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SOME RESULTS ON A GRAPH ASSOCIATED WITH A NON-QUASI-LOCAL ATOMIC DOMAIN

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Let R be an atomic domain which admits at least two maximal ideals. Let $Irr(R)$ denote the set of all irreducible elements of R and let $\mathcal{A}(R) = \{R\pi \mid \pi \in Irr(R)\}$. Let $\mathcal{I}(R)$ denote the subset of $\mathcal{A}(R)$ consisting of all $R\pi \in \mathcal{A}(R)$ such that π does not belong to the Jacobson radical of R. With R, we associate an undirected graph denoted by $\mathbb{G}(R)$ whose vertex set is $\mathcal{I}(R)$ and distinct vertices $R\pi_1$ and $R\pi_2$ are adjacent if and only if $R\pi_1 \cap R\pi_2 = R\pi_1\pi_2$. The aim of this article is to discuss some results on the connectedness of $\mathbb{G}(R)$ and on the girth of $\mathbb{G}(R)$.

1. Introduction

The rings considered in this article are commutative with identity. The graphs considered in this article are undirected and simple. For a graph G, we denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$. Let R be a ring. This article is motivated by the interesting results that are found in [11], in which Sharma and Bhatwadekar associated and investigated an undirected graph denoted by $G(R)$ such that $V(G(R)) = R$ and distinct vertices x and y are adjacent if and only if $Rx + Ry = R$. Several inspiring results on the coloring of $G(R)$ were

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proved in [11]. Maimani et al. investigated the graph properties of several subgraphs of $G(R)$ in [7]. The graph $G(R)$ is called the *comaximal graph* of R and it is denoted by $\Gamma(R)$ in [7]. We denote the set of all units of R by $U(R)$ and the set of all non-units of R by $NU(R)$. The set of all maximal ideals of R is denoted by $Max(R)$ and the set of all prime ideals of R is denoted by $Spec(R)$. We denote the Jacobson radical of R by $J(R)$. The cardinality of a set A is denoted by $|A|$. Many properties of the subgraphs $\Gamma_1(R)$, $\Gamma_2(R)$, and $\Gamma_2(R)\backslash J(R)$ of $\Gamma(R)$ were discussed in [7], where $\Gamma_1(R)$ is induced by $U(R)$, $\Gamma_2(R)$ is induced by $NU(R)$, and with the assumption $|Max(R)| \geq 2$, $\Gamma_2(R)\backslash J(R)$ is induced by $NU(R)\backslash J(R)$. The interplay between the graph-theoretic properties of the comaximal graph of R and the algebraic properties of R can be found in the literature (see for example, [4, 8, 10]).

Let R be an integral domain. We denote the set of all irreducible elements of R by $Irr(R)$. We recall that R is said to be *atomic* if any non-zero non-unit of R can be expressed as the product of a finite number of irreducible elements of R. It is well-known that any integral domain that satisfies the ascending chain condition on principal ideals is an atomic domain [6, Proposition 1.1.1, page 156]. Hence, any Noetherian domain is atomic. If a set A is a subset of a set B and $A \neq B$, then we denote it by $A \subset B$ (or by $B \supset A$). For any atomic domain T, we denote $\{T\pi \mid \pi \in Irr(T)\}$ by $\mathcal{A}(T)$.

Let R be an atomic domain with $|Max(R)| \geq 2$. Let $\mathcal{A}(R) = \{R\pi \mid \pi \in Irr(R)\}$. Let $\mathcal{I}(R)$ denote the subset of $\mathcal{A}(R)$ consisting of all $R\pi \in \mathcal{A}(R)$ such that $\pi \notin J(R)$. With R, in this article, we introduce an undirected graph denoted by $\mathbb{G}(R)$ such that $V(\mathbb{G}(R)) = \mathcal{I}(R)$ and distinct vertices $R\pi$ and $R\pi'$ are adjacent if and only if $R\pi \cap R\pi' = R\pi\pi'$. The aim of this article is to discuss some results on the connectedness of $\mathbb{G}(R)$ and on the girth of $\mathbb{G}(R)$.

Let G be a graph. A subgraph H of G is said to be a *spanning subgraph* of G if $V(H) = V(G)$. Let T be a ring. We recall that the ideals I, J of T are said to be *coprime* (or *comaximal*) if $I + J = T$ [1, page 7]. Let $n \geq 2$. If the ideals $I_1, I_2, \ldots, I_n$ of T are such that I_i, I_j are comaximal for all distinct $i, j \in \{1, 2, \ldots, n\}$, then $\prod_{i=1}^{n} I_i = \bigcap_{i=1}^{n} I_i$ [1, Proposition 1.10(*i*)]. Thus if $a, b \in T$ are such that $Ta + Tb = T$, then $Ta \cap Tb = Tab$. For an atomic domain R with $|Max(R)| \geq 2$, in this article, we also consider another undirected graph denoted by $\mathbb{CGI}(R)$ such that $V(\mathbb{CGI}(R)) = \mathcal{I}(R)$ and distinct vertices $R\pi$ and $R\pi'$ are adjacent if and only if $R\pi + R\pi' = R$. It is clear that $\mathbb{CGI}(R)$ is a spanning subgraph of $\mathbb{G}(R)$. We use some graph properties of $\mathbb{CGI}(R)$ in our study on the properties of $\mathbb{G}(R)$.

A ring R is said to be *quasi-local* (respectively, *semi-quasi-local*) if $|Max(R)| = 1$ (respectively, $|Max(R)| < \infty$). A Noetherian quasi-local (respectively, semi-quasi-local) ring is referred to as a *local* (respectively, *semi-local*) ring. The Krull dimension of R is simply denoted by $dimR$. For definitions and terminologies from commutative ring theory that are not defined in this article, the reader can refer any of the standard text books in commutative ring theory (for example, [1, 5, 9]).

It is useful to recall the following definitions from graph theory before we give an account of the results that are proved in this article. Let $G = (V,E)$ be a graph. Let $a,b \in V$ with $a \neq b$. Suppose that there exists a path in G between a and b. Recall that the *distance* between a and b, denoted by $d(a,b)$ is defined as the length of a shortest path in G between a and b. We define $d(a,b) = \infty$ if there exists no path in G between a and b. We define $d(a,a) = 0$ [2, Definition 1.5.5]. A graph $G = (V,E)$ is said to be *connected* if for any distinct $a,b \in V$, there exists a path in G between a and b [2, Definition 1.5.4]. For a connected graph $G = (V,E)$, the *diameter of* G denoted by $diam(G)$ is defined as $diam(G) = sup\{d(a,b) \mid a,b \in V\}$ [2, Definition 4.3.1(1)]. Let $a \in V$. The *eccentricity of* a, denoted by $e(a)$ is defined as $e(a) = sup\{d(a,b) \mid b \in V\}$ [2, Definition 4.3.1(2)]. The *radius of* G, denoted by $r(G)$ is defined as $r(G) = min\{e(a) \mid a \in V\}$ [2, Definition 4.3.1(3)]. A simple graph $G = (V,E)$ is said to be *complete* if every pair of distinct vertices of G are adjacent in G [2, Definition 1.2.11].

A graph $G = (V,E)$ is said to be *bipartite* if V can be partitioned into two non-empty subsets V_1 and V_2 such that each edge of G has one end in V_1 and the other in V_2. A bipartite graph with vertex partition V_1 and V_2 is said to be *complete* if each element of V_1 is adjacent to every element of V_2. A complete bipartite graph with vertex partition V_1 and V_2 is called a *star* if either $|V_1| = 1$ or $|V_2| = 1$ [2, Definition 1.2.12]. Suppose that a graph $G = (V,E)$ contains a cycle. Recall that the *girth* of G, denoted by $gr(G)$ is defined as the length of a shortest cycle in G. If G does not contain any cycle, then we set $gr(G) = \infty$.

Let R be an atomic domain with $|Max(R)| \geq 2$. This article consists of three sections including the introduction. In Section 2 of this article, we discuss some results on the connectedness of $\mathbb{G}(R)$. It is proved in Proposition 2.3 that $\mathbb{CGI}(R)$ is connected and $diam(\mathbb{CGI}(R)) \leq 3$ and it is deduced in Corollary 2.4 that $\mathbb{G}(R)$ is connected and $diam(\mathbb{G}(R)) \leq 3$. If π is a prime element of R (R can possibly be quasi-local), then it is verified in Lemma 2.5 that $R\pi \cap R\pi' = R\pi\pi'$ for all $R\pi' \in \mathcal{A}(R)$ such that $R\pi \neq R\pi'$. As a consequence of Lemma 2.5, it is proved in Corollary 2.6 that if R is a unique factorization domain (UFD) with $|Max(R)| \geq 2$, then $\mathbb{G}(R)$ is complete. We do not know whether $\mathbb{G}(R)$ is complete implies that R is a unique factorization domain.

