigcup\limits_{i=1}^k E(G_i) \bigcup\limits_{e \in E(H)} E(T[\mathcal{V}_e])$.

Theorem 4.5. *Definitions* 4.1, 4.3 *and* 4.4 *are equivalent.*

Proof. Let $\mathcal{G} = (G_i(V_i,E_i))_{i=1}^k$ be a sequence of k hypergraphs with $|V_i| = n_i$ for $i = 1,2,\dots,k$.

(1) Consider an indicating tensor $T[\mathcal{V}]$, where $\mathcal{V} = (V_i)_{i=1}^k$ and assume that we have constructed the hypergraph $\bigvee\limits_{T[\mathcal{V}]} \mathcal{G}$ as per Definition 4.1. We show that this hypergraph can be viewed as the hypergraph $\bigvee\limits_{\mathcal{T}^*} \mathcal{G}$ for some suitable $\mathcal{T}^*$ as per Definition 4.3. For each $S \in \widehat{[k]}$, let $\mathcal{V}_S = (V_i)_{i \in S}$ and $w(S) =$

$\sum_{r\in S} n_r$. Take $\mathcal{T}^* = \{T^*[\mathcal{V}_S] \mid S \in \widehat{[k]}\}$, where $T^*[\mathcal{V}_S]$ is the indicating tensor of type-2 with

$$T^*[\mathcal{V}_S]_{p_1p_2\ldots p_{w(S)}} = T[\mathcal{V}]_{q_1q_2\ldots q_N},$$

where $q_1,q_2,\ldots,q_N$ are such that $\mathcal{R}^{\blacktriangledown}(q_1,q_2,\ldots,q_N) = \mathcal{R}(p_1,p_2,\ldots,p_{w(S)})$. Now construct the hypergraph $\bigvee_{\mathcal{T}^*} \mathcal{G}$ as per Definition 4.1. Then this hypergraph is the same as the hypergraph $\bigvee_{T[\mathcal{V}]} \mathcal{G}$.

(2) Let $\mathcal{T}^* = \{T^*[\mathcal{V}_S] \mid S \in \widehat{[k]}\}$ be a set of indicating tensors of type-2, where $\mathcal{V}_S = (V_i)_{i\in S}$ for all $S \in \widehat{[k]}$. Assume that we have constructed the hypergraph $\bigvee_{\mathcal{T}^*} \mathcal{G}$ as per Definition 4.3. We show that this hypergraph is the same as the hypergraph $\mathcal{G}(H,\mathcal{T})$ for some suitable hypergraph H and a set of indicating tensors $\mathcal{T}$ as per Definition 4.4. First construct the hypergraph H by using $\mathcal{T}^*$ as follows: Take $V(H) = [k]$. For each $T^*[\mathcal{V}_S] \in \mathcal{T}^*$, make $S \subseteq V(H)$ as an edge in H if and only if $T^*[\mathcal{V}_S]$ is non-zero. Now, for each $e \in E(H)$, let $N_e = \sum_{r\in e} n_r$. Take $\mathcal{T} = \{T[\mathcal{V}_e] \mid e \in E(H)\}$, where $T[\mathcal{V}_e]$ is the indicating tensor with

$$T[\mathcal{V}_e]_{p_1p_2\ldots p_{N_e}} = T^*[\mathcal{V}_e]_{q_1q_2\ldots q_{N_e}},$$

where $q_1,q_2,\ldots,q_{N_e}$ are such that $\mathcal{R}(q_1,q_2,\ldots,q_{N_e}) = \mathcal{R}^{\blacktriangledown}(p_1,p_2,\ldots,p_{N_e})$. Now, construct the hypergraph $\mathcal{G}(H,\mathcal{T})$ as per Definition 4.4. Then this hypergraph is the same as the hypergraph $\bigvee_{\mathcal{T}^*} \mathcal{G}$.

(3) Let H be a hypergraph with $V(H) = [k]$. For each $e \in E(H)$, let $\mathcal{V}_e = (V_i)_{i\in e}$. Let $\mathcal{T} = \{T[\mathcal{V}_e] \mid e \in E(H)\}$. Assume that we have constructed $\mathcal{G}(H,\mathcal{T})$ as per Definition 4.4. We show that this hypergraph can be viewed as $\bigvee_{T[\mathcal{V}]} \mathcal{G}$ for some suitable indicating tensor $T[\mathcal{V}]$, where $\mathcal{V} = (V_i)_{i=1}^k$. Take the indicating tensor $T[\mathcal{V}]$ with

$$T[\mathcal{V}]_{p_1p_2\ldots p_N} = \begin{cases} T[\mathcal{V}_e]_{q_1q_2\ldots q_{N_e}} & \text{if there exists } e \in E(H) \text{ such that } \mathcal{R}^{\blacktriangledown}(p_1,p_2,\ldots,p_N) \cap V_i \neq \Phi \text{ for all } i \in e \text{ with } \mathcal{R}^{\blacktriangledown}(p_1,p_2,\ldots,p_N) = \mathcal{R}^{\blacktriangledown}(q_1,q_2,\ldots,q_{N_e}); \\ 0 & \text{otherwise.} \end{cases}$$

Construct the hypergraph $\bigvee_{T[\mathcal{V}]} \mathcal{G}$ as per Definition 4.1, which becomes the same as the hypergraph $\mathcal{G}(H,\mathcal{T})$.

$\square$

In view of Theorem 4.5, hereafter we say 'the tensor join of hypergraphs' to mean the hypergraph obtained by any one of the operations defined in Definitions 4.1, 4.3 and 4.4, unless we specifically mentioned otherwise.

Note 4.6. Any hypergraph can be viewed as a tensor join of some hypergraphs. For, let H be a hypergraph with $|V(H)| = n$. Take a partition $V_i,\ i = 1,2,\ldots,k$ of $V(H)$, where $k \le n$. For each $i = 1,2,\ldots,k$, let G_i be the subhypergraph of H induced by the vertex subset V_i. Let $\mathcal{G} = (G_i)_{i=1}^k$ and $\mathcal{V} = (V_i)_{i=1}^k$. Now consider the indicating tensor $T[\mathcal{V}]$ with

$$T[\mathcal{V}]_{p_1p_2\ldots p_n} = \begin{cases} 1 & \text{if } \mathcal{R}^{\blacktriangledown}(p_1,p_2,\ldots,p_n) \in E(H); \\ 0 & \text{otherwise.} \end{cases}$$

Then it is clear that H is the same as the hypergraph $\bigvee\limits_{T[\mathcal{V}]} \mathcal{G}$.

In the following theorem, we assert that for a given sequence $\mathcal{M}$ of matrices, the $\mathcal{M}$-join of graphs in a sequence $\mathcal{G}$ defined in Definition 2.2 can be viewed as a $T[\mathcal{A}]$-join of graphs in $\mathcal{G}$ for some suitable $T[\mathcal{A}]$ and vice versa.

Theorem 4.7. Let $\mathcal{G} = (G_i)_{i=1}^k$ be a sequence of graphs with $V(G_i) = \{u_{i1}, u_{i2}, \ldots, u_{in_i}\}$ for $i = 1,2,\ldots,k$ and let $\mathcal{V} = (V_i)_{i=1}^k$. Then corresponding to a given sequence $\mathcal{M} = (M_{12}, M_{13}, \ldots, M_{1k}, M_{23}, M_{24}, \ldots, M_{2k}, \ldots, M_{(k-1)k})$, where M_{ij} is a $0-1$ matrix of size $n_i \times n_j$, there exist an indicating tensor $T[\mathcal{V}]$ such that the graph $\bigvee_{\mathcal{M}} \mathcal{G}$ is the same as the graph $\bigvee\limits_{T[\mathcal{V}]} \mathcal{G}$ and vice versa.

Proof. Assume that the graph $\bigvee_{\mathcal{M}} \mathcal{G}$ is constructed as per Definition 2.2. Let us denote the (r,t)-th entry of M_{ij} by $(M_{ij})_{rt}$. Now consider the indicating tensor $T[\mathcal{V}]$ with

$$T[\mathcal{V}]_{p_1p_2\ldots p_N} = \begin{cases} (M_{ij})_{rt} & \text{if } \mathcal{R}^{\blacktriangledown}(p_1,p_2,\ldots,p_N) = \{u_{ir}, u_{jt}\}; \\ 0 & \text{otherwise.} \end{cases}$$

Then the graph $\bigvee\limits_{T[\mathcal{V}]} \mathcal{G}$ constructed as per Definition 4.1 is the same as the graph $\bigvee_{\mathcal{M}} \mathcal{G}$.

Conversely, assume that an indicating tensor $T[\mathcal{V}]$ corresponding to $\mathcal{V}$ is given and the hypergraph $\bigvee\limits_{T[\mathcal{V}]} \mathcal{G}$ is constructed as per Definition 4.1. For, $1 \le i \le j \le k$, consider the matrix M_{ij} whose (r,t)-th entry is defined as $(M_{ij})_{rt} = T[\mathcal{V}]_{p_1p_2\ldots p_N}$, where $p_{n_1+n_2+\cdots+n_{i-1}+1} = \cdots = p_{n_1+n_2+\cdots+n_i} = u_{ir}$, $p_{n_1+n_2+\cdots+n_{j-1}+1} = \cdots = p_{n_1+n_2+\cdots+n_j} = u_{jt}$ and all other indices are zero. Then the graph $\bigvee_{\mathcal{M}} \mathcal{G}$ constructed as per Definition 2.2 is the same as the graph $\bigvee\limits_{T[\mathcal{V}]} \mathcal{G}$. □

Naturally, there are several ways of constructing the matrix M_{ij} from the given indicating tensor $T[\mathcal{V}]$. In Theorem 4.7, we exhibit a way of constructing such matrices. Also, notice that the indicating tensor $T[\mathcal{V}]$ referred in Theorem 4.7 is especially the indicating tensor $T[\mathcal{V};2]$.

4.1. Some classes of hypergraphs as $T[\mathcal{A}]$-join of hypergraphs

In Table 1, we list some existing and new classes of hypergraphs which can be expressed as a ${}_BT[\mathcal{A}]$-join of hypergrphs in $\mathcal{G} = (H_i)_{i=1}^k$, by suitably taking the hypergraphs H_is, the set B and the value k as shown in the same table correspond to each class of hypergaphs, where $\mathcal{A} = (V(H_i))_{i=1}^k$.

