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# ON A QUESTION ABOUT NON-UNIQUENESS OF GLOBAL MINIMA

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Let $X$ be a topological space, $I$ and $J$ be sequentially l.s.c. real functions on $X$ such that $J$ is non-negative with sequentially compact sublevels, and $I/J$ is not bounded from below outside the sublevel $J^{-1}([0,c[)$ of $J$ for some $c>0$. Is it true that for any large enough $\lambda$ and any increasing l.s.c. real function $\varphi$ the function $x \mapsto I(x)+\lambda J(x)+\mu\varphi{\circ}J(x)$ has at least two global minima for some positive $\mu$ ? We give a positive answer to this question assuming that $I+\lambda J+\mu{\circ}\varphi$ has sequentially compact sublevels for some $\lambda$ and all $\mu>0$.

During my stay in Catania in May 2022, Prof. B. Ricceri asked me the above stated question related to his paper [1]. Theorem 6 of the present paper answers positively to this question. This improves Theorem 1.2 of [1] and gives a positive answer to Problem 1 of the same paper.

Let $X$ be a topological space, $I$ and $J$ be two real sequentially l.s.c. functions on $X$ satisfying the following three:

$(i)$ The sublevels $S_\alpha := J^{-1}(]-\infty,\alpha])$ of $J$ are sequentially compact and $\inf_{x\in X} J(x) = 0$,

$(ii)$ There exists $c>0$ such that $\inf_{J(x)>c} \dfrac{I(x)}{J(x)} = -\infty$,

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$(iii)$ There exists a l.s.c. increasing function $\varphi : J(X) \to \mathbb{R}^+$ such that for all $\mu > 0$, the function

$$\psi_\mu : x \mapsto I(x) + J(x) + \mu\,\varphi\circ J(x)$$

admits a unique global minimum[1] $x_\mu$,

Under conditions $(i)$ and $(ii)$, $X$ cannot be compact : the function $I$ would be bounded from below on $X$, thus also the quotient $I/J$ on $X \setminus S_c$.

Since $J$ attains its minimum, there exists $x^*$ such that $J(x^*) = 0$. And we can assume that $I$ attains at $x^*$ its minimum on the compact set $S_0 \neq \emptyset$. Up to replacing $\varphi$ by the increasing l.s.c. function : $t \mapsto \varphi(t) - \varphi(0)$, in what case $\psi_\mu$ is replaced by $\psi_\mu - \mu\varphi(0)$ which has the same minima, we will always assume $\varphi(0) = 0$.

Notice that the function $\varphi$ in $(iii)$ has to be unbounded : indeed condition $(ii)$ implies that $I + \lambda J$ cannot be bounded from below but so is $I + \lambda J + \mu\,\varphi\circ J$ which attains a finite global minimum.

**Lemma 1.** *There exist an $\varepsilon > 0$, an integer $n_0$ and some $\lambda^* > 0$ such that for $\lambda \geq \lambda^*$, we never have simultaneously for any $x \in X$:*

$$\lambda J(x) \leq n_0 \quad \text{and} \quad I(x) \leq I(x^*) - n_0\,.$$

*Proof.* Take $\varepsilon > 0$ in $J(X)$. Then the set $S_\varepsilon$ is sequentially compact and the set $F_n = \{x \in X : I(x) \leq I(x^*) - n\}$ is sequentially closed. Moreover $\bigcap_{n\in\mathbb{N}} F_n$ is empty, and it follows that the non-increasing sequence $(S_\varepsilon \cap F_n)_n$ of sequentially compact sets has empty intersection : thus there exists some integer $n_0 > 0$ such that $F_{n_0} \cap S_\varepsilon = \emptyset$. Take $\lambda^* = \dfrac{n_0}{\varepsilon}$. Then if $x \in F_{n_0}$ and $\lambda \geq \lambda^*$, we must have $x \notin S_\varepsilon$, hence

$$\lambda J(x) > \lambda\varepsilon \ \geq \lambda^*\varepsilon = n_0\,,$$

the wanted result. □

Up to taking $\lambda \geq \lambda^*$ and replacing $J$ by $J' = \lambda J$ and $\varphi$ by $\varphi' : t \mapsto \varphi(t/\lambda)$, hence

$$I(x) + J'(x) + \mu\,\varphi'\circ J'(x) = I(x) + \lambda J(x) + \mu\,\varphi\circ J(x)\,,$$