For an atomic domain R (R can possibly be quasi-local), in Remark 2.7, as suggested by the referee, we consider an undirected graph denoted by $\mathbb{A}(R)$ such that $V(\mathbb{A}(R)) = \mathcal{A}(R) = \{R\pi \mid \pi \in Irr(R)\}$ and distinct vertices $R\pi$ and $R\pi'$ are adjacent if and only if $R\pi \cap R\pi' = R\pi\pi'$. It is noted in Remark 2.7 that if R is a UFD, then $\mathbb{A}(R)$ is complete. If $\pi \in Irr(R)$ is such that $R\pi \cap R\pi' = R\pi\pi'$ for all $\pi' \in Irr(R)$ with $R\pi \neq R\pi'$, then it is proved in Proposition 2.10 that π is a prime element of R. As a corollary to Proposition 2.10 and Remark 2.7, it is shown in Corollary 2.11 that $\mathbb{A}(R)$ is complete if and only if R is a UFD. Note that for an atomic domain R with $|Max(R)| \geq 2$, $\mathbb{A}(R)$ is a supergraph of $\mathbb{G}(R)$. If $J(R) = (0)$, then it is deduced in Corollary 2.12 that if $\mathbb{G}(R)$ is complete, then R is a UFD.

Let R be an atomic domain with $|Max(R)| \geq 2$. If $J(R) \in Spec(R)$, then it is proved in Corollary 2.13 that $diam(\mathbb{G}(R)) \leq 2$. Let $R\pi \in \mathcal{I}(R)$. It is shown in Lemma 2.14 that $e(R\pi) = 1$ in $\mathbb{G}(R)$ if and only if π is a prime element of R. It is deduced in Corollary 2.15 that $diam(\mathbb{G}(R)) = 2$ if there exist $R\pi, R\pi' \in \mathcal{I}(R)$ such that π is a prime element of R and π' is not a prime element of R. In Example 2.17(1), we provide an atomic domain R with $|Max(R)| \geq 2$ such that $diam(\mathbb{G}(R)) = 2$ and $r(\mathbb{G}(R)) = 1$ and in Example 2.17(2), we provide an atomic domain R with $|Max(R)| \geq 2$ such that $diam(\mathbb{G}(R)) = r(\mathbb{G}(R)) = 2$. It is proved in Proposition 2.18 that $diam(\mathbb{CGI}(R)) = 3$ if and only if there exist distinct $R\pi_1, R\pi_2 \in \mathcal{I}(R)$ such that $R\pi_1 + R\pi_2 \subseteq \mathfrak{m}$ for some $\mathfrak{m} \in Max(R)$ and $\pi_1\pi_2 \in J(R)$. In Example 2.19, we provide a UFD R with $|Max(R)| = 3$ such that $diam(\mathbb{CGI}(R)) = 3$. It is proved in Proposition 2.21 that $diam(\mathbb{G}(R)) = 3$ if there exist distinct $R\pi_1, R\pi_2 \in \mathcal{I}(R)$ such that $R\pi_1 + R\pi_2 \subseteq \mathfrak{p}$ for some maximal t-deal $\mathfrak{p}$ of R and $\pi_1\pi_2$ belongs to every maximal t-ideal of R. We are not able to provide an example to illustrate Proposition 2.21.

Let R be an atomic domain with $|Max(R)| \geq 2$. In Section 3 of this article, we discuss some results on the girth of $\mathbb{G}(R)$ and some related results. If $|Max(R)| \geq 3$, then it is proved in Proposition 3.1 that $gr(\mathbb{CGI}(R)) = 3$ and it is deduced in Corollary 3.2 that $gr(\mathbb{G}(R)) = 3$. If $|Max(R)| = 2$, then it is shown in Lemma 3.4 that $\mathbb{CGI}(R)$ is a complete bipartite graph. For an atomic domain R with $|Max(R)| \geq 2$, necessary and sufficient conditions are determined in Proposition 3.5 (respectively, in Proposition 3.7) in order that $\mathbb{G}(R)$ to be a complete bipartite graph (respectively, a star graph). If $\mathbb{G}(R)$ contains a cycle with $gr(\mathbb{G}(R)) \neq 3$, then it is proved in Proposition 3.8 that $gr(\mathbb{G}(R)) = 4$ and moreover, in such a case, it is shown that $\mathbb{CGI}(R) = \mathbb{G}(R)$ is a complete bipartite graph but not a star graph. It is deduced in Corollary 3.9 that $gr(\mathbb{G}(R)) = \infty$ if and only if $|Max(R)| = 2$ and at least one maximal ideal of R is principal. If $gr(\mathbb{G}(R)) = \infty$, then it is deduced in Corollary 3.9 that $\mathbb{G}(R)$ is a star graph. Some examples are given to illustrate the results proved in this section (see Examples 3.11 and 3.12).

2. Some results on the connectedness of $\mathbb{G}(R)$

Let R be an atomic domain with $|Max(R)| \geq 2$. The aim of this section is to discuss some results on the connectedness of $\mathbb{G}(R)$. It is already noted in Section 1 that $\mathbb{CGI}(R)$ is a spanning subgraph of $\mathbb{G}(R)$. In Proposition 2.3, we show that $\mathbb{CGI}(R)$ is connected and $diam(\mathbb{CGI}(R)) \leq 3$. We deduce in Corollary 2.4 that $\mathbb{G}(R)$ is connected and $diam(\mathbb{G}(R)) \leq 3$. We use Lemma 2.1 often in our discussion. We use Lemma 2.2 in the proof of Proposition 2.3.

Lemma 2.1. *Let R be an atomic domain with $|Max(R)| \geq 2$ and let $\mathfrak{p} \in Spec(R)$ be such that $\mathfrak{p} \not\subseteq J(R)$. Then there exists $R\pi \in \mathcal{I}(R)$ such that $\pi \in \mathfrak{p}$.*

Proof. Assume that $\mathfrak{p} \in Spec(R)$ is such that $\mathfrak{p} \not\subseteq J(R)$. Let $a \in \mathfrak{p} \backslash J(R)$. Since R is atomic, a can be expressed as the product of finite number of irreducible elements of R. By the choice of a, it follows that no irreducible factor of a belongs to $J(R)$ and at least one irreducible factor π of a such that $\pi \in \mathfrak{p}$. It is clear that $R\pi \in \mathcal{I}(R)$ and $\pi \in \mathfrak{p}$. $\square$

Let R be an atomic domain with $|Max(R)| \geq 2$. Let $\mathfrak{m} \in Max(R)$. Then $\mathfrak{m} \in Spec(R)$ and $\mathfrak{m} \not\subseteq J(R)$. Hence, it follows from Lemma 2.1 that there exists $R\pi \in \mathcal{I}(R)$ such that $\pi \in \mathfrak{m}$. Given $R\pi \in \mathcal{I}(R)$, we claim that there exists $R\pi' \in \mathcal{I}(R)$ such that $R\pi$ and $R\pi'$ are adjacent in $\mathbb{CGI}(R)$. Let $\mathfrak{m} \in Max(R)$ be such that $\pi \in \mathfrak{m}$. Note that $\pi \notin \mathfrak{m}'$ for some $\mathfrak{m}' \in Max(R)$. Hence, $R\pi + \mathfrak{m}' = R$. So, there exist $r \in R$ and $a' \in \mathfrak{m}'$ such that $r\pi + a' = 1$. It is clear that $a' \notin \mathfrak{m}$. Since R is atomic, it follows that there exists $R\pi' \in \mathcal{I}(R)$ such that π' is a factor of a' in R and $\pi' \in \mathfrak{m}'$. It is now evident that $R\pi + R\pi' = R$. Hence, $R\pi$ and $R\pi'$ are adjacent in $\mathbb{CGI}(R)$ and so, they are adjacent in $\mathbb{G}(R)$.

Lemma 2.2. *Let R be an atomic domain with $|Max(R)| \geq 2$ and let $R\pi_1, R\pi_2 \in \mathcal{I}(R)$ be distinct. If $\pi_1\pi_2 \notin J(R)$, then there exists a path of length at most two between $R\pi_1$ and $R\pi_2$ in $\mathbb{CGI}(R)$.*

Proof. Assume that $R\pi_1, R\pi_2 \in \mathcal{I}(R)$ are such that $R\pi_1 \neq R\pi_2$ and $\pi_1\pi_2 \notin J(R)$. We can assume that $R\pi_1$ and $R\pi_2$ are not adjacent in $\mathbb{CGI}(R)$. Note that there exists $\mathfrak{m} \in Max(R)$ such that $\pi_1\pi_2 \notin \mathfrak{m}$ and so, $R\pi_1\pi_2 + \mathfrak{m} = R$. Hence, there exist $r \in R$ and $a \in \mathfrak{m}$ such that $r\pi_1\pi_2 + a = 1$. It is clear that $a \in \mathfrak{m} \backslash J(R)$. It follows that there exists $R\pi_3 \in \mathcal{I}(R)$ such that π_3 is a factor of a in R and $\pi_3 \in \mathfrak{m}$. Observe that $R\pi_1\pi_2 + Ra = R = R\pi_i + R\pi_3$ for each $i \in \{1,2\}$. Therefore, $R\pi_1 - R\pi_3 - R\pi_2$ is a path of length two between $R\pi_1$ and $R\pi_2$ in $\mathbb{CGI}(R)$. $\square$

The proof of Proposition 2.3 is motivated by the proof of [7, Theorem 3.1].