S. No.	Name of the hypergraph	H_i	k	B
1.	Complete m-uniform m-partite hypergraph [23]	$K_{n_i}^c$	m	$\{m\}$
2.	Complete m-uniform weak k-partite hypergraph, $k \leq m$ [22]	$K_{n_i}^c$	k	$\{m\}$
3.	Complete weak k-partite hypergraph	$K_{n_i}^c$	k	$\{k, k+1, \ldots, N\}$
4.	Join of a set $\mathcal{G}$ of non-uniform hypergraphs [22]	H_i	k	a subset of $\{k, k+1, \ldots, N\}$
5.	Join of a set $\mathcal{G}$ of m-uniform hypergraphs [22]	H_i	$k(\leq m)$	$\{m\}$

Table 1: Viewing some existing and new class of hypergraphs as a $T[\mathcal{A}]$-join of hypergraphs in $\mathcal{G}$

Notice that if $m \geq 2$, the complete m-uniform weak 2-partite hypergraph becomes the complete m-uniform bipartite hypergraph. Also the complete weak 2-partite hypergraph becomes the complete bipartite hypergraph.

4.2. Some unary hypergraph operations as $T[\mathcal{A}]$-join of hypergraphs

First we define a new type of complement of a hypergraph.

Definition 4.8. Let $H(V,E)$ be a hypergraph. We define the *total complement of* H, denoted by $\overline{H}(V,\overline{E})$, as the hypergraph with vertex set V and the edge set $\overline{E} = \mathcal{P}^*(V) \backslash (E \cup S)$, where S is the set of all singletons of V.

In Table 2, we define several new unary hypergraph operations and name them analogous to the unary operations on graphs defined in Section 4.1 of [8]. For the operations given in S.Nos. 37-126 of this table, we assume that H contains no loops.

S. No.	Description	Name of the hypergraph
1.	$H \underset{_rT[\mathcal{V}]}{\bigvee} H$	r-Mirror hypergraph of H
2.	$H \underset{_rT[\mathcal{V}]}{\bigvee} H^c$	r-Mirror complemented neighbourhood hypergraph of H
3.	$H \underset{_rT[\mathcal{V}]}{\bigvee} K_n$	C-r-complete hypergraph of H
4.	$H \underset{_rT[\mathcal{V}]}{\bigvee} K_n^c$	C-r-hypergraph of H
5.	$H \underset{_rT[\mathcal{V}]}{\bigvee} \overline{H}$	r-Mirror total complemented neighbourhood hypergraph of H
6.	$H \underset{J[\mathcal{V}]}{\bigvee} H$	Join neighbourhood hypergraph of H
7.	$H \underset{J[\mathcal{V}]}{\bigvee} H^c$	Join complemented neighbourhood hypergraph of H
8.	$H \underset{J[\mathcal{V}]}{\bigvee} K_n$	Join complete hypergraph of H
9.	$H \underset{J[\mathcal{V}]}{\bigvee} K_n^c$	Join hypergraph of H
10.	$H \underset{J[\mathcal{V}]}{\bigvee} \overline{H}$	Join total complemented neighbourhood hypergraph of H
11.	$H \underset{\Im_r[\mathcal{V}]}{\bigvee} H$	VC-r-neighbourhood hypergraph of H
12.	$H \underset{\Im_r[\mathcal{V}]}{\bigvee} H^c$	VC-r-complemented neighbourhood hypergraph of H
13.	$H \underset{\Im_r[\mathcal{V}]}{\bigvee} K_n$	VC-r-complete hypergraph of H
14.	$H \underset{\Im_r[\mathcal{V}]}{\bigvee} K_n^c$	VC-r-hypergraph of H
15.	$H \underset{\Im_r[\mathcal{V}]}{\bigvee} \overline{H}$	VC-r-total complemented neighbourhood hypergraph of H
16.	$H^c \underset{_rT[\mathcal{V}]}{\bigvee} H^c$	r-Mirror-complement hypergraph of H
17.	$H^c \underset{_rT[\mathcal{V}]}{\bigvee} K_n$	C-r-complete complement hypergraph of H
18.	$H^c \underset{_rT[\mathcal{V}]}{\bigvee} K_n^c$	C-r-complement hypergraph of H
19.	$H^c \underset{J[\mathcal{V}]}{\bigvee} H^c$	Join neighbourhood-complement hypergrph of H
20.	$H^c \underset{J[\mathcal{V}]}{\bigvee} K_n$	Join complete-complement hypergrph of H

21.	$H^c \underset{J[\mathcal{V}]}{\bigvee} K_n^c$	Join-complement hypergrph of H
22.	$H^c \underset{\Im_r[\mathcal{V}]}{\bigvee} H^c$	VC-r-neighbourhood-complement hypergraph of H
23.	$H^c \underset{\Im_r[\mathcal{V}]}{\bigvee} K_n$	VC-r-complete-complement hypergraph of H
24.	$H^c \underset{\Im_r[\mathcal{V}]}{\bigvee} K_n^c$	VC-r-complement hypergraph of H
25.	$\overline{H} \underset{_rT[\mathcal{V}]}{\bigvee} H^c$	Total r-mirror complement hypergraph of H
26.	$\overline{H} \underset{_rT[\mathcal{V}]}{\bigvee} K_n$	C-r-complete total complement hypergraph of H
27.	$\overline{H} \underset{_rT[\mathcal{V}]}{\bigvee} K_n^c$	C-r-total complement hypergraph of H
28.	$\overline{H} \underset{_rT[\mathcal{V}]}{\bigvee} \overline{H}$	r-Mirror total complemented hypergraph of H
29.	$\overline{H} \underset{J[\mathcal{V}]}{\bigvee} H^c$	Total join neighbourhood complement hypergraph of H
30.	$\overline{H} \underset{J[\mathcal{V}]}{\bigvee} K_n$	Join complete total complement hypergraph of H
31.	$\overline{H} \underset{J[\mathcal{V}]}{\bigvee} K_n^c$	Join total complement hypergrph of H
32.	$\overline{H} \underset{J[\mathcal{V}]}{\bigvee} \overline{H}$	Join neighbourhood-total complement hypergraph of H
33.	$\overline{H} \underset{\Im_r[\mathcal{V}]}{\bigvee} H^c$	Total VC-r-neighbourhood complement hypergraph of H
34.	$\overline{H} \underset{\Im_r[\mathcal{V}]}{\bigvee} K_n$	VC-r-complete total complement hypergraph of H
35.	$\overline{H} \underset{\Im_r[\mathcal{V}]}{\bigvee} K_n^c$	VC-r-total complement hypergraph of H
36.	$\overline{H} \underset{\Im_r[\mathcal{V}]}{\bigvee} \overline{H}$	VC-r-neighbourhood total complement hypergraph of H
37.	$H \underset{N_H[\mathcal{V}]}{\bigvee} H$	N-neighbourhood hypergraph of H
38.	$H \underset{N_H[\mathcal{V}]}{\bigvee} H^c$	N-complemented neighbourhood hypergraph of H
39.	$H \underset{N_H[\mathcal{V}]}{\bigvee} K_n$	N-complete hypergraph of H
40.	$H \underset{N_H[\mathcal{V}]}{\bigvee} K_n^c$	N-hypergraph of H
41.	$H \underset{N_H[\mathcal{V}]}{\bigvee} \overline{H}$	N-total complemented neighbourhood hypergraph of H

42.	$H \bigvee_{H_r N[\mathcal{V}]} H$	$\overline{N}$-r-neighbourhood hypergraph of H
43.	$H \bigvee_{H_r N[\mathcal{V}]} H^c$	$\overline{N}$-r-complemented neighbourhood hypergraph of H
44.	$H \bigvee_{H_r N[\mathcal{V}]} K_n$	$\overline{N}$-r-complete hypergraph of H
45.	$H \bigvee_{H_r N[\mathcal{V}]} K_n^c$	$\overline{N}$-r-hypergraph of H
46.	$H \bigvee_{H_r N[\mathcal{V}]} \overline{H}$	$\overline{N}$-r-total complemented neighbourhood hypergraph of H
47.	$H \bigvee_{N_{H^c}[\mathcal{V}]} H$	NC-neighbourhood hypergraph of H
48.	$H \bigvee_{N_{H^c}[\mathcal{V}]} H^c$	NC-complemented neighbourhood hypergraph of H
49.	$H \bigvee_{N_{H^c}[\mathcal{V}]} K_n$	NC-complete hypergraph of H
50.	$H \bigvee_{N_{H^c}[\mathcal{V}]} K_n^c$	NC-hypergraph of H
51.	$H \bigvee_{N_{H^c}[\mathcal{V}]} \overline{H}$	NC-total complemented neighbourhood hypergraph of H
52.	$H \bigvee_{N_{\overline{H}}[\mathcal{V}]} H$	NTC-neighbourhood hypergraph of H
53.	$H \bigvee_{N_{\overline{H}}[\mathcal{V}]} H^c$	NTC-complemented neighbourhood hypergraph of H
54.	$H \bigvee_{N_{\overline{H}}[\mathcal{V}]} K_n$	NTC-complete hypergraph of H
55.	$H \bigvee_{N_{\overline{H}}[\mathcal{V}]} K_n^c$	NTC-hypergraph of H
56.	$H \bigvee_{N_{\overline{H}}[\mathcal{V}]} \overline{H}$	NTC-total complemented neighbourhood hypergraph of H
57.	$H \bigvee_{H_r^c N[\mathcal{V}]} H$	$\overline{N}C$-r-neighbourhood hypergraph of H
58.	$H \bigvee_{H_r^c N[\mathcal{V}]} H^c$	$\overline{N}C$-r-complemented neighbourhood hypergraph of H
59.	$H \bigvee_{H_r^c N[\mathcal{V}]} K_n$	$\overline{N}C$-r-complete hypergraph of H
60.	$H \bigvee_{H_r^c N[\mathcal{V}]} K_n^c$	$\overline{N}C$-r-hypergraph of H
61.	$H \bigvee_{H_r^c N[\mathcal{V}]} \overline{H}$	$\overline{N}C$-r-total complemented neighbourhood hypergraph of H