[1]In fact we can assume $\varphi$ l.s.c., increasing on $J(X)$ and defined on $[0,+\infty[$ By hypothesis $J$ is unbounded, and if $t \in \mathbb{R}^+ \setminus J(X)$, there exists $\alpha, \beta \in J(X)$ with $\alpha < t < \beta$. We then put $\alpha^* = \sup J(X) \cap [0,t[$ and $\beta^* = \inf J(X) \cap ]t,\infty[$. Si $\alpha^* = t = \beta^*$, we define $\varphi(t) = \sup\{\varphi(\alpha) : \alpha \in J(X),\ \alpha < t\}$, what ensures the semi-continuity at $t$.

In contrary if $\alpha^* < \beta^*$, we put $\varphi(\alpha^*) = \sup\{\varphi(\alpha) : \alpha \in J(X),\ \alpha \leq \alpha^*\}$, and for $\alpha^* < t \leq \beta^*$, $\varphi(t) = \inf\{\varphi(\beta) : \beta \in J(X),\ \beta \geq \beta^*\}$. It is easily checked that this extends $\varphi$ to an increasing l.s.c. function. It follows that, so defined on $\mathbb{R}^*$, $\varphi$ is increasing on $J(X)$.

we will later on assume that

$$(iv) \qquad J(x) \le n_0 \Longrightarrow I(x) > I(x^*) - n_0 .$$

**Theorem 2.** *The function $\rho : \mu \mapsto x_\mu$ is continuous from $\mathbb{R}_+^*$ to $X$.*

*Proof.* It is enough to prove the continuity of $\rho$ on each interval $[\alpha,\beta]$ where $0 < \alpha \le \beta < \infty$. Recall that $x_\mu \in X$ is the unique global minimum of the function $\psi_\mu : x \mapsto I(x) + J(x) + \mu\varphi\circ J(x)$.

We prove first that there exists some $\theta > 0$ such that $\rho([\alpha,\beta]) \subset S_\theta$.

If $\alpha \le \nu < \mu \le \beta$, we have $\psi_\mu(x_\mu) \le \psi_\mu(x_\nu)$ and $\psi_\nu(x_\mu) \ge \psi_\nu(x_\nu)$, hence

$$(\mu - \nu)\varphi\circ J(x_\mu) = \psi_\mu(x_\mu) - \psi_\nu(x_\mu) \le \psi_\mu(x_\nu) - \psi_\nu(x_\nu) = (\mu - \nu)\varphi\circ J(x_\nu)$$

whence $\varphi\circ J(x_\mu) \le \varphi\circ J(x_\nu)$, and since $\varphi^{-1}$ is increasing [2] : $J(x_\mu) \le J(x_\nu)$. And if we put $\theta = J(x_\alpha)$, we get $\rho(\mu) = x_\mu \in S_\theta$.

So we have $\rho([\alpha,\beta]) \subset S_\theta$. If $V$ is any open neighborhood of $y = \rho(\mu)$ for $\mu \in [\alpha,\beta]$ and if $\rho^{-1}(V)$ is not a neighborhood of $\mu$, there exists a sequence $(\mu_n)$ in $[\alpha,\beta] \setminus \rho^{-1}(V)$ which converges to $\mu$ . And by sequential compactness of $S_\theta$, the sequence $y_n = x_{\mu_n}$ possesses a cluster value $y^* \in S_\theta \setminus V$, and in particular $y^* \ne y$. Since $(\mu,x) \mapsto \psi_\mu(x)$ is sequentially l.s.c., and since $\psi_{\mu_n}(y_n) \le \psi_{\mu_n}(y)$, we have

$$\begin{aligned}\psi_\mu(y^*) \le \liminf_n \psi_{\mu_n}(y_n) \le \liminf_n \psi_{\mu_n}(y) &= \liminf_n \Big(I(y) + J(y) + \mu_n\varphi\circ J(y)\Big) \\ &= \psi_\mu(y)\end{aligned}$$

from what follows that $y^*$ is a global minimum of $\psi_\mu$, hence that $y^* = y$, a contradiction. This proves the continuity of $\rho$. □