Proposition 2.3. *Let R be an atomic domain with $|Max(R)| \geq 2$. Then $\mathbb{CGI}(R)$ is connected and $diam(\mathbb{CGI}(R)) \leq 3$.*

Proof. Let $R\pi_1, R\pi_2 \in \mathcal{I}(R)$ be such that $R\pi_1 \neq R\pi_2$. We can assume that $R\pi_1$ and $R\pi_2$ are not adjacent in $\mathbb{CGI}(R)$. If $\pi_1\pi_2 \notin J(R)$, then there exists a path of length two between $R\pi_1$ and $R\pi_2$ in $\mathbb{CGI}(R)$ by Lemma 2.2. Assume that $\pi_1\pi_2 \in J(R)$. Note that for each $i \in \{1,2\}$, there exists $\mathfrak{m}_i \in Max(R)$ such that $\pi_i \notin \mathfrak{m}_i$. From $\pi_1\pi_2 \in J(R)$, it follows that $\pi_1 \in \mathfrak{m}_2$ and $\pi_2 \in \mathfrak{m}_1$. Observe that $R\pi_1 + \mathfrak{m}_1 = R$ and so, there exist $r \in R$ and $a_1 \in \mathfrak{m}_1$ such that $r\pi_1 + a_1 = 1$. It is clear that $a_1 \notin \mathfrak{m}_2$. Since R is atomic, there exists a factor π_3 of a_1 in R such that $R\pi_3 \in \mathcal{I}(R)$ and $\pi_3 \in \mathfrak{m}_1$. It is clear that $R\pi_1 + R\pi_3 = R$ and $\pi_3 \notin \mathfrak{m}_2$. Therefore, $\pi_3\pi_2 \notin \mathfrak{m}_2$. Hence, it follows from Lemma 2.2 that there exists a path P of length at most two between $R\pi_3$ and $R\pi_2$ in $\mathbb{CGI}(R)$. As $R\pi_1$ and $R\pi_3$ are adjacent in $\mathbb{CGI}(R)$, it follows that the union of the edge $R\pi_1 - R\pi_3$ and the path P gives a path of length at most three between $R\pi_1$ and $R\pi_2$ in $\mathbb{CGI}(R)$.

This proves that $\mathbb{CGI}(R)$ is connected and $diam(\mathbb{CGI}(R)) \leq 3$. □

Corollary 2.4. *Let R be an atomic domain with $|Max(R)| \geq 2$. Then $\mathbb{G}(R)$ is connected and $diam(\mathbb{G}(R)) \leq 3$.*

Proof. As $\mathbb{CGI}(R)$ is a spanning subgraph of $\mathbb{G}(R)$, it follows from Proposition 2.3 that $\mathbb{G}(R)$ is connected and $diam(\mathbb{G}(R)) \leq 3$. □

Let R be an atomic domain with $|Max(R)| \geq 2$. We next try to characterize R such that $diam(\mathbb{G}(R))$ is equal to $1, 2$ or 3. If R is a unique factorization domain (UFD), then we verify in Corollary 2.6 that $\mathbb{G}(R)$ is complete. We use Lemma 2.5 in the proof of Corollary 2.6. Recall that a non-zero non-unit π of an integral domain T is called a *prime element* if $T\pi \in Spec(T)$.

Lemma 2.5. *Let R be an atomic domain (R can possibly be quasi-local). If π is a prime element of R, then $R\pi \cap R\pi' = R\pi\pi'$ for all $\pi' \in Irr(R)$ such that $R\pi \neq R\pi'$.*

Proof. Assume that π is a prime element of R. Let $\pi' \in Irr(R)$ be such that $R\pi \neq R\pi'$. It is clear that $R\pi\pi' \subseteq R\pi \cap R\pi'$. Let $x \in R\pi \cap R\pi'$. Then $x = r\pi = s\pi'$ for some $r, s \in R$. Since π is a prime element of R and π, π' are non-associates in R, it follows that π divides s in R. Hence, $s = \pi s_1$ for some $s_1 \in R$ and so, $x = \pi\pi' s_1$. This proves that $R\pi \cap R\pi' \subseteq R\pi\pi'$ and so, $R\pi \cap R\pi' = R\pi\pi'$. □

Corollary 2.6. *Let R be a UFD with $|Max(R)| \geq 2$. Then $\mathbb{G}(R)$ is complete.*

Proof. Assume that R is a UFD with $|Max(R)| \geq 2$. Let $R\pi, R\pi' \in \mathcal{I}(R)$ be such that $R\pi \neq R\pi'$. As any irreducible element of a UFD is a prime element by [6, Proposition 1.2.1, page 158], it follows from Lemma 2.5 that $R\pi$ and $R\pi'$ are adjacent in $\mathbb{G}(R)$ and so, $\mathbb{G}(R)$ is complete. □

Let R be an atomic domain with $|Max(R)| \geq 2$. We do not know the status of the converse of Corollary 2.6. If $J(R) = (0)$ and $\mathbb{G}(R)$ is complete, then we prove in Corollary 2.12 that R is a UFD. The following Remark 2.7 is due to the referee.

Remark 2.7. Let R be an atomic domain (R can possibly be quasi-local). Let $\mathcal{A}(R) = \{R\pi \mid \pi \in Irr(R)\}$. With $\mathcal{A}(R)$, we can associate an undirected graph denoted by $\mathbb{A}(R)$ such that $V(\mathbb{A}(R)) = \mathcal{A}(R)$ and distinct vertices $R\pi$ and $R\pi'$ are adjacent in $\mathbb{A}(R)$ if and only if $R\pi \cap R\pi' = R\pi\pi'$. Let R be a UFD. As any $\pi \in Irr(R)$ is a prime element of R, we obtain from Lemma 2.5 that $\mathbb{A}(R)$ is complete. We regard graph with a single vertex to be complete. We mention an example of a principal ideal domain (PID) R such that $|\mathcal{A}(R)| = 1$. If $R = K[[X]]$, the power series ring in one variable X over a field K, then R is a local PID and $\mathcal{A}(R) = \{RX\}$. If R is an atomic domain with $|\mathcal{A}(R)| = 1$, then we verify in the proof of Corollary 2.11 that R is a local PID. Let R be an atomic domain (R can possibly be quasi-local). We prove in Corollary 2.11 that $\mathbb{A}(R)$ is complete if and only if R is a UFD.

Let R be an atomic domain (R can possibly be quasi-local). We prove in Proposition 2.10 that the converse of Lemma 2.5 is true. We use Lemma 2.8 in the proof of Proposition 2.10. We are very much grateful to the referee for pointing out Proposition 2.10 and its proof.

First, it is useful to recall the following definition from [3] before we mention the result from [12] relevant to this article. Let D be an integral domain with quotient field K. Let A be a D-submodule of K. A is said to be a *fractional ideal* of D if there exists $d \in D\backslash\{0\}$ such that $dA \subseteq D$ [3, page 24]. Let us denote the set of all non-zero fractional ideals of D by $F(D)$. We recall that a mapping $F \to F^*$ of $F(D)$ into $F(D)$ is called a $*$*-operation on D* if the following conditions $(1)-(3)$ hold for each non-zero $a \in K$ and all $A, B \in F(D)$: (1) $(Da)^* = Da$; $(aA)^* = aA^*$, (2) $A \subseteq A^*$; if $A \subseteq B$, then $A^* \subseteq B^*$, and (3) $(A^*)^* = A^*$ [3, page 392]. For any $A \in F(D)$, recall that $(D :_K A) = \{\alpha \in K \mid \alpha A \subseteq D\}$ is denoted by A^{-1} [3, page 416]. It is not hard to verify that $A^{-1} \in F(D)$. For any $A \in F(D)$, $(A^{-1})^{-1}$ is denoted by A_v. For any $F \in F(D)$, let us denote $\{\alpha \in K \mid D\alpha \supseteq F\}$ by $\mathcal{A}$. It is known that $F_v = \bigcap_{\alpha \in \mathcal{A}} D\alpha$ [3, Theorem 34.1(1)]. The mapping $F \to F_v$ from $F(D)$ into $F(D)$ is called the *v-operation on D* [3, page 416]. It is known that the v-operation on D is a $*$-operation [3, Theorem 34.1(2)]. We recall that two non-zero elements $x, y \in D$ are called *v-coprime* if $(Dx + Dy)_v = D$ (i.e. $Dx \cap Dy = Dxy$ or equivalently $(Dx + Dy)^{-1} = D$) [12, Definition 2.1].

Lemma 2.8. *Let R be an integral domain. Let $a, b_1, \ldots, b_k \in R\backslash\{0\}$ be such that $Ra \cap Rb_i = Rab_i$ for each $i \in \{1, \ldots, k\}$. Then $Ra \cap R(\prod_{i=1}^{k} b_i) = R(a\prod_{i=1}^{k} b_i)$.*

Proof. For a proof of this lemma, one can refer the remark mentioned in [12, Definition 2.1] and [12, Proposition 2.2(2)]. $\square$

We include Remark 2.9 which we need in the proof of Lemma 2.14.

Remark 2.9. Let R be an integral domain. Let $a,b \in R\backslash\{0\}$. As $(Ra+Rb)_v = (Ra+R(a+b))_v$, it follows from the remark mentioned in [12, Definition 2.1] that $Ra \cap Rb = Rab$ if and only if $Ra \cap R(a+b) = R(a(a+b))$.

Proposition 2.10. *Let R be an atomic domain (R can possibly be quasi-local). Let $\pi \in Irr(R)$ be such that $R\pi \cap R\pi' = R\pi\pi'$ for all $\pi' \in Irr(R)$ with $R\pi \neq R\pi'$. Then π is a prime element of R.*

Proof. Assume that $\pi \in Irr(R)$ is such that $R\pi \cap R\pi' = R\pi\pi'$ for all $\pi' \in Irr(R)$ such that $R\pi \neq R\pi'$. Let $a,b \in R$ be such that π divides ab in R. We verify that either π divides a in R or π divides b in R. We can assume that both a and b are non-zero non-units of R.