62.	$H \bigvee_{\overline{H}_r N[\mathcal{V}]} H$	$\overline{N}TC$-r-neighbourhood hypergraph of H
63.	$H \bigvee_{\overline{H}_r N[\mathcal{V}]} H^c$	$\overline{N}TC$-r-complemented neighbourhood hypergraph of H
64.	$H \bigvee_{\overline{H}_r N[\mathcal{V}]} K_n$	$\overline{N}TC$-r-complete hypergraph of H
65.	$H \bigvee_{\overline{H}_r N[\mathcal{V}]} K_n^c$	$\overline{N}TC$-r-hypergraph of H
66.	$H \bigvee_{\overline{H}_r N[\mathcal{V}]} \overline{H}$	$\overline{N}TC$-r-total complemented neighbourhood hypergraph of H
67.	$H^c \bigvee_{N_H[\mathcal{V}]} H^c$	N-neighbourhood complement hypergraph of H
68.	$H^c \bigvee_{N_H[\mathcal{V}]} K_n$	N-complete complement hypergraph of H
69.	$H^c \bigvee_{N_H[\mathcal{V}]} K_n^c$	N-complement hypergraph of H
70.	$\overline{H} \bigvee_{N_H[\mathcal{V}]} H^c$	Total N-neighbourhood complement hypergraph of H
71.	$\overline{H} \bigvee_{N_H[\mathcal{V}]} K_n$	N-complete total complement hypergraph of H
72.	$\overline{H} \bigvee_{N_H[\mathcal{V}]} K_n^c$	N-total complement hypergraph of H
73.	$\overline{H} \bigvee_{N_H[\mathcal{V}]} \overline{H}$	N-neighbourhood total complement hypergraph of H
74.	$H^c \bigvee_{N_{\overline{H}}[\mathcal{V}]} H^c$	NTC-neighbourhood-complement hypergraph of H
75.	$H^c \bigvee_{N_{\overline{H}}[\mathcal{V}]} K_n$	NTC-complete-complement hypergraph of H
76.	$H^c \bigvee_{N_{\overline{H}}[\mathcal{V}]} K_n^c$	NTC-complement hypergraph of H
77.	$\overline{H} \bigvee_{N_{\overline{H}}[\mathcal{V}]} H^c$	Total NTC-neighbourhood-complement hypergraph of H
78.	$\overline{H} \bigvee_{N_{\overline{H}}[\mathcal{V}]} K_n$	NTC-complete total complement hypergraph of H
79.	$\overline{H} \bigvee_{N_{\overline{H}}[\mathcal{V}]} K_n^c$	NTC-total complement hypergraph of H
80.	$\overline{H} \bigvee_{N_{\overline{H}}[\mathcal{V}]} \overline{H}$	NTC-neighbourhood total complement hypergraph of H
81.	$H^c \bigvee_{N_{H^c}[\mathcal{V}]} H^c$	NC-neighbourhood-complement hypergraph of H

82.	$H^c \bigvee\limits_{N_{H^c}[\mathcal{V}]} K_n$	NC-complete-complement hypergraph of H
83.	$H^c \bigvee\limits_{N_{H^c}[\mathcal{V}]} K_n^c$	NC-complement hypergraph of H
84.	$\overline{H} \bigvee\limits_{N_{H^c}[\mathcal{V}]} H^c$	Total NC-neighbourhood-complement hypergraph of H
85.	$\overline{H} \bigvee\limits_{N_{H^c}[\mathcal{V}]} K_n$	NC-complete total complement hypergraph of H
86.	$\overline{H} \bigvee\limits_{N_{H^c}[\mathcal{V}]} K_n^c$	NC-total complement hypergraph of H
87.	$\overline{H} \bigvee\limits_{N_{H^c}[\mathcal{V}]} \overline{H}$	NC-neighbourhood total complement hypergraph of H
88.	$H^c \bigvee\limits_{{}_{H_r}N[\mathcal{V}]} H^c$	$\overline{N}$-r-neighbourhood-complement hypergraph of H
89.	$H^c \bigvee\limits_{{}_{H_r}N[\mathcal{V}]} K_n$	$\overline{N}$-r-complete-complement hypergraph of H
90.	$H^c \bigvee\limits_{{}_{H_r}N[\mathcal{V}]} K_n^c$	$\overline{N}$-r-complement hypergraph of H
91.	$\overline{H} \bigvee\limits_{{}_{H_r}N[\mathcal{V}]} H^c$	Total $\overline{N}$-r-neighbourhood complement hypergraph of H
92.	$\overline{H} \bigvee\limits_{{}_{H_r}N[\mathcal{V}]} K_n$	$\overline{N}$-r-complete total complement hypergraph of H
93.	$\overline{H} \bigvee\limits_{{}_{H_r}N[\mathcal{V}]} K_n^c$	$\overline{N}$-r-total complement hypergraph of H
94.	$\overline{H} \bigvee\limits_{{}_{H_r}N[\mathcal{V}]} \overline{H}$	$\overline{N}$-r-neighbourhood total complement hypergraph of H
95.	$H^c \bigvee\limits_{{}_{H_r^c}N[\mathcal{V}]} H^c$	$\overline{N}C$-r-neighbourhood-complement hypergraph of H
96.	$H^c \bigvee\limits_{{}_{H_r^c}N[\mathcal{V}]} K_n$	$\overline{N}C$-r-complete-complement hypergraph of H
97.	$H^c \bigvee\limits_{{}_{H_r^c}N[\mathcal{V}]} K_n^c$	$\overline{N}C$-r-complement hypergraph of H
98.	$\overline{H} \bigvee\limits_{{}_{H_r^c}N[\mathcal{V}]} H^c$	Total $\overline{N}C$-r-neighbourhood-complement hypergraph of H
99.	$\overline{H} \bigvee\limits_{{}_{H_r^c}N[\mathcal{V}]} K_n$	$\overline{N}C$-r-complete-total complement hypergraph of H
100.	$\overline{H} \bigvee\limits_{{}_{H_r^c}N[\mathcal{V}]} K_n^c$	$\overline{N}C$-r-total complement hypergraph of H
101.	$\overline{H} \bigvee\limits_{{}_{H_r^c}N[\mathcal{V}]} \overline{H}$	$\overline{N}C$-r-neighbourhood total complement hypergraph of H

102.	$H^c \bigvee_{\overline{H}_r N[\mathcal{V}]} H^c$	$\overline{N}TC$-r-neighbourhood-complement hypergraph of H
103.	$H^c \bigvee_{\overline{H}_r N[\mathcal{V}]} K_n$	$\overline{N}TC$-r-complete-complement hypergraph of H
104.	$H^c \bigvee_{\overline{H}_r N[\mathcal{V}]} K_n^c$	$\overline{N}TC$-r-complement hypergraph of H
105.	$\overline{H} \bigvee_{\overline{H}_r N[\mathcal{V}]} H^c$	Total $\overline{N}TC$-r-neighbourhood-complement hypergraph of H
106.	$\overline{H} \bigvee_{\overline{H}_r N[\mathcal{V}]} K_n$	$\overline{N}TC$-r-complete total complement hypergraph of H
107.	$\overline{H} \bigvee_{\overline{H}_r N[\mathcal{V}]} K_n^c$	$\overline{N}TC$-r-total complement hypergraph of H
108.	$\overline{H} \bigvee_{\overline{H}_r N[\mathcal{V}]} \overline{H}$	$\overline{N}TC$-r-neighbourhood-total complement hypergraph of H
109.	$K_n^c \bigvee_{N_H[\mathcal{V}]} K_n^c$	Duplicate hypergraph of H
110.	$K_n \bigvee_{N_H[\mathcal{V}]} K_n^c$	Duplicate complete hypergraph of H
111.	$K_n \bigvee_{N_H[\mathcal{V}]} K_n$	Fully complete duplicate hypergraph of H
112.	$K_n^c \bigvee_{H_r N[\mathcal{V}]} K_n^c$	r-$D\overline{N}$-hypergraph of H
113.	$K_n \bigvee_{H_r N[\mathcal{V}]} K_n^c$	r-$D\overline{N}$-complete hypergraph of H
114.	$K_n \bigvee_{H_r N[\mathcal{V}]} K_n$	Fully complete r-$D\overline{N}$-hypergraph of H
115.	$K_n^c \bigvee_{N_{H^c}[\mathcal{V}]} K_n^c$	Complemented duplicate hypergraph of H
116.	$K_n \bigvee_{N_{H^c}[\mathcal{V}]} K_n^c$	Complemented duplicate complete hypergraph of H
117.	$K_n \bigvee_{N_{H^c}[\mathcal{V}]} K_n$	Fully complete complemented duplicate hypergraph of H
118.	$K_n^c \bigvee_{N_{\overline{H}}[\mathcal{V}]} K_n^c$	Total complemented duplicate hypergraph of H
119.	$K_n \bigvee_{N_{\overline{H}}[\mathcal{V}]} K_n^c$	Total complemented duplicate complete hypergraph of H
120.	$K_n \bigvee_{N_{\overline{H}}[\mathcal{V}]} K_n$	Fully complete total complemented duplicate hypergraph of H
121.	$K_n^c \bigvee_{\overline{H}_r N[\mathcal{V}]} K_n^c$	Closed duplicate r-total complemented hypergraph of H

122.	$K_n \underset{\overline{H}_r N[\mathcal{V}]}{\bigvee} K_n^c$	Closed duplicate complete r-total complemented hypergraph of H
123.	$K_n \underset{\overline{H}_r N[\mathcal{V}]}{\bigvee} K_n$	Fully complete closed duplicate r-total complemented hypergraph of H
124.	$K_n^c \underset{H_r^c N[\mathcal{V}]}{\bigvee} K_n^c$	Closed r-duplicate hypergraph of H
125.	$K_n \underset{H_r^c N[\mathcal{V}]}{\bigvee} K_n^c$	Closed r-duplicate complete hypergraph of H
126.	$K_n \underset{H_r^c N[\mathcal{V}]}{\bigvee} K_n$	Fully complete closed r-duplicate hypergraph of H

Table 2: New unary hypergraph operations defined as tensor join of two hypergraphs

When $r = 1$, the hypergraph given in S.No.1 of Table 2 becomes $H \underset{I[\mathcal{V}]}{\bigvee} H$ and we call it simply as *the mirror hypergraph of* H. Similarly, the rest of the hypergraph operations defined in Table 2 in which ${}_rT[\mathcal{V}]$ is involved can be renamed.

Now, we show that the hypergraph operations listed in Table 8 are unary. Consider the r-Mirror hypergraph of H. It is constructed from the hypergraph H as follows: First take H and corresponds to each of its vertex, add a new vertex. Now, make each set S of new vertices as an edge in the r-Mirror hypergraph of H if and only if the set of vertices in H corresponding to the vertices of S forms an edge in H. Then for each r-subset S_r of vertices of H, make the set of all vertices in S_r together with all the new vertices corresponding to each vertices in S_r as an edge in the r-Mirror hypergraph of H. The resulting hypergraph is the desired one. Similarly, the rest of the operations can be viewed.