**Lemma 3.** *As $\mu$ tends to $+\infty$, $\rho(\mu)$ tends to $J^{-1}(0)$, in the sense that $\rho([1,+\infty[)$ is conditionnally compact in $X$ and that the cluster values of $\rho$ at $+\infty$ belong to $S_\theta$, where $\theta = \inf(J(X) \setminus \{0\}$. More precisely, if $\varepsilon \in J(X) \setminus \{0\}$, we have $\rho(\mu) \in S_\varepsilon$ for all large enough $\mu$.*

*Proof.* Let $\varepsilon \in J(X) \setminus \{0\}$ and $\delta = \varphi(\varepsilon)$. We have $\delta > 0$ since $\varphi$ is increasing. If $x_\mu = \rho(\mu) \notin S_\varepsilon$, fix some $\nu < \mu$; we have

$$\begin{aligned}I(x^*) &= I(x^*) + J(x^*) + \varphi\circ J(x^*) = \psi_\mu(x^*) \\ &\ge \psi_\mu(x_\mu) = I(x_\mu) + J(x_\mu) + \mu\varphi\circ J(x_\mu) \\ &\ge I(x_\mu) + J(x_\mu) + \nu\varphi\circ J(x_\mu) + (\mu - \nu)\varphi\circ J(x_\mu) \\ &\ge \psi_\nu(x_\nu) + (\mu - \nu)\varphi\circ J(x_\mu) \\ &> \psi_\nu(x_\nu) + (\mu - \nu)\delta\end{aligned}$$

[2]if we assume only $\varphi$ non-decreasing, we can find $\theta$ such that $\varphi(\theta) > \varphi\circ J(x_\alpha)$ since $\varphi$ is unbounded, whence $\varphi\circ J(x_\mu) \le \varphi\circ J(x_\alpha) < \varphi(\theta)$ and $J(x_\mu) < \theta$ for all $\mu \in [\alpha,\beta]$.

whence $\mu < \nu + \dfrac{1}{\delta}\Big(I(x^*) - \psi_\nu(x_\nu)\Big)$, which is contradictory for large $\mu$. Thus for all $\varepsilon > 0$ we have $\rho(\mu) \in S_\varepsilon$ for large enough $\mu$. Since the set $S_\varepsilon$ is compact, we see that $\rho([1, +\infty[)$ is contained in $S_\varepsilon \cup \rho([1, \mu])$ for a convenient $\mu$, whence we get the conditional compactness. And if $x$ is a cluster value of $\rho$ at $+\infty$, we get[3] $x \in S_\varepsilon$ for all $\varepsilon > 0$, hence $x \in \bigcap_{\varepsilon>0,\ \varepsilon\in J(X)} S_\varepsilon = S_\theta$.

If 0 belongs to the closure of $J(X) \setminus \{0\}$, we have $\theta = 0$ and $S_\theta = J^{-1}(0)$. □

**Lemma 4.** *We have* $\liminf_{\mu\to 0} I\circ\rho(\mu) = -\infty$.

*Proof.* Let $\eta > 0$. There exists $c > 0$ such that $\inf_{x\notin S_c} \dfrac{I(x)}{J(x)} = -\infty$. Thus, for each integer $q > 1$, we can find some $z_q \in X$ such that $I(z_q) < -(q+2)J(z_q)$ and $J(z_q) \geq c$. We the have

$$I(z_q) + J(z_q) < -(1+q)J(z_q) \leq -(1+q)c$$

and one can find some $\mu \in\, ]0, \eta[$ such that $\mu\,\varphi\circ J(z_q) < c$, and by definition of $x_\mu = \rho(\mu)$,

$$\begin{aligned} I\circ\rho(\mu) = I(x_\mu) &\leq I(x_\mu) + J(x_\mu) + \mu\,\varphi\circ J(x_\mu) \\ &\leq I(z_q) + J(z_q) + \mu\,\varphi\circ J(z_q) \\ &< -(1+q)J(z_q) + \mu\,\varphi\circ J(z_q) \\ &< -(1+q)c + c = -qc\,, \end{aligned}$$