Now, $ab = \pi r$ for some $r \in R$. Suppose that π does not divide a in R. Since R is an atomic domain, there exist atoms $\pi_1, \ldots, \pi_k$ of R such that $a = \prod_{i=1}^{k} \pi_i$. Hence, $(\prod_{i=1}^{k} \pi_i)b = \pi r$. Let $i \in \{1, \ldots, k\}$. As π does not divide a in R by assumption, we obtain that $R\pi \neq R\pi_i$. Hence, $R\pi \cap R\pi_i = R\pi\pi_i$. So, $R\pi \cap Ra = R\pi \cap R(\prod_{i=1}^{k} \pi_i) = R\pi a$ by Lemma 2.8. Now, from $ab = \pi r$, we get that $ab \in R\pi \cap Ra = R\pi a$ and so, $ab = \pi a s$ for some $s \in R$. Therefore, $b = \pi s$. This shows that π divides b in R. Hence, π is a prime element of R. $\square$

Corollary 2.11. *Let R be an atomic domain (R can possibly be quasi-local). Then $\mathbb{A}(R)$ is complete if and only if R is a UFD.*

Proof. By hypothesis, R is an atomic domain. Assume that $\mathbb{A}(R)$ is complete.

Suppose that $|\mathcal{A}(R)| = 1$. Let $\mathcal{A}(R) = \{R\pi\}$. Hence, if π' is any irreducible element of R, then $R\pi = R\pi'$ and so, π and π' are associates in R. If a is any non-zero non-unit of R, then $a \in R\pi$, since R is an atomic domain. Therefore, $R\pi = NU(R)$ and so, $Max(R) = \{R\pi\}$. Since any non-zero prime ideal of R contains at least one irreducible element, it follows that $Spec(R)\backslash\{(0)\} = Max(R) = \{R\pi\}$. It follows from [5, Exercise 10, page 8] that R is a PID and so, R is a local PID.

Suppose that $|\mathcal{A}(R)| \geq 2$. Let $\pi \in R$ be irreducible. Let $R\pi' \in \mathcal{A}(R)$ be such that $R\pi \neq R\pi'$. Since $\mathbb{A}(R)$ is complete by assumption, $R\pi \cap R\pi' = R\pi\pi'$. Hence, π is a prime element of R by Proposition 2.10. This shows that any $\pi \in Irr(R)$ is a prime element of R. Since any non-zero non-unit of R can be expressed as the product of a finite number of irreducible elements of R, it follows that R is a UFD by [6, Proposition 1.2.1, page 158].

Conversely, assume that R is a UFD. It is already noted in Remark 2.7 that $\mathbb{A}(R)$ is complete. $\square$

Let R be an atomic domain with $J(R) = (0)$. Then $Max(R)$ is necessarily infinite. If $\mathbb{G}(R)$ is complete, then with the help of Corollary 2.11, we deduce in Corollary 2.12 that R is a UFD.

Corollary 2.12. *Let R be an atomic domain with $J(R) = (0)$. If $\mathbb{G}(R)$ is complete, then R is a UFD.*

Proof. By hypothesis, R is an atomic domain with $J(R) = (0)$. Then it is clear that $\mathcal{A}(R) = \mathcal{I}(R)$ and $\mathbb{G}(R) = \mathbb{A}(R)$.

Assume that $\mathbb{G}(R)$ is complete. Then $\mathbb{A}(R)$ is complete and so, we obtain from Corollary 2.11 that R is a UFD. □

Let R be an atomic domain with $|Max(R)| \geq 2$. If $J(R) \in Spec(R)$, then we prove in Corollary 2.13 that $diam(\mathbb{G}(R)) \leq 2$. We use Lemma 2.2 in the proof of Corollary 2.13.

Corollary 2.13. *Let R be an atomic domain with $|Max(R)| \geq 2$. If $J(R) \in Spec(R)$, then $diam(\mathbb{G}(R)) \leq 2$. In particular if $J(R) = (0)$, then $diam(\mathbb{G}(R)) \leq 2$.*

Proof. Assume that $J(R) \in Spec(R)$. Let $R\pi_1, R\pi_2 \in \mathcal{I}(R)$ with $R\pi_1 \neq R\pi_2$. Since $\pi_i \notin J(R)$ for each $i \in \{1,2\}$, it follows that $\pi_1\pi_2 \notin J(R)$. Hence, there exists a path of length at most two between $R\pi_1$ and $R\pi_2$ in $\mathbb{CGI}(R)$ by Lemma 2.2 and so, $diam(\mathbb{CGI}(R)) \leq 2$. Therefore, $diam(\mathbb{G}(R)) \leq 2$, since $\mathbb{CGI}(R)$ is a spanning subgraph of $\mathbb{G}(R)$.

If $J(R) = (0)$, then it is clear that $J(R) \in Spec(R)$ and hence, we obtain that $diam(\mathbb{G}(R)) \leq 2$. □

Let R be an atomic domain with $|Max(R)| \geq 2$. We next try to characterize R such that $diam(\mathbb{G}(R)) = 2$. It is clear that $diam(\mathbb{G}(R)) \geq 2$ if and only if there exist distinct $R\pi, R\pi' \in \mathcal{I}(R)$ such that $R\pi$ and $R\pi'$ are not adjacent in $\mathbb{G}(R)$. That is, $R\pi \cap R\pi' \neq R\pi\pi'$. Hence, by Lemma 2.5, it follows that $diam(\mathbb{G}(R)) \geq 2$ if and only if there exist at least two distinct $R\pi, R\pi' \in \mathcal{I}(R)$ such that π and π' are not prime elements of R. In Corollary 2.15, we provide a sufficient condition on the behaviour of $\mathcal{I}(R)$ such that $diam(\mathbb{G}(R)) = 2$. We use Proposition 2.10 and Lemma 2.14 in the proof of Corollary 2.15. It is already noted in the paragraph which appears just preceding the statement of Lemma 2.2 that $|\mathcal{I}(R)| \geq 2$.

Lemma 2.14. *Let R be an atomic domain with $|Max(R)| \geq 2$. Let $R\pi \in \mathcal{I}(R)$. Then the following statements are equivalent:*
(1) *$e(R\pi) = 1$ in $\mathbb{G}(R)$.*
(2) *π is a prime element of R.*

Proof. (1) $\Rightarrow$ (2) Assume that $e(R\pi) = 1$ in $\mathbb{G}(R)$. We claim that $R\pi \cap R\pi' = R\pi\pi'$ for all $R\pi' \in \mathcal{A}(R)$ such that $R\pi \neq R\pi'$. This is clear if $R\pi' \in \mathcal{I}(R)$, since $e(R\pi) = 1$ in $\mathbb{G}(R)$. Suppose that $\pi' \in Irr(R)$ be such that $\pi' \in J(R)$. Note that $\pi + \pi' \in NU(R) \backslash J(R)$. Since R is an atomic domain, it follows that there exist $k \in \mathbb{N}$ and $\pi_1, \ldots, \pi_k \in Irr(R)$ such that $\pi + \pi' = \prod_{i=1}^{k} \pi_i$. Let $i \in \{1, \ldots, k\}$. Note that $\pi_i \notin J(R)$ and $R\pi \neq R\pi_i$. Hence, $R\pi \cap R\pi_i = R\pi\pi_i$. Hence, we obtain from Lemma 2.8 that $R\pi \cap R(\pi + \pi') = R\pi \cap R(\prod_{i=1}^{k} \pi_i) = R(\pi(\prod_{i=1}^{k} \pi_i)) = R\pi(\pi + \pi')$. Hence, we obtain from Remark 2.9 that $R\pi \cap R\pi' = R\pi\pi'$. Thus $R\pi \cap R\pi' = R\pi\pi'$ for all $R\pi' \in \mathcal{I}(R)$ such that $R\pi \neq R\pi'$. Therefore, we obtain from Proposition 2.10 that π is a prime element of R.
(2) $\Rightarrow$ (1) Assume that π is a prime element of R. Then $R\pi$ and $R\pi'$ are adjacent in $\mathbb{G}(R)$ for all $R\pi' \in \mathcal{I}(R) \backslash \{R\pi\}$ by Lemma 2.5. Hence, $e(R\pi) = 1$ in $\mathbb{G}(R)$. □

Corollary 2.15. *Let R be an atomic domain with $|Max(R)| \geq 2$. If there are elements $R\pi, R\pi' \in \mathcal{I}(R)$ such that π is a prime element of R and π' is not a prime element of R, then $diam(\mathbb{G}(R)) = 2$.*

Proof. Assume that there are $R\pi, R\pi' \in \mathcal{I}(R)$ such that π is a prime element of R and π' is not a prime element of R. Note that $e(R\pi) = 1$ in $\mathbb{G}(R)$ by (2) $\Rightarrow$ (1) of Lemma 2.14 and so, $r(\mathbb{G}(R)) = 1$. Therefore, $diam(\mathbb{G}(R)) \leq 2$. Since π' is not a prime element of R, it follows from (1) $\Rightarrow$ (2) of Lemma 2.14 that $e(R\pi') \geq 2$ in $\mathbb{G}(R)$ and so, $diam(\mathbb{G}(R)) \geq 2$. Therefore, $diam(\mathbb{G}(R)) = 2$. □

In Example 2.17(1), we provide an atomic domain R such that $diam(\mathbb{G}(R)) = 2$ and $r(\mathbb{G}(R)) = 1$ and in Example 2.17(2), we provide an atomic domain R such that $diam(\mathbb{G}(R)) = 2 = r(\mathbb{G}(R))$. We use Lemma 2.16 in the proof of Example 2.17(1) and (2).