4.3. Some unary hypergraph operations as $\mathcal{T}^*$-join of hypergraphs in $\mathcal{G}$

Let $G_i(V_i,E_i), i = 1,2,\ldots,k$ be $k(>1)$ copies of a hypergraph H with$|V(H)| = n$. Let $\mathcal{G} = (G_i(V_i,E_i,W_i))_{i=1}^{k}$. For each $S \in \widehat{[k]}$, let $\mathcal{V}_S = (V_i)_{i\in S}$. Let $\mathcal{T}^* = \{T^*[\mathcal{V}_S] \mid S \in \widehat{[k]}\}$ be a set of indicating tensors of type-2. In Table 3, we list some new classes of unary hypergraph operations as $\mathcal{T}^*$-join of hypergraphs in $\mathcal{G}$, for some suitable $\mathcal{T}^*$ as mentioned in the same table. In this table, we take $1 < l \leq k$ and $\mathbf{0}$ denotes a zero tensor of appropriate order and dimension.

S. No.	**Name of the hypergraph**	$T^*[\mathcal{V}_S]$

1.	(l,r)-mirror hypergraph of H	$\begin{cases} {}_rT[\mathcal{V}_S] & \text{if } \lvert S\rvert = l, \\ \mathbf{0} & \text{otherwise} \end{cases}$
2.	Join l-neighbourhood hypergraph of H	$\begin{cases} J[\mathcal{V}_S] & \text{if } \lvert S\rvert = l, \\ \mathbf{0} & \text{otherwise} \end{cases}$
3.	VC-(l,r)-neighbourhood hypergraph of H	$\begin{cases} \Im_r[\mathcal{V}_S] & \text{if } \lvert S\rvert = l, \\ \mathbf{0} & \text{otherwise} \end{cases}$

Table 3: Viewing some new unary hypergraph operations as $\mathcal{T}^*$-join of hypergraphs in $\mathcal{G}$.

4.4. Some classes of hypergraphs as $(H,\mathcal{T})$-join of hypergraphs

Whenever we consider the $(H,\mathcal{T})$-weighted/unweighted join of weighted/ unweighted hypergraphs, without loss of generality, we take the vertex set of H of cardinality k as $[k]$.

Let H be a hypergraph with $|V(H)| = k$ and let $\mathcal{G} = (G_i(V_i,E_i))_{i=1}^k$ be a sequence of k hypergraphs. For each $e \in E(H)$, let $\mathcal{V}_e = (V_i)_{i\in e}$. In Table 4, we list some classes of hypergraphs that can be viewed as a $(H,\mathcal{T})$-join of hypergraphs in $\mathcal{G}$, for some suitable H, G_i and $\mathcal{T}$.

S. No.	**Name of the hypergraph**	H	G_i	$\mathcal{T}$
1.	Join of set $\mathcal{G}$ of m-uniform hypergraphs on a backbone hypergraph H, $r(H) \leq m$ [22]	H	G_i	$\{{}_{B_e}T[\mathcal{V}_e] \mid e \in E(H)\}$, where $B_e = \{m\}$.
2.	Join of set $\mathcal{G}$ of non-uniform hypergraphs on a backbone hypergraph H [22]	H	G_i	$\{{}_{B_e}T[\mathcal{V}_e] \mid e \in E(H)\}$, where $B_e \subseteq \{\lvert e\rvert, \lvert e\rvert+1, \ldots, N_e\}$.
3.	Complete m-uniform strong k-partite hypergraph $(k \geq m)$[22]	K_k^m	$K_{n_i}^c$	$\{{}_{B_e}T[\mathcal{V}_e] \mid e \in E(H)\}$, where $B_e = \{m\}$.
4.	Complete strong k-partite hypergraph	K_k	$K_{n_i}^c$	$\{{}_{B_e}T[\mathcal{V}_e] \mid e \in E(H)\}$, where $B_e = \{\lvert e\rvert\}$.
5.	Lexicographic product of the hypergraphs H_1 and H_2 [12]	H_1	H_2	$\{{}_{B_e}T[\mathcal{V}_e] \mid e \in E(H)\}$, where $B_e = \{\lvert e\rvert\}$.

6.	Cartesian product of the hypergraphs H_1 and H_2 [2]	H_1	H_2	$\{I[\mathcal{V}_e] \mid e \in E(H)\}$

Table 4: Viewing some existing and new class of hypergraphs as $(H,\mathcal{T})$-join of hypergraphs.

5. Spectra of the tensor join of weighted hypergraphs

In this section, we obtain the characteristic polynomial of the adjacency, the Laplacian, the normalized Laplacian matrices of some classes of hypergraphs constructed by the tensor join operations defined in Section 4. For the computation of the normalized Laplacian spectrum of the tensor join of hypergraphs, it is assumed that the constituting hypergraphs do not have isolated vertices.

5.1. Spectra of the $T[\mathcal{A}]$-join of hypergraphs

Let $\mathcal{G} = (G_i(V_i,E_i,W_i))_{i=1}^k$ be a sequence of k weighted hypergraphs. Consider an indicating tensor $T[\mathcal{V}]$, where $\mathcal{V} = (V_i)_{i=1}^k$. We construct the hypergraph $\bigvee\limits_{T[\mathcal{V}]} \mathcal{G}(V,E,W)$ with a weight function $W : E \to \mathbb{R}_{\geq 0}$ defined by,

$$W(e) = \begin{cases} W_i(e) & \text{if } e \in E_i; \\ w_c & \text{if } e \notin E_i \text{ with } |e| = c, \text{ for } i = 1,2,\ldots,k. \end{cases}, \tag{1}$$

where w_c is a non-negative real number corresponding to a new edge of cardinality c.

Throughout this section, we consider the weight function as defined above for any $T[\mathcal{V}]$-join of weighted hypergraphs, unless, we specifically mentioned otherwise.

Theorem 5.1. *Let $\mathcal{G} = (G_i(V_i,E_i,W_i))_{i=1}^k$ be a sequence of weighted r_i-regular hypergraphs G_i with $|V_i| = n_i$, let $\mathcal{V} = (V_i)_{i=1}^k$ and let $X = \{2,3,\ldots,N\}$. Consider an indicating tensor $T[\mathcal{V}]$ such that for every $p \in V_i$, $q \in V_j$ and $c \in X$, $|E^c_{p,q}(T[\mathcal{V}])|$ is a constant, say $n^{(c)}_{ij}$ for all $1 \leq i \leq j \leq k$. Then the characteristic polynomial of the adjacency (resp. the Laplacian, the normalized Laplacian) matrix of the weighted hypergraph $\bigvee\limits_{T[\mathcal{V}]} \mathcal{G}$ is*

$$\left\{ \prod_{i=1}^{k} \prod_{j=1; j\neq i}^{n_i} (x - \alpha_i \lambda_{ij} - \beta_i) \right\} \times P_R(x),$$

where λ_{ij} is a non-Perron adjacency eigenvalue of G_i for $i = 1, 2, \ldots, k$; $j = 1, 2, \ldots, n_i$ and

$$R = \begin{bmatrix} r_1\alpha_1 + \beta_1 + n_1\gamma_1 & n_2\delta_{12}\sum\limits_{c\in X}\frac{w_c \cdot n_{12}^{(c)}}{c-1} & \cdots & n_k\delta_{1k}\sum\limits_{c\in X}\frac{w_c \cdot n_{1k}^{(c)}}{c-1} \\ n_1\delta_{12}\sum\limits_{c\in X}\frac{w_c \cdot n_{12}^{(c)}}{c-1} & r_2\alpha_2 + \beta_2 + n_2\gamma_2 & \cdots & n_k\delta_{2k}\sum\limits_{c\in X}\frac{w_c \cdot n_{2k}^{(c)}}{c-1} \\ \vdots & \vdots & \ddots & \vdots \\ n_1\delta_{1k}\sum\limits_{c\in X}\frac{w_c \cdot n_{1k}^{(c)}}{c-1} & n_2\delta_{2k}\sum\limits_{c\in X}\frac{w_c \cdot n_{2k}^{(c)}}{c-1} & \cdots & r_k\alpha_k + \beta_k + n_k\gamma_k \end{bmatrix}_{k\times k}$$

and for $1 \leq i \leq j \leq k$, the values α_i, β_i, γ_i, δ_{ij} are given in Table 5 corresponding to the respective matrices, where

$$z_i = r_i + (n_i - 1)\sum_{c\in X}\frac{n_{ii}^{(c)} \cdot w_c}{c-1} + \sum_{j=1, j\neq i}^{k} n_j \sum_{c\in X}\frac{n_{ij}^{(c)} \cdot w_c}{c-1}$$

with $n_{ij}^{(c)} = n_{ji}^{(c)}$ for $i, j = 1, 2, \ldots, k$.

Name of the matrix	α_i	γ_i	β_i	δ_{ij}
Adjacency matrix	1	$\sum\limits_{c\in X}\frac{w_c \cdot n_{ii}^{(c)}}{c-1}$	$-\gamma_i$	1
Laplacian matrix	-1	$-\sum\limits_{c\in X}\frac{w_c \cdot n_{ii}^{(c)}}{c-1}$	$z_i - \gamma_i$	-1
Normalized Laplacian matrix	$\frac{-1}{z_i}$	$\alpha_i \sum\limits_{c\in X}\frac{w_c \cdot n_{ii}^{(c)}}{c-1}$	$1 - \gamma_i$	$\frac{-1}{\sqrt{z_i z_j}}$

Table 5: Necessary values to compute the spectrum of the matrices associated with $\bigvee\limits_{T[\mathcal{V}]} \mathcal{G}$

Proof. The adjacency (resp. the Laplacian, the normalized Laplacian) matrix of $\bigvee\limits_{T[\mathcal{V}]} \mathcal{G}$ is a $k \times k$ symmetric block matrix of order $N \times N$ in which the $(i,i)^{th}$ block is

$$\alpha_i A(G_i) + \beta_i I_{n_i} + \gamma_i J_{n_i}$$

and for $i \neq j$, the $(i,j)^{th}$ block is

$$\delta_{ij} \sum_{c \in X} \frac{w_c \cdot n_{ij}^{(c)}}{c-1} J_{n_i \times n_j},$$

where $N := \sum_{i=1}^{k} n_i$ and the values $\alpha_i, \beta_i, \gamma_i, \delta_{ij}$ for $i,j = 1,2,\ldots,k$ are given in Table 5