and this completes the proof since $qc$ est arbitrarily large. □

**Theorem 5.** *Hypotheses (i), (ii), (iii) and (iv) cannot hold simultaneously.*

*Proof.* Consider the two sets $H_1 = \rho^{-1}\big(\{x \in X : I(x) \leq I(x^*) - n_0\}\big)$ and $H_2 = \rho^{-1}\big(S_{n_0}\big)$. Since $\rho$ is continuous both sets are sequentially closed, hence closed in $\mathbb{R}_+^*$ . Il follows from condition $(iv)$ that $H_1 \cap H_2 = \emptyset$, from lemma 3 that $H_2 \neq \emptyset$ and from lemma 4 that $H_1 \neq \emptyset$. By connectedness of $\mathbb{R}_+^*$, we cannot have $H_1 \cup H_2 = \mathbb{R}_+^*$. Thus there exists $\mu^*$ such that $\mu^* \notin H_1 \cup H_2$. Then we have, for $z^* = \rho(\mu^*)$, $J(z^*) > n_0$ and $I(z^*) > I(x^*) - n_0$, thus

$$\begin{aligned} I(x^*) < I(z^*) + J(z^*) &\leq I(z^*) + J(z^*) + \mu^*\,\varphi\circ J(z^*) \\ &\leq I(x^*) + J(x^*) + \mu^*\,\varphi\circ J(x^*) = I(x^*)\,, \end{aligned}$$

since $J(x^*) = 0$ and $\varphi(0) = 0$, a contradiction. □

[3] if $\varphi$ is only assumed to be non-decreasing, we can find $\eta > 0$ such that $\delta = \varphi(\eta) > 0$. We will then have $\rho(\mu) \in S_\eta$ for all large $\mu$, and the cluster values of $\rho$ at $+\infty$ will belong to $S_\zeta$, where $\zeta = \inf\{\varepsilon \in J(X) : \varphi(\varepsilon) > 0\}$.

**Theorem 6.** *Let $X$ be a topological space, $I$ and $J$ two real sequentially l.s.c. functions on $X$ satisfying $(i)$ and $(ii)$. Then there exists some $\lambda^* > 0$ such that for all $\lambda \geq \lambda^*$ and all increasing l.s.c. function $\varphi$ from $J(X)$ to $\mathbb{R}^+$ the function $\psi_\mu : x \mapsto I(x) + \lambda J(x) + \mu\varphi\circ J(x)$ has necessarily at least two global minima on $X$ for some $\mu > 0$ if it has sequentially compact sublevels.*

*Proof.* Notice that if the l.s.c. function $I + \lambda J + \mu\varphi\circ J$ has sequentially compact sublevels for some $\lambda$, $\mu$, then for $\lambda' \geq \lambda$ and $\mu' \geq \mu$,

$$S'_\alpha = \{x : \big(I + \lambda' J + \mu'\varphi\circ J\big)(x) \geq \alpha\} \subset S_\alpha = \{x : \big(I + \lambda J + \mu\varphi\circ J\big)(x) \geq \alpha\}$$

hence $S'_\alpha$ which is closed in the sequentially compact set $S_\alpha$ is itself sequentially compact. For example this is the case if $\varepsilon I + \varphi\circ J$ is bounded from below for every $\varepsilon > 0$ : $S_\alpha$ is contained in some sublevel of $J$.

It follows from what precedes that one can find $\lambda^*$ such that for every $\lambda \geq \lambda^*$ the extra condition $(iv)$ be satisfied for $I$ and $\lambda J$. Then the result follows immediately from theorem 5 : the condition $(iii)$ cannot hold ; for every increasing sequentially l.s.c. function $\varphi$ there are some $\mu > 0$ for which $\psi_\mu$ does not have a unique global minimum. But since this function $\psi_\mu$ has sequentially compact sublevels, the global minimum is attained, necessarily at several points: $\psi_\mu$ has several global minima. $\square$

## REFERENCES

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