Lemma 2.16. *Let $T = K[X]$ be the polynomial ring in one variable X over a field K and let R be the subring of T given by $R = K[X^2, X^3]$. Then R is a Noetherian domain with $dimR = 1$ and $diam(\mathbb{G}(R)) = 2$.*

Proof. Since R is a subring of an integral domain T, it follows that R is an integral domain. Note that R is Noetherian by [1, Corollary 7.7]. It is clear that $T = K[X] = R + RX$ is a finitely generated R-module. Therefore, T is integral over R. As $dimT = 1$, $dimR = 1$ by [3, 11.8]. Since $J(T) = (0)$, it follows from the lying-over theorem that $J(R) = (0)$ and so, $diam(\mathbb{G}(R)) \leq 2$ by Corollary 2.13. Since $X \notin R$, T is a UFD (indeed, a principal ideal domain), and $U(T) = U(R)$, it follows that X^2 is an irreducible element of R. Similarly, it follows that X^3 is an irreducible element of R. It is clear that X^2 and X^3 are not associates in R. Therefore, $RX^2, RX^3 \in \mathcal{I}(R)$ with $RX^2 \neq RX^3$. Note that $X^6 \in RX^2 \cap RX^3$

but $X^6 \notin RX^5$. Therefore, $RX^2 \cap RX^3 \neq RX^5$ and so, $d(RX^2, RX^3) \geq 2$ in $\mathbb{G}(R)$. This proves that $diam(\mathbb{G}(R)) \geq 2$. Hence, $diam(\mathbb{G}(R)) = 2$. $\square$

Example 2.17. (1) Let R be as in Lemma 2.16 with $K = \mathbb{R}$. Then $diam(\mathbb{G}(R)) = 2$ and $r(\mathbb{G}(R)) = 1$.
(2) Let R be as in Lemma 2.16 with $K = \mathbb{C}$. Then $diam(\mathbb{G}(R)) = 2 = r(\mathbb{G}(R))$.

Proof. (1) Note that $diam(\mathbb{G}(R)) = 2$ by Lemma 2.16. Observe that X^2+1 is irreducible over $\mathbb{R}$ and so, $\mathfrak{m} = (X^2+1)\mathbb{R}[X] \in Max(\mathbb{R}[X])$. It is not hard to verify that $\mathfrak{m} \cap R = (X^2+1)R$. Thus $(X^2+1)R \in Spec(R)$ (indeed, it belongs to $Max(R)$, since $dim R = 1$). Therefore, X^2+1 is a prime element of R. It is noted in the proof of Lemma 2.16 that $J(R) = (0)$. Hence, $R(X^2+1) \in \mathcal{I}(R)$ is such that X^2+1 is a prime element of R. Therefore, $e(R(X^2+1)) = 1$ in $\mathbb{G}(R)$ by (2) $\Rightarrow$ (1) of Lemma 2.14 and so, $r(\mathbb{G}(R)) = 1$.
(2) Observe that $diam(\mathbb{G}(R)) = 2$ by Lemma 2.16. We claim that no maximal ideal of R is principal. Let $\mathfrak{m} \in Max(R)$. Then there exists $\mathfrak{p} \in Spec(\mathbb{C}[X])$ such that $\mathfrak{p} \cap R = \mathfrak{m}$ by [1, Theorem 5.10]. As $dim\mathbb{C}[X] = 1$, $\mathfrak{p} \in Max(\mathbb{C}[X])$. Since $\mathbb{C}$ is algebraically closed, it follows that $\mathfrak{p} = (X-\alpha)\mathbb{C}[X]$ for some $\alpha \in \mathbb{C}$. If $\alpha = 0$, then $\mathfrak{m} = X^2\mathbb{C}[X]$. Since X^2, X^3 are non-associate irreducible elements belonging to $\mathfrak{m}$, it follows that $\mathfrak{m}$ is not principal. Suppose that $\alpha \neq 0$. Then as $X - \alpha \notin R$, $\mathbb{C}[X]$ is a UFD, and $U(R) = U(\mathbb{C}[X])$, it follows that $(X-\alpha)X^2, (X-\alpha)X^3$ are irreducible elements of R and it is clear that they are non-associates in R. Since $(X-\alpha)X^2, (X-\alpha)X^3 \in \mathfrak{m}$, we get that $\mathfrak{m}$ is not principal. This shows that no maximal ideal of R is principal. As any prime element of R generates a prime ideal of R and $dim R = 1$, we obtain that R has no prime element. Therefore, by (1) $\Rightarrow$ (2) of Lemma 2.14, we get that $e(R\pi) \geq 2$ in $\mathbb{G}(R)$ for each $R\pi \in \mathcal{I}(R)$ and so, $r(\mathbb{G}(R)) \geq 2$. Since $diam(\mathbb{G}(R)) = 2$, it follows that $diam(\mathbb{G}(R)) = 2 = r(\mathbb{G}(R))$. $\square$

Let R be an atomic domain with $|Max(R)| \geq 2$. It is shown in Corollary 2.4 that $\mathbb{G}(R)$ is connected and $diam(\mathbb{G}(R)) \leq 3$. We next try to characterize R such that $diam(\mathbb{G}(R)) = 3$. In Proposition 2.21, we are able to provide a sufficient condition on the behaviour of $\mathcal{I}(R)$ such that $diam(\mathbb{G}(R)) = 3$. Since $\mathbb{CGI}(R)$ is a spanning subgraph of $\mathbb{G}(R)$, it follows that $diam(\mathbb{CGI}(R)) \geq 3$ and hence by Proposition 2.3, we get that $diam(\mathbb{CGI}(R)) = 3$. This happens if and only if there exist distinct $R\pi_1, R\pi_2 \in \mathcal{I}(R)$ such that $d(R\pi_1, R\pi_2) = 3$ in $\mathbb{CGI}(R)$. In Proposition 2.18, we determine necessary and sufficient conditions in order that $diam(\mathbb{CGI}(R))$ to be equal to 3.

Proposition 2.18. *Let R be an atomic domain with $|Max(R)| \geq 2$. Then $diam(\mathbb{CGI}(R)) = 3$ if and only if there exist distinct $R\pi_1, R\pi_2 \in \mathcal{I}(R)$ such that $R\pi_1 + R\pi_2 \subseteq \mathfrak{m}$ for some $\mathfrak{m} \in Max(R)$ and $\pi_1\pi_2 \in J(R)$.*

Proof. Assume that $diam(\mathbb{CGI}(R)) = 3$. Hence, there exist distinct $R\pi_1, R\pi_2 \in \mathcal{I}(R)$ such that $d(R\pi_1, R\pi_2) = 3$ in $\mathbb{CGI}(R)$. Therefore, $R\pi_1$ and $R\pi_2$ are not adjacent in $\mathbb{CGI}(R)$. Hence, $R\pi_1 + R\pi_2 \neq R$ and so, there exists $\mathfrak{m} \in Max(R)$ such that $R\pi_1 + R\pi_2 \subseteq \mathfrak{m}$ by [1, Corollary 1.4]. It follows from Lemma 2.2 that $\pi_1\pi_2 \in J(R)$.

Conversely, assume that there exist distinct $R\pi_1, R\pi_2 \in \mathcal{I}(R)$ such that $R\pi_1 + R\pi_2 \subseteq \mathfrak{m}$ for some $\mathfrak{m} \in Max(R)$ and $\pi_1\pi_2 \in J(R)$. It is clear that $R\pi_1$ and $R\pi_2$ are not adjacent in $\mathbb{CGI}(R)$. We claim that there exists no path of length two between $R\pi_1$ and $R\pi_2$ in $\mathbb{CGI}(R)$. Let $R\pi_3 \in \mathcal{I}(R)$ with $R\pi_3 \neq R\pi_1$ be such that $R\pi_1$ and $R\pi_3$ are adjacent in $\mathbb{CGI}(R)$. Hence, $R\pi_1 + R\pi_3 = R$. This implies that $R\pi_1\pi_2 + R\pi_3\pi_2 = R\pi_2$. Let $\mathfrak{m}' \in Max(R)$ be such that $\pi_3 \in \mathfrak{m}'$. As $\pi_1\pi_2, \pi_3 \in \mathfrak{m}'$, we get that $\pi_2 \in \mathfrak{m}'$. Therefore, $R\pi_3 + R\pi_2 \neq R$ and so, $R\pi_3$ and $R\pi_2$ are not adjacent in $\mathbb{CGI}(R)$. This shows that there exists no path of length two between $R\pi_1$ and $R\pi_2$ in $\mathbb{CGI}(R)$ and so, $d(R\pi_1, R\pi_2) \geq 3$ in $\mathbb{CGI}(R)$. Hence, $diam(\mathbb{CGI}(R)) = 3$ by Proposition 2.3. □

In Example 2.19, we provide a UFD R with $|Max(R)| = 3$ to illustrate Proposition 2.18.