Notice that for each $i = 1,2,\ldots,k$, the adjacency matrix $A(G_i)$ of G_i is real symmetric of order n_i with the constant row sum r_i. Thus each $A(G_i)$ has an orthogonal basis of $\mathbb{R}^{n_i}$ consisting of its eigenvectors, including the all-one vector $J_{n_i \times 1}$ corresponds to the eigenvalue r_i. Let us denote the eigenvectors of $A(G_i)$ by $X_{i1}(=J_{n_{i\times 1}}), X_{i2}, \cdots, X_{in_i}$ corresponds to the eigenvalues $\lambda_{i1}(=r_i), \lambda_{i2}, \cdots, \lambda_{in_i}$, for all $i = 1,2,\ldots,n_i$. Let

$$\mathcal{X}_{\mathbf{ij}} := [\mathbf{0}, \mathbf{0}, \cdots, \underbrace{X_{ij}}_{\text{i-th place}}, \mathbf{0}, \cdots \mathbf{0}]^{\mathbf{T}}_{\mathbf{1 \times N}}$$

for all $i = 1,2,\ldots,k,\ j = 2,\ldots,n_i$. Then for each $i = 1,2,\ldots,k,\ j = 2,\ldots,n_i$, $\alpha_i \lambda_{ij} + \beta_i$ is an eigenvalue of $\mathcal{A}$ corresponds to the eigenvector $\mathcal{X}_{\mathbf{ij}}$. Since, the span of the remaining k eigenvectors of $\mathcal{A}$ is same as the span of vectors

$$[\mathbf{0}, \mathbf{0}, \ldots, \underbrace{J_{n_i \times 1}}_{\text{i-th place}}, \mathbf{0}, \ldots, \mathbf{0}]^{\mathbf{T}}_{\mathbf{1 \times N}}, i = 1,2,\ldots,k,$$

so let μ be an eigenvalue of $\mathcal{A}$ corresponds to the eigenvector

$$\mathcal{Y} = [a_1 J_{n_1 \times 1}, a_2 J_{n_2 \times 1}, \cdots, a_k J_{n_k \times 1}],$$

where $(a_1, a_2, \ldots, a_k)$ is a non-zero vector in $\mathbb{R}^k$. Then the system of equations $(\mathcal{A} - \mu)\mathcal{Y} = \mathbf{0}$ reduces to the system of equations $(R - \mu)y' = \mathbf{0}$, where $y' = (a_1, a_2, \ldots, a_k)$ and the matrix R is as mentioned in the statement of this theorem. Thus the remaining eigenvalues of $\mathcal{A}$ are the eigenvalues of the matrix R. This completes the proof. □

In the following corollary, we establish infinite families of cospectral hypergraphs by using the $T[\mathcal{A}]$-join operation on hypergraphs.

Corollary 5.2. Let $G_i(V_i, E_i, W_i)$ and $G_i'(V_i', E_i', W_i')$ be r_i-regular $A-$cospectral weighted hypergraphs for $i = 1,2,\ldots,k$. Let $\mathcal{G} = (G_i(V_i, E_i, W_i))_{i=1}^k$, $\mathcal{G}' = (G_i'(V_i', E_i', W_i'))_{i=1}^k$ and let $X = \{2,3,\ldots,N\}$. Let $\mathcal{V} = (V_i)_{i=1}^k$ and $\mathcal{V}' = (V_i')_{i=1}^k$. Consider an indicating tensor $T[\mathcal{V}]$ such that for every $p \in V_i$ and $q \in V_j$, $|E_{p,q}^c(T[\mathcal{V}])| = n_{ij}^{(c)}$ for all $c \in X,\ 1 \leq i \leq j \leq k$. Let $T[\mathcal{V}']$ be an indicating tensor such that $T[\mathcal{V}'] = T[\mathcal{V}]$. Then the weighted $T[\mathcal{V}]$-join of hypergraphs in $\mathcal{G}$ and the weighted $T[\mathcal{V}']$-join of hypergraphs in $\mathcal{G}'$ are simultaneously $A-$cospectral, $L-$cospectral and $\mathcal{L}-$cospectral.

Proof. Since G_i and G_i' are r_i-regular and have the same adjaceny spectrum, the result directly follows from Theorem 5.1. $\square$

Corollary 5.3. Let $\mathcal{G} = (G_i(V_i,E_i,W_i))_{i=1}^k$, where G_i is a weighted r_i-regular m-uniform hypergraph with $|V_i| = n_i$ for $i = 1,2,\ldots,k$. Let $\mathcal{V} = (V_i)_{i=1}^k$. Consider an indicating tensor $T[\mathcal{V};m]$ such that for every $p \in V_i$ and $q \in V_j$, $|E_{p,q}^m(T[\mathcal{V};m])|$ is a constant, say $n_{ij}^{(m)}$, for $1 \leq i \leq j \leq k$. Then the characteristic polynomial of the adjacency (resp. the Laplacian, the normalized Laplacian) matrix of the weighted hypergraph $\bigvee\limits_{T[\mathcal{V};m]} \mathcal{G}$ is

$$\left\{ \prod_{i=1}^{k} \prod_{j=1;j\neq i}^{n_i} (x - \alpha_i\lambda_{ij} - \beta_i) \right\} \times P_R(x),$$

where λ_{ij} is a non-Perron adjacency eigenvalue of G_i for $i = 1,2,\ldots,k;\ j = 1,2,\ldots,n_i$ and

$$R = \begin{bmatrix} r_1\alpha_1 + \beta_1 + n_1\gamma_1 & \delta_{12}\cdot n_2 \frac{w_m \cdot n_{12}^{(m)}}{m-1} & \cdots & \delta_{1k}\cdot n_k \frac{w_m \cdot n_{1k}^{(m)}}{m-1} \\ \delta_{12}\cdot n_1 \frac{w_m \cdot n_{12}^{(m)}}{m-1} & r_2\alpha_2 + \beta_2 + n_2\gamma_2 & \cdots & \delta_{2k}\cdot n_k \frac{w_m \cdot n_{2k}^{(m)}}{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{1k}\cdot n_1 \frac{w_m \cdot n_{1k}^{(m)}}{m-1} & \delta_{2k}\cdot n_2 \frac{w_m \cdot n_{2k}^{(m)}}{m-1} & \cdots & r_k\alpha_k + \beta_k + n_k\gamma_k \end{bmatrix}_{k\times k}$$

and for $1 \leq i \leq j \leq k$, the values $\alpha_i,\ \beta_i,\ \gamma_i,\ \delta_{ij}$ are given in Table 6;

$$z_i = r_i + (n_i - 1)\frac{n_{ii}^{(m)} w_m}{m-1} + \sum_{j=1,j\neq i}^{k} n_j \frac{n_{ij}^{(m)} w_m}{m-1}$$

with $n_{ij}^{(m)} = n_{ji}^{(m)}$ for $i,j = 1,2,\ldots,k$.

Name of the matrix	α_i	γ_i	β_i	δ_{ij}
Adjacency matrix	1	$\frac{w_m \cdot n_{ii}^{(m)}}{m-1}$	$-\gamma_i$	1
Laplacian matrix	-1	$-\frac{w_m \cdot n_{ii}^{(m)}}{m-1}$	$z_i - \gamma_i$	-1

Normalized Laplacian matrix	$\frac{-1}{z_i}$	$\alpha_i \frac{w_m \cdot n_{ii}^{(m)}}{m-1}$	$1-\gamma_i$	$\frac{-1}{\sqrt{z_i z_j}}$

Table 6: Necessary values to compute the spectrum of the matrices associated with $\bigvee\limits_{T[\mathcal{V};m]} \mathcal{G}$.

Proof. If we take $X = \{m\}$ in Theorem 5.1, then

$$z_i = r_i + (n_i - 1)\frac{n_{ii}^{(m)} \cdot w_m}{m-1} + \sum_{j=1, j\neq i}^{k} n_j \frac{n_{ij}^{(m)} \cdot w_m}{m-1}$$

for all $i = 1, 2, \ldots, k$ and so the proof follows. □

Notation 5.1. Let S be a family of k finite sets $A_1, A_2, \ldots, A_k$ and let $c \in \{2, 3, \ldots, |A_1| + |A_2| + \cdots + |A_k|\}$. For $1 \le i \le j \le k$, we denote,

$$n_{ij}^{c}(S) = \begin{cases} \displaystyle\sum_{\substack{l_i \ge 0,\ l_t > 0\ (t\neq i) \\ l_1+l_2+\cdots+l_k = c-2}} \binom{|A_1|}{l_1}\binom{|A_2|}{l_2}\cdots\binom{|A_{i-1}|}{l_{i-1}}\binom{|A_i|-2}{l_i}\cdots\binom{|A_k|}{l_k} & \text{if } i = j; \\ \displaystyle\sum_{\substack{l_i, l_j \ge 0,\ l_t > 0\ (t\neq i,j) \\ l_1+l_2+\cdots+l_k = c-2}} \binom{|A_1|}{l_1}\cdots\binom{|A_{i-1}|}{l_{i-1}}\binom{|A_i|-1}{l_i}\cdots\binom{|A_{j-1}|}{l_{j-1}}\binom{|A_j|-1}{l_j}\cdots\binom{|A_k|}{l_k} & \text{if } i \neq j; \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 5.4. Assume additionally that the hypergraphs given in S.Nos. 4 and 5 of Table 1 be constructed by r_i-regular weighted hypergraph $H_i(V_i, E_i, W_i)$ for all $i = 1, 2, \ldots, k$. Then the characteristic polynomial of the adjacency (resp. the Laplacian, the normalized Laplacian) matrix of the weighted hypergraphs given in Table 1 are obtained from Theorem 5.1 by taking the values $\alpha_i, \beta_i, \gamma_i, \delta_i$ as in that theorem, the values $n_{ij}^{(c)}$, r_i as given in Table 7 and taking $X = B$ given in Table 1 for the respective hypergraph.

S. No.	**Name of the hypergraph**	r_i	c	$n_{ij}^{(c)}$
1.	Complete m-uniform m-partite hypergraph [22]	0	m	$n_{ij}^{(m)} = \begin{cases} 0 & \text{if } i = j \\ \prod\limits_{p=1, p\neq i,j}^{m} n_p & \text{if } i \neq j \end{cases}$

2.	Complete m-uniform weak k-partite hypergraph, $k \leq m$ [22, Example 3.1.2]	0	m	$n_{ij}^{m}(S)$, where $S = \{V(K_{n_i}^{c})\}_{i=1}^{k}$.
3.	Complete weak k-partite hypergraph	0	c	$n_{ij}^{c}(S)$, where $S = \{V(K_{n_i}^{c})\}_{i=1}^{k}$.
4.	Join of a collection $\mathcal{G}$ of non-uniform hypergraphs [22, Theorem 3.2.1]	r_i	c	$n_{ij}^{c}(S)$, where $S = \{V_i\}_{i=1}^{k}$.
5.	Join of a collection $\mathcal{G}$ of m-uniform hypergraphs [22]	r_i	m	$n_{ij}^{m}(S)$, where $S = \{V_i\}_{i=1}^{k}$.