Example 2.19. Let $T = \mathbb{Z}[X]$ be the polynomial ring in one variable X over $\mathbb{Z}$. Let $\mathfrak{m}_1 = T2 + TX, \mathfrak{m}_2 = T2 + T(X-1)$, and $\mathfrak{m}_3 = T3 + TX$. Let $S = T \backslash (\bigcup_{i=1}^{3} \mathfrak{m}_i)$ and let $R = S^{-1}T$. Then R is a UFD, $|Max(R)| = 3$, $diam(\mathbb{CGI}(R))$ $= 3$, and $diam(\mathbb{G}(R)) = 1$.

Proof. It is clear that $\mathfrak{m}_i \in Max(T)$ for all $i \in \{1,2,3\}$ and $\mathfrak{m}_i \neq \mathfrak{m}_j$ for all distinct $i, j \in \{1,2,3\}$. Observe that $T \backslash (\bigcup_{i=1}^{3} \mathfrak{m}_i)$ is a multiplicatively closed subset (m.c. subset) of T. Since T is UFD, it follows from [5, Theorem 5] and [1, Proposition 3.11(iv)] that $R = S^{-1}T$ is a UFD. It follows from [1, Proposition 3.11(iv)] and [1, Proposition 1.11(i)] that $Max(R) = \{S^{-1}\mathfrak{m}_i \mid i \in \{1,2,3\}\}$. It is clear that $2, X$ are prime elements of R, they are non-associate in R, they do not belong to $J(R)$. and $2X \in J(R)$. Note that $R2, RX \in \mathcal{I}(R)$. Observe that $R2 + RX = S^{-1}\mathfrak{m}_1$ and $2X \in J(R)$. Hence, we obtain from Proposition 2.18 that $diam(\mathbb{CGI}(R)) = 3$. Since R is a UFD, it follows from Corollary 2.6 that $diam(\mathbb{G}(R)) = 1$. □

For an atomic domain R with $|Max(R)| \geq 2$, in Proposition 2.21, we provide a sufficient condition on the behaviour of $\mathcal{I}(R)$ such that $diam(\mathbb{G}(R)) = 3$. First, it is useful to recall the following definitions and results from multiplicative ideal theory.

Let $*$ be a star operation on an integral domain D with quotient field K. Recall that with $*$, we can associate $*_f$ defined by $A^{*_f} = \bigcup_{F \in \mathcal{B}} F^*$, where $\mathcal{B} = \{F \mid F \text{is a non-zero finitely generated } D\text{-submodule of } A\}$ [12, page 387]. It is not hard to verify that $*_f$ is a star operation on D. The well-known t-operation on D is given by $t = v_f$. A star operation $*$ on D is said to be of *finite type* if $A^* = A^{*_f}$ for each $A \in F(D)$. Observe that for any star operation $*$ on D, $*_f$ is of finite type. If $*$ is a star operation on D, then $A \in F(D)$ is said to be a *$*$-ideal* if $A = A^*$. The ideals of D are referred to as *integral ideals*. If $*$ is a finite type star operation on D, then a proper integral ideal of D that is maximal with respect to being a $*$-ideal is called a *maximal $*$-ideal* and is necessarily a prime ideal of D. Moreover, every proper $*$-ideal is contained in at least one maximal $*$-ideal. As the t-operation on D is a star operation of finite type, it follows that any proper t-ideal of D is contained in at least one maximal t-ideal [12, pages 387-388]. We denote the set of all maximal t- ideals of D by $t - Max(D)$.

Let R be an atomic domain with $|Max(R)| \geq 2$. Let $R\pi_1, R\pi_2 \in \mathcal{I}(R)$ be distinct. As suggested by the referee, we include Proposition 2.20 which describes when $R\pi_1$ and $R\pi_2$ are adjacent in $\mathbb{G}(R)$, in terms of the v-operation on R.

Proposition 2.20. *Let R be an atomic domain with $|Max(R)| \geq 2$. Let $R\pi_1, R\pi_2 \in \mathcal{I}(R)$ be distinct. Then $R\pi_1$ and $R\pi_2$ are adjacent in $\mathbb{G}(R)$ if and only if $(R\pi_1 + R\pi_2)_v = R$.*

Proof. We include a proof of this proposition for the sake of completeness. Note that for any non-zero ideal A of R, $A^{-1} = (A_v)^{-1}$ and so, $A^{-1} = R$ if and only if $A_v = R$. Let $R\pi_1, R\pi_2 \in \mathcal{I}(R)$ be distinct. Let us denote the ideal $R\pi_1 + R\pi_2$ by I. Note that $I^{-1} = \frac{1}{\pi_1\pi_2}(R\pi_1 \cap R\pi_2)$. Observe that $R\pi_1$ and $R\pi_2$ are adjacent in $\mathbb{G}(R)$ if and only if $R\pi_1 \cap R\pi_2 = R\pi_1\pi_2$ if and only if $\frac{1}{\pi_1\pi_2}(R\pi_1 \cap R\pi_2) = R$ if and only if $I^{-1} = R$ if and only if $I_v = R$. This shows that $R\pi_1$ and $R\pi_2$ are adjacent in $\mathbb{G}(R)$ if and only if $(R\pi_1 + R\pi_2)_v = R$. □

Proposition 2.21. *Let R be an atomic domain with $|Max(R)| \geq 2$. If there exist distinct $R\pi_1, R\pi_2 \in \mathcal{I}(R)$ such that $R\pi_1 + R\pi_2 \subseteq \mathfrak{p}$ for some maximal t-ideal $\mathfrak{p}$ of R and $\pi_1\pi_2$ belongs to every maximal t-ideal of R, then $diam(\mathbb{G}(R)) = 3$.*

Proof. Assume that there exist distinct $R\pi_1, R\pi_2 \in \mathcal{I}(R)$ such that $R\pi_1 + R\pi_2 \subseteq \mathfrak{p}$, where $\mathfrak{p}$ is a maximal t-ideal of R and $\pi_1\pi_2$ belongs to every maximal t-ideal of R. Observe that $(R\pi_1 + R\pi_2)_v = (R\pi_1 + R\pi_2)_t \subseteq \mathfrak{p}_t = \mathfrak{p}$. Therefore, $(R\pi_1 + R\pi_2)_v \neq R$ and so, it follows from Proposition 2.20 that $R\pi_1$ and $R\pi_2$ are not adjacent in $\mathbb{G}(R)$. We claim that there exists no path of length two between $R\pi_1$ and $R\pi_2$ in $\mathbb{G}(R)$. Suppose that there exists a path of length two between $R\pi_1$ and $R\pi_2$ in $\mathbb{G}(R)$. Let $R\pi \in \mathcal{I}(R)$ be such that $R\pi_1 - R\pi - R\pi_2$ is a path of length two between $R\pi_1$ and $R\pi_2$ in $\mathbb{G}(R)$. Hence, we obtain that $(R\pi_1 + R\pi)_v = R = (R\pi_2 + R\pi)_v$. Note that $(R\pi)_v = (R\pi)_t = R\pi$ and so, there exists a maximal t-ideal $\mathfrak{q}$ of R such that $R\pi \subseteq \mathfrak{q}$. By assumption, $\pi_1\pi_2 \in \mathfrak{q}$ and as any maximal t-ideal of R is a prime ideal of R, we obtain that $\pi_i \in \mathfrak{q}$

for some $i \in \{1,2\}$. In such a case, $(R\pi_i + R\pi)_v = (R\pi_i + R\pi)_t \subseteq \mathfrak{q}_t = \mathfrak{q}$. This is impossible, since $(R\pi_i + R\pi)_v = R$. This proves that there exists no path of length two between $R\pi_1$ and $R\pi_2$ in $\mathbb{G}(R)$. Hence, $d(R\pi_1, R\pi_2) \geq 3$ in $\mathbb{G}(R)$ and as $diam(\mathbb{G}(R)) \leq 3$ by Corollary 2.4, we obtain that $diam(\mathbb{G}(R)) = 3$. $\square$

We are not able to provide an atomic domain R such that $diam(\mathbb{G}(R)) = 3$ and we are not able to decide whether the sufficient condition mentioned in Proposition 2.21 is also necessary.

3. Some results on the girth of $\mathbb{G}(R)$

Let R be an atomic domain with $|Max(R)| \geq 2$. The aim of this section is to discuss some results on the girth of $\mathbb{G}(R)$ and some related results. If $|Max(R)| \geq 3$, then we prove in Proposition 3.1 that $gr(\mathbb{CGI}(R)) = 3$ and deduce in Corollary 3.2 that $gr(\mathbb{G}(R)) = 3$.