Table 7: Necessary values for determining the spectrum of the matrices associated with the hypergraphs given in Table 1.

5.2. Spectra of hypergraphs constructed by unary hypergraph operations

Notations 5.1. Let $X = \{2,3,\dots,nl\}$ and $r \in \{1,2,\dots,n\}$, where $l,k \in \mathbb{N}\setminus\{1\}$, $l \leq k$; $n \in \mathbb{N}$.

(i) For $c \in X$, let us denote

$$p_1^{(c)} = \begin{cases} \displaystyle\sum_{\substack{t_1+t_2+\cdots+t_l=c-2,\\ t_j\geq 0,\ t_p>0\\ \text{for some } p(p\neq 1)}} \binom{n-2}{t_1}\binom{n}{t_2}\cdots\binom{n}{t_i}\cdots\binom{n}{t_l} & \text{if } c-2>0 \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

and

$$p_2^{(c)} = \begin{cases} \displaystyle\sum_{\substack{t_1+t_2+\cdots+t_l=c-2,\\ t_j\geq 0}} \binom{n-1}{t_1}\binom{n-1}{t_2}\binom{n}{t_3}\binom{n}{t_4}\cdots\binom{n}{t_l} & \text{if } c-2\geq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

(iii) Let $x_1 := \frac{1}{2r-1}\binom{n-1}{r-1}$;

Let $x_2 := \begin{cases} 0 & \text{if } r=1; \\ \frac{1}{2r-1}\binom{n-2}{r-2} & \text{otherwise.} \end{cases}$

Let $H(V(H),E(H))$ be a hypergraph. Consider a weight function $W : E(H) \to \mathbb{R}_{\geq 0}$ defined by,

$$W(e) = w_{|e|} \quad \text{for all } e \in E(H). \tag{4}$$

In the following theorem, we obtain the characteristic polynomial of the adjacency, the Laplacian, the normalized Laplacian matrices of the weighted hypergraphs given in S.Nos.1-36 of Table 2 by assuming a weight function given in (4) on each of the constituting hypergraphs.

Theorem 5.5. *Let H be a hypergraph on n vertices. Consider the hypergraphs $H, H^c, K_n, \overline{H}$ with the weight function given in* (4). *Let $G_1, G_2 \in \{H, H^c, K_n, K_n^c, \overline{H}\}$. Let $\mathcal{V} = (V(G_i))_{i=1}^2$ and $T \in \{{}_rT[\mathcal{V}], I[\mathcal{V}], J[\mathcal{V}], \Im_r[\mathcal{V}], \Im[\mathcal{V}]\}$. If H is r'-regular, then the characteristic polynomial of the adjacency (resp. the Laplacian, the normalized Laplacian) matrix of the weighted hypergraph $G_1 \bigvee_T G_2$ is*

$$\prod_{t=1}^{n}\big(x^2 - x[\lambda_t(M(G_1)+\theta_1\beta I_n+\theta_1'\gamma J_n)+\lambda_t(M(G_2)+\theta_2\beta I_n+\theta_2'\gamma J_n)] \\ +\lambda_t(M(G_1)+\theta_1\beta I_n+\theta_1'\gamma J_n)\times\lambda_t(M(G_2)+\theta_2\beta I_n+\theta_2'\gamma J_n) \\ -(\lambda_t(\delta a I_n+\delta b J_n))^2\big),$$

where for $i = 1,2$, $t = 1,2,\ldots,n$, $\lambda_t(M(G_i)+\theta_i\beta I_n+\theta_i'\gamma J_n)$ and $\lambda_t(\delta a I_n + \delta b J_n)$ are the co-eigenvalues of the matrices $M(G_i)+\theta_i\beta I_n+\theta_i'\gamma J_n$ and $\delta a I_n + \delta b J_n$, respectively and the values $\theta_i, \theta_i', \delta$ and $M(G_i)$ are given in Table 8;

$$r_i = \begin{cases} r' & \text{if } G_i = H; \\ m'-r' & \text{if } G_i = H^c; \\ m-r' & \text{if } G_i = \overline{H}; \\ m & \text{if } G_i = K_n; \\ 0 & \text{if } G_i = K_n^c, \end{cases}$$

where $m' = \sum_{i\in K} w_i\binom{n-1}{i-1}$, $K = \{|e| \mid e \in E(H)\}$ and $m = \sum_{i=2}^{n} w_i\binom{n-1}{i-1}$.
For $i = 1,2$, let $z_i = r_i + \beta + z$; $z = n\gamma + a + nb$. The values β, γ, a and b are given in Table 9 corresponding to the tensor T and the values $p_1^{(c)}$ and $p_2^{(c)}$ are given in (2) *and* (3)*, respectively when $k = l = 2$.*

Name of the matrix	θ_i	θ_i'	δ	$M(G_i)$
Adjacency matrix	1	1	1	$A(G_i)$
Laplacian matrix	$\begin{cases} \frac{z}{\beta} & \text{if } \beta \neq 0, \\ z & \text{if } \beta = 0. \end{cases}$	-1	-1	$L(G_i)$
normalized Laplacian matrix	$\begin{cases} \frac{1}{\beta}-\frac{1}{z_i} & \text{if } \beta \neq 0, \\ -\frac{1}{z_i} & \text{if } \beta = 0. \end{cases}$	$-\frac{1}{z_i}$	$-\frac{1}{\sqrt{z_1 z_2}}$	$-\frac{1}{z_i}A(G_i)$

Table 8: Necessary values for determining the spectrum of the matrices associated with the hypergraphs given in Table 2.

Tensor T	β	γ	a	b
${}_rT[\mathcal{V}]$	$-x_2 \cdot w_{2r}$	$x_2 \cdot w_{2r}$	$w_{2r}(x_1 - x_2)$	x_2
$I[\mathcal{V}]$	0	0	1	0
$J[\mathcal{V}]$	$-\sum\limits_{c=2}^{2n} \frac{p_1^{(c)} \cdot w_c}{c-1}$	$\sum\limits_{c=2}^{2n} \frac{p_1^{(c)} \cdot w_c}{c-1}$	0	$\sum\limits_{c=2}^{2n} \frac{p_2^{(c)} \cdot w_c}{c-1}$
$\mathfrak{I}_r[\mathcal{V}]$	$x_2 \cdot w_{2r} - \sum\limits_{c=2}^{2n} \frac{p_1^{(c)} \cdot w_c}{c-1}$	$-x_2 \cdot w_{2r} + \sum\limits_{c=2}^{2n} \frac{p_1^{(c)} \cdot w_c}{c-1}$	$w_{2r}(x_2 - x_1)$	$-x_2 \cdot w_{2r} + \sum\limits_{c=2}^{2n} \frac{p_2^{(c)} \cdot w_c}{c-1}$
$\mathfrak{I}[\mathcal{V}]$	$-\sum\limits_{c=2}^{2n} \frac{p_1^{(c)} \cdot w_c}{c-1}$	$\sum\limits_{c=2}^{2n} \frac{p_1^{(c)} \cdot w_c}{c-1}$	$-x_1 \cdot w_2$	$-x_2 \cdot w_2 + \sum\limits_{c=2}^{2n} \frac{p_2^{(c)} \cdot w_c}{c-1}$

Table 9: The values of β, γ, a and b corresponding to the indicating tensor T.

Proof. The adjacency (resp. the Laplacian, the normalized Laplacian) matrix of $G_1 \bigvee\limits_T G_2$ is of the form,

$$\mathcal{A} = \begin{bmatrix} M(G_1) + \theta_1 \beta I_n + \theta_1' \gamma J_n & \delta(aI_n + bJ_n) \\ \delta(aI_n + bJ_n) & M(G_2) + \theta_2 \beta I_n + \theta_2' \gamma J_n \end{bmatrix}_{2n \times 2n}$$

where for $i = 1, 2$ the values β, γ, a and b corresponding to the indicating tensor T are given in Table 9 and $M(G_i)$, θ_i, θ_i', δ are given in the statement of Theorem 5.5. Since, G_is are regular hypergraphs, any pair of blocks of $\mathcal{A}$ commute with each other. Thus, the proof follows from Theorem 2.4. $\square$

Corollary 5.6. In Theorem 5.5, let $G_1 = G_2$ (= G, say) be r-regular and let $\mu_1 = c, \mu_2, \ldots, \mu_n$ be the eigenvalues of $M(G)$. Then the characteristic polynomial of the adjacency (resp. the Laplacian, the normalized Laplacian) matrix of $G \bigvee\limits_T G$ is

$$\begin{aligned}(x^2 - (2x - (c + \theta\beta + n\theta'\gamma))(c + \theta\beta + n\theta'\gamma) - \delta^2(a + nb)^2)\\ \times \prod_{i=2}^{n} \left(x^2 - 2(\mu_i + \theta\beta)x + (\mu_i + \theta\beta)^2 - \delta^2 a^2\right),\end{aligned}$$

where

$$c = \begin{cases} r & \text{for the characteristic polynomial of } A\left(G\underset{T}{\bigvee}G\right); \\ 0 & \text{for the characteristic polynomial of } L\left(G\underset{T}{\bigvee}G\right); \\ -\frac{r}{z'} & \text{for the characteristic polynomial of } \mathcal{L}\left(G\underset{T}{\bigvee}G\right). \end{cases}$$

and the values $\beta,\gamma,\delta,a,b,\theta(=\theta_1=\theta_2),\theta'(=\theta_1'=\theta_2'),r(=r_1=r_2),z'(=z_1'=z_2')$ are as given in Theorem 5.5.