Proposition 3.1. *Let R be an atomic domain. If $|Max(R)| \geq 3$, then $gr(\mathbb{CGI}(R))$ $= 3$.*

Proof. Assume that R is an atomic domain with $|Max(R)| \geq 3$. It is already noted in the paragraph which appears just preceding the statement of Lemma 2.2 that there exist $R\pi_1, R\pi_2 \in \mathcal{I}(R)$ such that $R\pi_1 + R\pi_2 = R$. Let $\mathfrak{m}_i \in Max(R)$ be such that $\pi_i \in \mathfrak{m}_i$ for each $i \in \{1,2\}$. It is clear that $\mathfrak{m}_1 \neq \mathfrak{m}_2$. Since $|Max(R)| \geq 3$, there exists $\mathfrak{m}_3 \in Max(R)$ such that $\mathfrak{m}_3 \neq \mathfrak{m}_i$ for each $i \in \{1,2\}$. If $\pi_1\pi_2 \notin \mathfrak{m}_3$, then $R\pi_1\pi_2 + \mathfrak{m}_3 = R$. It can be shown as in the proof of Lemma 2.2 that there exists $R\pi_3 \in \mathcal{I}(R)$ such that $\pi_3 \in \mathfrak{m}_3$ and $R\pi_1\pi_2 + R\pi_3 = R$. It is then clear that $R\pi_1 - R\pi_2 - R\pi_3 - R\pi_1$ is a cycle of length three in $\mathbb{CGI}(R)$. Suppose that $\pi_1\pi_2 \in \mathfrak{m}_3$. As $R\pi_1 + R\pi_2 = R$, it follows that exactly one between π_1 and π_2 can belong to $\mathfrak{m}_3$. Without loss of generality, we can assume that $\pi_1 \in \mathfrak{m}_3$. Then $\mathfrak{m}_1\pi_2 \not\subseteq \mathfrak{m}_3$ and so, $\mathfrak{m}_1\pi_2 + \mathfrak{m}_3 = R$. There exist $a_1 \in \mathfrak{m}_1, a_3 \in \mathfrak{m}_3$ such that $Ra_1\pi_2 + Ra_3 = R$. It is clear that $a_1, a_3 \notin J(R)$. Since R is atomic, there exist $R\pi_1' \in \mathcal{I}(R)$ and $R\pi_3 \in \mathcal{I}(R)$, $\pi_1' \in \mathfrak{m}_1$, $\pi_3 \in \mathfrak{m}_3$ such that π_1' is a divisor of a_1 and π_3 is a divisor of a_3 in R. It is now clear that $R\pi_1' + R\pi_3 = R = R\pi_2 + R\pi_3$. Observe that $\pi_2\pi_3 \notin \mathfrak{m}_1$. Hence, $R\pi_2\pi_3 + \mathfrak{m}_1 = R$ and so, there exists $R\pi_1'' \in \mathcal{I}(R)$ such that $\pi'' \in \mathfrak{m}_1$ and $R\pi_2\pi_3 + R\pi_1'' = R$. Note that $R\pi_1'' - R\pi_2 - R\pi_3 - R\pi_1''$ is a cycle of length three in $\mathbb{CGI}(R)$.

This proves that $gr(\mathbb{CGI}(R)) = 3$. $\square$

Corollary 3.2. *Let R be an atomic domain with $|Max(R)| \geq 3$. Then $gr(\mathbb{G}(R)) =$ 3.*

Proof. Assume that R is an atomic domain with $|Max(R)| \geq 3$. Since $gr(\mathbb{CGI}(R)) = 3$ by Proposition 3.1 and $\mathbb{CGI}(R)$ is a spanning subgraph of $\mathbb{G}(R)$, it follows that $gr(\mathbb{G}(R)) = 3$. $\square$

Let R be an atomic domain with $|Max(R)| \geq 2$ and suppose that $gr(\mathbb{G}(R)) \neq 3$. It then follows from Corollary 3.2 that $|Max(R)| = 2$. With the assumption that $|Max(R)| = 2$, we next try to determine $gr(\mathbb{G}(R))$. If $\mathbb{G}(R)$ contains a cycle and $gr(\mathbb{G}(R)) \neq 3$, then we prove in Proposition 3.8 that $gr(\mathbb{G}(R)) = 4$ and in this case, we verify that $\mathbb{CGI}(R) = \mathbb{G}(R)$ is a complete bipartite graph but not a star graph.

Let R be an atomic domain with $|Max(R)| \geq 2$. In Proposition 3.5, we determine necessary and sufficient conditions in order that $\mathbb{G}(R)$ to be a complete bipartite graph. We use Lemmas 3.3 and 3.4 in the proof of Proposition 3.5.

Lemma 3.3. *Let $G = (V,E)$ be a graph and let H be a spanning subgraph of G such that H is a complete bipartite graph. If G is bipartite, then $H = G$.*

Proof. Assume that G is a bipartite graph with vertex partition $V = V_1 \cup V_2$ and its spanning subgraph H is a complete bipartite graph with vertex partition $V = W_1 \cup W_2$. Let $x \in V_1$. It follows from $V = W_1 \cup W_2$ and $W_1 \cap W_2 = \emptyset$ that x is in exactly one between W_1 and W_2. Without loss of generality, we can assume that $x \in W_1$. Let $x' \in V_1$ be such that $x' \neq x$. Now, x and x' are not adjacent in G and so, they are not adjacent in H. As any element of W_1 is adjacent to every element of W_2 in H, it follows that $x' \in W_1$. This shows that $V_1 \subseteq W_1$. Let $y \in W_2 \subset V = V_1 \cup V_2$. As $V_1 \subseteq W_1$ and $W_1 \cap W_2 = \emptyset$, it follows that $y \in V_2$. This shows that $W_2 \subseteq V_2$. Let $a \in W_1$. Fix an element $b \in W_2$. Since a and b are adjacent in H, we obtain that a and b are adjacent in G. As $b \in V_2$, we get that $a \in V_1$. This proves that $W_1 \subseteq V_1$ and so, it follows that $V_1 = W_1$. Let $y \in V_2 \subset V = W_1 \cup W_2$ and from $V_1 = W_1$, we obtain that $y \in W_2$. Therefore, $V_2 \subseteq W_2$ and so, $V_2 = W_2$. Since H is a subgraph of G, it is clear that $E(H) \subseteq E(G)$. Let $x - y$ be an edge of G. We can assume without loss of generality that $x \in V_1 = W_1$ and $y \in V_2 = W_2$. As H is a complete bipartite graph with vertex partition $V = W_1 \cup W_2$, it follows that $x - y$ is an edge of H. This shows that $E(G) \subseteq E(H)$ and so, $E(G) = E(H)$. Therefore, we obtain that $H = G$. $\square$

Lemma 3.4. *Let R be an atomic domain with $|Max(R)| = 2$. Let $Max(R) = \{\mathfrak{m}_i \mid i \in \{1,2\}\}$. Then $\mathbb{CGI}(R)$ is a complete bipartite graph with vertex partition $\mathcal{I}(R) = V_1 \cup V_2$, where $V_1 = \{R\pi \in \mathcal{I}(R) \mid \pi \in \mathfrak{m}_1 \backslash \mathfrak{m}_2\}$ and $V_2 = \{R\pi' \in \mathcal{I}(R) \mid \pi' \in \mathfrak{m}_2 \backslash \mathfrak{m}_1\}$.*

Proof. Assume that $Max(R) = \{\mathfrak{m}_i \mid i \in \{1,2\}\}$. It is clear that $V(\mathbb{CGI}(R)) = \mathcal{I}(R) = V_1 \cup V_2$, where $V_1 = \{R\pi \in \mathcal{I}(R) \mid \pi \in \mathfrak{m}_1 \backslash \mathfrak{m}_2\}$ and $V_2 = \{R\pi' \in \mathcal{I}(R) \mid$

$\pi' \in \mathfrak{m}_2 \backslash \mathfrak{m}_1\}$. Note that no two vertices of V_i are adjacent in $\mathbb{CGI}(R)$ for each $i \in \{1,2\}$. If $R\pi \in V_1$ and $R\pi' \in V_2$, then $R\pi + R\pi' = R$ and so, $R\pi$ and $R\pi'$ are adjacent in $\mathbb{CGI}(R)$. Therefore, $\mathbb{CGI}(R)$ is a complete bipartite graph with vertex partition $\mathcal{I}(R) = V_1 \cup V_2$, where $V_1 = \{R\pi \in \mathcal{I}(R) \mid \pi \in \mathfrak{m}_1 \backslash \mathfrak{m}_2\}$ and $V_2 = \{R\pi' \in \mathcal{I}(R) \mid \pi' \in \mathfrak{m}_2 \backslash \mathfrak{m}_1\}$. □

Proposition 3.5. *Let R be an atomic domain with $|Max(R)| \geq 2$. The following statements are equivalent:*
(1) $\mathbb{G}(R)$ *is a complete bipartite graph.*
(2) $\mathbb{G}(R)$ *is a bipartite graph.*
(3) $|Max(R)| = 2$ *and* $\mathbb{G}(R)$ *is a bipartite graph.*
(4) $\mathbb{CGI}(R) = \mathbb{G}(R)$ *is a complete bipartite graph with vertex partition* $\mathcal{I}(R) = V_1 \cup V_2$*, where* $Max(R) = \{\mathfrak{m}_i \mid i \in \{1,2\}\}$*,* $V_1 = \{R\pi \in \mathcal{I}(R) \mid \pi \in \mathfrak{m}_1 \backslash \mathfrak{m}_2\}$*, and* $V_2 = \{R\pi' \in \mathcal{I}(R) \mid \pi' \in \mathfrak{m}_2 \backslash \mathfrak{m}_1\}$*.*