Proof. From Theorem 5.5, the characteristic polynomial of the adjacency (resp. the Laplacian, the normalized Laplacian) matrix of the weighted hypergraph $G\underset{T}{\bigvee}G$ is

$$\prod_{t=1}^{n}\big(x^2-2x[\lambda_t(M(G)+\theta\beta I_n+\theta'\gamma J_n)] \\ +(\lambda_t(M(G)+\theta\beta I_n+\theta'\gamma J_n))^2-(\lambda_t(\delta a I_n+\delta b J_n))^2\big), \tag{5}$$

where $\theta(=\theta_1=\theta_2),\theta'(=\theta_1'=\theta_2'),r(=r_1=r_2),z'(=z_1'=z_2')$ are as given in Theorem 5.5 Since $M(G)$ is a real symmetric matrix of order n with the row sum c, there exists an orthogonal basis of $\mathbb{R}^n$ consisting of its eigenvectors, including the all-one vector $J_{n\times 1}$ corresponds to the eigenvalue c. Let us denote the eigenvectors of $M(G)$ by $X_1(=J_{n\times 1}),X_2,\ldots,X_n$ corresponding to the eigenvalues $\mu_1(=c),\mu_2,\ldots,\mu_n$.

Notice that, $\lambda_1(M(G)+\theta\beta I_n+\theta'\gamma J_n)=c+\theta\beta+n\theta'\gamma$ and $\lambda_1(\delta a I_n+\delta b J_n)=\delta(a+nb)$ are the co-eigenvalues corresponding to the common eigenvector X_1.

For $i=2,\ldots,n$, $\lambda_i(M(G)+\theta\beta I_n+\theta'\gamma J_n)=\mu_i+\theta\beta$ and $\lambda_i(\delta a I_n+\delta b J_n)=\delta a$ are the co-eigenvalues corresponding to the common eigenvector X_i. Thus from equation (5), we have

$$\big(x^2-2x(c+\theta\beta+n\theta'\gamma)+(c+\theta\beta+n\theta'\gamma)^2-\delta^2(a+nb)^2\big) \\ \times\prod_{i=2}^{n}\big(x^2-2x(\mu_i+\theta\beta)+(\mu_i+\theta\beta)^2-\delta^2a^2\big).$$

This completes the proof. $\square$

Theorem 5.7. *Let $G_i(V_i,E_i,W_i), i=1,2,\ldots,k$ be $k(>1)$ copies of a weighted r'-regular hypergraph H with $|V(H)|=n$. Then the characteristic polynomial of the adjacency (resp. the Laplacian, the normalized Laplacian) matrix of the* *weighted hypergraphs given in Table 3 is*

$$\prod_{t=1}^{n}\Big[k\,(x-\lambda_t(\alpha A(H)+\beta I_n+\gamma J_n)+\lambda_t(aI_n+bJ_n))^k \\ -\lambda_t(aI_n+bJ_n)(x-\lambda_t(\alpha A(H)+\beta I_n+\gamma J_n)+\lambda_t(aI_n+bJ_n))^{k-1}\Big], \tag{6}$$

where the values α,β,γ,a,b are given in Table 10 and for $t=1,2,\ldots,n$, $\lambda_t(\alpha A(H)+\beta I_n+\gamma J_n)$, $\lambda_t(aI_n+bJ_n)$ are the co-eigenvalues of the matrices $\alpha A(H)+\beta I_n+\gamma J_n$, aI_n+bJ_n, respectively. Let $z=r'\alpha+\beta+n\gamma+(k-1)a+k(n-1)b$, where α,β,γ,a,b are taken corresponding to the matrix of the respective graphs given in Table 10.
Also, for $X=\{2,3,\ldots,ln\}$, $r\in\{1,2,\ldots,n\}$, let

$$p_1'=\binom{k-1}{l-1}\sum_{c\in X}\frac{w_c\cdot p_1^{(c)}}{c-1},\qquad p_2'=\binom{k-2}{l-2}\sum_{c\in X}\frac{w_c\cdot p_2^{(c)}}{c-1},$$

$$p_{21}=\tfrac{w_{lr}}{lr-1}\tbinom{k-2}{l-2}\tbinom{n-1}{r-1},\qquad p_{22}=\begin{cases}0 & \text{if } r=1;\\ \tfrac{w_{lr}}{lr-1}\tbinom{k-2}{l-2}\tbinom{n-2}{r-2} & \text{otherwise},\end{cases}$$

and $$p_{12}=\begin{cases}0 & \text{if } r=1;\\ \tfrac{w_{lr}}{lr-1}\tbinom{k-1}{l-1}\tbinom{n-2}{r-2} & \text{otherwise}.\end{cases}$$

Name of the hypergraph	***Name of the matrix***	α	β	γ	a	b
(l,r)-*mirror hypergraph of H*	*Adjacency matrix*	1	$-p_{12}$	p_{12}	$p_{21}-p_{22}$	p_{22}
	Laplacian matrix	-1	$z+p_{12}$	$-p_{12}$	$p_{22}-p_{21}$	$-p_{22}$
	normalized Laplacian matrix	$-\frac{1}{z}$	$1+\frac{p_{12}}{z}$	$-\frac{p_{12}}{z}$	$\frac{p_{22}-p_{21}}{z}$	$\frac{-p_{22}}{z}$
Join l-neighbourhood hypergraph of H	*Adjacency matrix*	1	$-p_1'$	p_1'	0	p_2'
	Laplacian matrix	-1	$z+p_1'$	$-p_1'$	0	$-p_2'$
	normalized Laplacian matrix	$-\frac{1}{z}$	$1+\frac{p_1'}{z}$	$-\frac{p_1'}{z}$	0	$\frac{-p_2'}{z}$

VC-(l,r)-neighbourhood hypergraph of H	Adjacency matrix	1	$p_{12}-p_1'$	$p_1'-p_{12}$	$p_{22}-p_{21}$	$p_2'-p_{22}$
	Laplacian matrix	-1	$z+p_1'-p_{12}$	$p_{12}-p_1'$	$p_{21}-p_{22}$	$p_{22}-p_2'$
	normalized Laplacian matrix	$-\frac{1}{z}$	$1+\frac{p_1'-p_{12}}{z}$	$\frac{p_{12}-p_1'}{z}$	$\frac{p_{21}-p_{22}}{z}$	$\frac{p_{22}-p_2'}{z}$

Table 10: Necessary values for determining the spectrum of the matrices associated with the hypergraphs given in Table 3

Proof. The adjacency (resp. The Laplacian, The normalized Laplacian) matrix of the hypergraphs given in Table 10 is of the form

$$\mathcal{A} = I_k \otimes (\alpha A(H) + \beta I_n + \gamma J_n) + (J_k - I_k) \otimes (aI_n + bJ_n)$$

with the values $\alpha, \beta, \gamma, a, b$ corresponding to the hypergraphs as given in Table 10. Let

$$\begin{aligned}\mathcal{D} &= I_k \otimes [\lambda_t(\alpha A(H) + \beta I_n + \gamma J_n) - \lambda_t(aI_n + bJ_n)];\\ M_t &= \mathcal{D} + [\lambda_t(aI_n + bJ_n) \times J_{k\times 1} \times J_{1\times k}].\end{aligned}$$

By Theorem 2.3,

$$\begin{aligned}P_{M_t}(x) &= det(xI_k - \mathcal{D} - \lambda_t(aI_n + bJ_n)J_{k\times 1}J_{1\times k})\\ &= det(xI_k - \mathcal{D})det(1 - \lambda_t(aI_n + bJ_n)J_{1\times k}(xI_k - \mathcal{D})^{-1}J_{k\times 1})\\ &= P_{\mathcal{D}}(x)det\left(1 - \frac{\lambda_t(aI_n + bJ_n)\cdot k}{x - \lambda_t(\alpha A(H) + \beta I_n + \gamma J_n) + \lambda_t(aI_n + bJ_n)}\right),\end{aligned}$$

where $P_{\mathcal{D}}(x) = (x - \lambda_t(\alpha A(H) + \beta I_n + \gamma J_n) + \lambda_t(aI_n + bJ_n))^k$. Therefore,

$$\begin{aligned}P_{M_t}(x) \quad = \quad & k(x - \lambda_t(\alpha A(H) + \beta I_n + \gamma J_n) + \lambda_t(aI_n + bJ_n))^k\\ & -\lambda_t(aI_n + bJ_n)(x - \lambda_t(\alpha A(H) + \beta I_n + \gamma J_n) + \lambda_t(aI_n + bJ_n))^{k-1}.\end{aligned}$$

Applying Theorem 2.4, we have $P_{\mathcal{A}}(x) = \prod_{t=1}^{n} P_{M_t}(x)$, as desired. □

5.3. Spectra of the $(H,\mathcal{T})$-join of hypergraphs

Let H be a hypergraph and let $\mathcal{G} = (G_i(V_i, E_i, W_i))_{i=1}^{k}$ be a sequence of weighted hypergraphs. Let E be the edge set of the hypergraph $\mathcal{G}(H,\mathcal{T})$. We define a weight function $W : E \to \mathbb{R}_{\geq 0}$ as follows:

$$W(e') = \begin{cases} W_i(e') & \text{if } e' \in E_i; \\ w_{|e|} & \text{if } e' \in E(T[\mathcal{V}_e]), \end{cases} \tag{7}$$

where $\mathcal{V}_e = (V_i)_{i\in e}$ for each $e \in E(H)$. We denote the hypergraph $\mathcal{G}(H,\mathcal{T})$ together with a weight function W given in (7) by $\mathcal{G}(H,\mathcal{T},W)$.

Throughout this subsection, we consider a weight function as defined above for any $(H,\mathcal{T})$-join of hypergraphs in $\mathcal{G}$.

Theorem 5.8. *Let H be a hypergraph on k vertices. Let $\mathcal{G} = (G_i(V_i,E_i,W_i))_{i=1}^{k}$ be a sequence of r_i-regular weighted hypergraphs G_i with $|V_i| = n_i$ and let $X = \{2,3,\ldots,N\}$. For each $e \in E(H)$, let $\mathcal{V}_e = (V_i)_{i\in e}$ and let $\mathcal{T} = \{T[\mathcal{V}_e] \mid e \in E(H)\}$ be such that for each $p \in V_i$, $q \in V_j$ and $c \in X$, $|E^c_{p,q}(T[\mathcal{V}_e])|$ is a constant, say $n^c_{ij}(e)$ for all $i, j \in e$ and $1 \leq i \leq j \leq k$. Then the characteristic polynomial of the adjacency (resp. the Laplacian, the normalized Laplacian) matrix of the weighted hypergraph $\mathcal{G}(H,\mathcal{T},W)$ is*

$$\prod_{i=1}^{k} \prod_{j=1;j\neq i}^{n_i} (x-\alpha_i\lambda_{ij}-\beta_i) \times P_R(x),$$

where λ_{ij} is a non-Perron adjacency eigenvalue of G_i for all $j = 1,2,\ldots,n_i$, $i = 1,2,\ldots,k$ and

$$R = \begin{bmatrix} r_1\alpha_1+\beta_1+n_1\gamma_1 & n_2\delta_{12}\Delta_{12} & \cdots & n_k\delta_{1k}\Delta_{1k} \\ n_1\delta_{12}\Delta_{12} & r_2\alpha_2+\beta_2+n_2\gamma_2 & \cdots & n_k\delta_{2k}\Delta_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ n_1\delta_{1k}\Delta_{1k} & n_2\delta_{2k}\Delta_{2k} & \cdots & r_k\alpha_k+\beta_k+n_k\gamma_k \end{bmatrix}$$

where, $\Delta_{ij} = \sum_{c\in X} \sum_{\substack{i,j\in e,\\ e\in E(H)}} \frac{w_c \cdot n^c_{ij}(e)}{c-1}$ and the values $\alpha_i, \beta_i, \gamma_i$, δ_{ij} can be computed using Table 5 by taking $n^{(c)}_{ij} = \sum_{\substack{i,j\in e,\\ e\in E(H)}} n^c_{ij}(e)$ for all $1 \leq i \leq j \leq k$ in Theorem 5.1.

Proof. As in Theorem 4.5, $(H,\mathcal{T})$-join of hypergraphs in $\mathcal{G}$ can be viewed as a $T[\mathcal{V}]$-join of hypergraphs in $\mathcal{G}$ for some suitable indicating tensor $T[\mathcal{V}]$, where $\mathcal{V} = (V_i)_{i=1}^{k}$. Since, $p \in V_i$, $q \in V_j$ and $c \in X$, $|E^c_{p,q}(T[\mathcal{V}_e])| = n^c_{ij}(e)$, for all $i, j \in e$ and $1 \leq i \leq j \leq k$, we have $n^{(c)}_{ij}$ is a constant and is equal to $\sum_{\substack{i,j\in e,\\ e\in E(H)}} n^c_{ij}(e)$. Thus the proof follows from Theorem 5.1. $\square$

In the following corollary, we construct infinite families of cospectral hypergraphs by using the $(\mathcal{H},\mathcal{T})$-join operation on hypergraphs.

Corollary 5.9. Let H be a hypergraph on k vertices and let $G_i(V_i,E_i,W_i)$, $G_i'(V_i',E_i',W_i')$ be $A-$cospectral r_i-regular weighted hypergraphs for $i=1,2,\ldots,k$. Let $\mathcal{G}=(G_i)_{i=1}^k$, $\mathcal{G}'=(G_i')_{i=1}^k$ and $X=\{2,3,\ldots,N\}$. For each $e\in E(H)$, let $\mathcal{V}_e=(V_i)_{i\in e}$, $\mathcal{V}_e'=(V_i')_{i\in e}$. Let $\mathcal{T}=\{T[\mathcal{V}_e]\mid e\in E(H)\}$ be such that, for each $p\in V_i$, $q\in V_j$ and $c\in X$, $|E^c_{p,q}(T[\mathcal{V}_e])|$ is a constant, say $n^c_{ij}(e)$, for all $i,j\in e$ and $1\le i\le j\le k$. Let $\mathcal{T}'=\{T[\mathcal{V}_e']\mid e\in E(H)\}$, where $T[\mathcal{V}_e']=T[\mathcal{V}_e]$. Then the hypergraphs $\mathcal{G}(H,\mathcal{T},W)$ and $\mathcal{G}'(H,\mathcal{T}',W)$ are simultaneously $A-$cospectral, $L-$cospectral and $\mathcal{L}-$cospectral.

Proof. Since G_i and G_i' are r_i regular and the values α_i, β_i, γ_i, δ_i depend only upon the indicating tensor $T[\mathcal{V}_e]$, from Theorem 5.8, the matrix R is the same for the hypergraphs $\mathcal{G}(H,\mathcal{T},W)$ and $\mathcal{G}'(H,\mathcal{T}',W)$. Since G_i and G_i' have the same A-spectrum, the result follows. □

Now we proceed to obtain various spectrum of the hypergraphs given in Table 4 by viewing them as a $(H,\mathcal{T})$-join of hypergraphs. In the following corollary, we deduce some results on the spectra of hypergraphs in the literature.

Corollary 5.10. *([22, Theorems 3.2.1, 3.1.1])*

(i) *Let H be a hypergraph with $|V(H)|=k$ and $\mathcal{G}=(G_i(V_i,E_i,W_i))_{i=1}^k$ be a sequence of r_i-regular weighted hypergraphs G_i. Then the characteristic polynomial of the adjacency matrix of the weighted join of set $\mathcal{G}$ of non-uniform hypergraphs on a backbone hypergraph H given in Table 4 is obtained from Theorem 5.1 by taking the values of $z_i,\alpha_i,\beta_i,\gamma_i$ and δ_{ij} as given in Theorem 5.1 and taking*

$$n^{(c)}_{ij}=\begin{cases}\sum\limits_{\substack{i,j\in e\\ e\in E(H)}} n^c_{ij}(S_e) & \text{if } c\in B_e;\\ 0 & \text{otherwise.}\end{cases}\tag{8}$$

(ii) *Let H be a hypergraph with $|V(H)|=k$ and let $\mathcal{G}=(G_i(V_i,E_i,W_i))_{i=1}^k$ be a sequence of r_i-regular m-uniform weighted hypergraphs $G_i(V_i,E_i,W_i)$. Then the characteristic polynomial of the adjacency matrix of the weighted join of set $\mathcal{G}$ of m-uniform hypergraphs on a backbone hypergraph $\mathcal{H}$ given in Table 4 is obtained from Corollary 5.3 by taking the values of $z_i,\alpha_i,\beta_i,\gamma_i$ and δ_{ij} as in Corollary 5.3 and the value of $n^{(c)}_{ij}$ as given in* (8) *with $c=m$.*

Corollary 5.11. *The characteristic polynomial of the Laplacian matrix and the normalized Laplacian matrix of the weighted join of set $\mathcal{G}$ of weighted non-uniform hypergraphs on a backbone hypergraph $\mathcal{H}$ given in S.No.*1 *of Table 4 with the weight function given in* (7) *can be obtained from Theorem* 5.1 *by taking the values $z_i, \alpha_i, \beta_i, \gamma_i$ and δ_{ij} corresponds to the Laplacian, the normalized Laplacian matrices given in Theorem 5.1 and the value $n_{ij}^{(c)}$ as given in* (8).

Corollary 5.12. *The characteristic polynomial of the Laplacian matrix, the normalized Laplacian matrix of the weighted join of set $\mathcal{G}$ of weighted m-uniform hypergraphs on a backbone hypergraph $\mathcal{H}$ given in S.No.*2 *of Table 4 with the weight function given in* (7) *can be obtained from Corollary 5.3 by taking the values $z_i, \alpha_i, \beta_i, \gamma_i$ and δ_{ij} corresponds to the Laplacian, the normalized Laplacian matrices given in Corollary 5.3 and the value $n_{ij}^{(c)}$ as given in* (8) *with $c = m$.*

Notation 5.2. For $1 \leq i \leq j \leq k$, let

$$q_{ij}^{(c)} = \begin{cases} \displaystyle\sum_{\substack{\{p_1,p_2,\ldots,p_{c-2}\}\subseteq\{1,2,\ldots,k\}\setminus\{i,j\},\\ |\{p_1,p_2,\ldots,p_{c-2}\}|=c-2}} n_{p_1}n_{p_2}\ldots n_{p_{c-2}} & \text{if } i \neq j; \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 5.13. (i) The characteristic polynomials of the adjacency, the Laplacian, the normalized Laplacian matrices of the weighted complete m-uniform strong k-partite hypergraph and weighted complete strong k-partite hypergraph mentioned in Table 4 are derived from Theorem 5.1 by using the necessary values given in Table 11 and α_i, β_i, γ_i, δ_i are taken as given in Theorem 5.1.

(ii) The characteristic polynomials of the adjacency, the Laplacian, the normalized Laplacian matrices of the weighted lexicographic product of a hypergraph H and a r-regular weighted hypergraph H' mentioned in Table 4 are obtained from Theorem 5.8 by taking the values given in S.No.3 of Table 11.

S. No.	**Name of the hypergraph**	**Values**
1.	Complete m-uniform strong k-partite hypergraph	$X = \{m\};\ n_{ij}^{(m)} = q_{ij}^{(m)}$
2.	Complete strong k-partite hypergraph	$X = \{2,3,\ldots,k\};\ n_{ij}^{(c)} = q_{ij}^{(c)}$

3.	Lexicographic product of $H(V,E)$ and $H'(V',E')$	$r_i = r$; $n_i = n$; $\alpha_i = 1$, $\beta_i = \gamma_i = 0$; $X = \{\lvert e\rvert \mid e \in E\}$; $n^c_{ij}(e) = \begin{cases} \lvert V'\rvert^{\lvert e\rvert - 2} & \text{if } i \neq j \\ 0 & \text{otherwise.} \end{cases}$

Table 11: Necessary values to compute the spectrum of the hypergraphs given in Table 4

Corollary 5.14. *If H, H' are k-uniform hypergraphs with $|V| = n$, $|V'| = m$ and if H' is weighted r-regular, then the characteristic polynomial of the adjacency matrix of the weighted lexicographic product of H and H' is*

$$\prod_{\lambda}(x-\lambda)^n \prod_{\mu}(x - r - \mu m^{k-1} w_k),$$

where the products run over all the non-Perron eigenvalues λ of $A(H')$ and all the eigenvalues μ of $A(H)$ respectively. The weight function considered in this lexicographic product is as given in (7).

Proof. The lexicographic product of H and H' can be viewed as a $(H, \mathcal{T})$-join of hypergraphs as mentioned in Table 4. So we take $G_i = H'$, $n_i = m$, $r_i = r$, $\alpha_i = 1$, $\beta_i = \gamma_i = 0$, $\delta_{ij} = 1$ and $n^k_{ij}(e) = m^{k-2}$ for all $1 \leq i \leq j \leq k$ in Theorem 5.8. Then the matrix R becomes $rI_n + (w_k m^{k-1})A(H)$. Since it is a polynomial in $A(H)$, the proof follows. □

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REFERENCES

R. VISHNUPRIYA
Department of Mathematics,
The Gandhigram Rural Institute (Deemed to be University),
Gandhigram – 624 302, Tamil Nadu, India.
e-mail: rrvmaths@gmail.com

R. RAJKUMAR
Department of Mathematics,
The Gandhigram Rural Institute (Deemed to be University),
Gandhigram – 624 302, Tamil Nadu, India.
e-mail: rrajmaths@yahoo.co.in