Proof. (1) $\Rightarrow$ (2) Assume that $\mathbb{G}(R)$ is a complete bipartite graph. It is then clear that $\mathbb{G}(R)$ is a bipartite graph.
(2) $\Rightarrow$ (3) Assume that $\mathbb{G}(R)$ is a bipartite graph. As a bipartite graph does not contain any cycle of odd length by [2, Theorem 1.5.10], it follows from Corollary 3.2 that $|Max(R)| = 2$.
(3) $\Rightarrow$ (4) Assume that $\mathbb{G}(R)$ is a bipartite graph and $|Max(R)| = 2$. Let $Max(R) = \{\mathfrak{m}_i \mid i \in \{1,2\}\}$. We know from Lemma 3.4 that $\mathbb{CGI}(R)$ is a complete bipartite graph with vertex partition $\mathcal{I}(R) = V_1 \cup V_2$, where $V_1 = \{R\pi \in \mathcal{I}(R) \mid \pi \in \mathfrak{m}_1 \backslash \mathfrak{m}_2\}$ and $V_2 = \{R\pi' \in \mathcal{I}(R) \mid \pi' \in \mathfrak{m}_2 \backslash \mathfrak{m}_1\}$. Since $\mathbb{CGI}(R)$ is a spanning subgraph of $\mathbb{G}(R)$, it follows from Lemma 3.3 that $\mathbb{CGI}(R) = \mathbb{G}(R)$. Therefore, $\mathbb{CGI}(R) = \mathbb{G}(R)$ is a complete bipartite graph with vertex partition $\mathcal{I}(R) = V_1 \cup V_2$, where $V_1 = \{R\pi \in \mathcal{I}(R) \mid \pi \in \mathfrak{m}_1 \backslash \mathfrak{m}_2\}$ and $V_2 = \{R\pi' \in \mathcal{I}(R) \mid \pi' \in \mathfrak{m}_2 \backslash \mathfrak{m}_1\}$.
(4) $\Rightarrow$ (1) This is clear. □

Let R be an atomic domain with $|Max(R)| \geq 2$. In Proposition 3.7, we determine necessary and sufficient conditions in order that $\mathbb{G}(R)$ to be a star graph. We use Lemma 3.6 in the proof of Proposition 3.7.

Lemma 3.6. *Let R be an atomic domain with $|Max(R)| = 2$. Let $Max(R) = \{\mathfrak{m}_i \mid i \in \{1,2\}\}$. Let V_1, V_2 be as in the statement of Lemma 3.4. If $|V_i| = 1$ for some $i \in \{1,2\}$, then $\mathfrak{m}_i$ is principal.*

Proof. Assume that $Max(R) = \{\mathfrak{m}_i \mid i \in \{1,2\}\}$. Let $V_1 = \{R\pi \in \mathcal{I}(R) \mid \pi \in \mathfrak{m}_1 \backslash \mathfrak{m}_2\}$ and $V_2 = \{R\pi' \in \mathcal{I}(R) \mid \pi' \in \mathfrak{m}_2 \backslash \mathfrak{m}_1\}$. Note that $\mathcal{I}(R) = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$. Suppose that $|V_1| = 1$. Let $V_1 = \{R\pi\}$. Note that $R\pi \subseteq \mathfrak{m}_1$. We claim that $\mathfrak{m}_1 = R\pi$. Let $a \in \mathfrak{m}_1 \backslash \mathfrak{m}_2$. Since R is an atomic domain, there exist $k \in \mathbb{N}$ and $\pi_1, \ldots, \pi_k \in Irr(R)$ such that $a = \prod_{i=1}^{k} \pi_i$. From the choice of a, it is clear that $\pi_j \notin \mathfrak{m}_2$ for each $j \in \{1, \ldots, k\}$ and $\pi_i \in \mathfrak{m}_1$ for at least one $i \in \{1, \ldots, k\}$.

Hence, $R\pi_i \in \mathcal{I}(R)$ is such that $\pi_i \in \mathfrak{m}_1 \backslash \mathfrak{m}_2$. Hence, $R\pi_i \in V_1 = \{R\pi\}$ and so, $R\pi_i = R\pi$. Therefore, $a \in R\pi$. This shows that $\mathfrak{m}_1 \subseteq \mathfrak{m}_2 \cup R\pi$. Hence, $\mathfrak{m}_1 \subseteq R\pi$ and so, $\mathfrak{m}_1 = R\pi$ is principal. Similarly, if $|V_2| = 1$, then it can be shown that $\mathfrak{m}_2$ is principal. □

Proposition 3.7. *Let R be an atomic domain with $|Max(R)| \geq 2$. The following statements are equivalent:*
(1) $\mathbb{G}(R)$ *is a star graph.*
(2) $\mathbb{G}(R)$ *is a bipartite graph, $|Max(R)| = 2$, and at least one maximal ideal of R is principal.*
(3) $\mathbb{CGI}(R) = \mathbb{G}(R)$ *is a star graph.*

Proof. (1) ⇒ (2) Assume that $\mathbb{G}(R)$ is a star graph. Then it is clear that $\mathbb{G}(R)$ is a complete bipartite graph and so, we obtain from (1) ⇒ (4) of Proposition 3.5 that $\mathbb{CGI}(R) = \mathbb{G}(R)$ is a complete bipartite graph with vertex partition $\mathcal{I}(R) = V_1 \cup V_2$, where $Max(R) = \{\mathfrak{m}_i \mid i \in \{1,2\}\}$, $V_1 = \{R\pi \in \mathcal{I}(R) \mid \pi \in \mathfrak{m}_1 \backslash \mathfrak{m}_2\}$, and $V_2 = \{R\pi' \in \mathcal{I}(R) \mid \pi' \in \mathfrak{m}_2 \backslash \mathfrak{m}_1\}$. Since $\mathbb{G}(R)$ is a star graph by assumption, it follows that $|V_i| = 1$ for at least one $i \in \{1,2\}$. Without loss of generality, we can assume that $|V_1| = 1$. Then, we obtain from Lemma 3.6 that $\mathfrak{m}_1$ is principal.
(2) ⇒ (3) Assume that $\mathbb{G}(R)$ is a bipartite graph, $|Max(R)| = 2$, and at least one maximal ideal of R is principal. Let $Max(R) = \{\mathfrak{m}_i \mid i \in \{1,2\}\}$. Without loss of generality, we can assume that $\mathfrak{m}_1$ is principal. It is already noted in the proof of (3) ⇒ (4) of Proposition 3.5 that $\mathbb{CGI}(R) = \mathbb{G}(R)$ is a complete bipartite graph with vertex partition $\mathcal{I}(R) = \bigcup_{i=1}^{2} V_i$, where $V_1 = \{R\pi \in \mathcal{I}(R) \mid \pi \in \mathfrak{m}_1 \backslash \mathfrak{m}_2\}$ and $V_2 = \{R\pi' \in \mathcal{I}(R) \mid \pi' \in \mathfrak{m}_2 \backslash \mathfrak{m}_1\}$. Let $R\pi_1 \in \mathcal{I}(R)$ be such that $\mathfrak{m}_1 = R\pi_1$. If $R\pi \in V_1$, then $\pi = r\pi_1$ for some $r \in R$. It is clear that $r \in U(R)$. Hence, $R\pi = R\pi_1$ and so, $V_1 = \{R\pi_1\}$. Therefore, $\mathbb{CGI}(R) = \mathbb{G}(R)$ is a star graph.
(3) ⇒ (1) This is clear. □

Proposition 3.8. *Let R be an atomic domain with $|Max(R)| = 2$ such that $\mathbb{G}(R)$ contains a cycle. If $gr(\mathbb{G}(R)) \neq 3$, then $gr(\mathbb{G}(R)) = 4$. Moreover, $\mathbb{CGI}(R) = \mathbb{G}(R)$ is a complete bipartite graph but not a star graph.*

Proof. Let $Max(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$. We know from Lemma 3.4 that $\mathbb{CGI}(R)$ is a complete bipartite graph with vertex partition $\mathcal{I}(R) = V(\mathbb{CGI}(R)) = V_1 \cup V_2$, where $V_1 = \{R\pi \in \mathcal{I}(R) \mid \pi \in \mathfrak{m}_1 \backslash \mathfrak{m}_2\}$ and $V_2 = \{R\pi' \in \mathcal{I}(R) \mid \pi' \in \mathfrak{m}_2 \backslash \mathfrak{m}_1\}$. Since $\mathbb{CGI}(R)$ is a spanning subgraph of $\mathbb{G}(R)$, it follows that each element of V_1 is adjacent to every element of V_2 in $\mathbb{G}(R)$. We claim that $|V_i| \geq 2$ for each $i \in \{1,2\}$. By hypothesis, $\mathbb{G}(R)$ contains a cycle. Hence, $|V_i| \geq 2$ for at least one $i \in \{1,2\}$. Without loss of generality, we can assume that $|V_2| \geq 2$. Suppose that $|V_1| = 1$. If no two elements of V_2 are adjacent in $\mathbb{G}(R)$, then we get that $\mathbb{G}(R)$ does not contain any cycle. This contradicts the assumption $\mathbb{G}(R)$

contains a cycle. Therefore, there exist $R\pi_1', R\pi_2' \in V_2$ such that $R\pi_1'$ and $R\pi_2'$ are adjacent in $\mathbb{G}(R)$. Let $V_1 = \{R\pi\}$. Observe that $R\pi - R\pi_1' - R\pi_2' - R\pi$ is a cycle of length 3 in $\mathbb{G}(R)$ and this contradicts the assumption $gr(\mathbb{G}(R)) \neq 3$. Therefore, we obtain that $|V_i| \geq 2$ for each $i \in \{1,2\}$. Let $\{R\pi_1, R\pi_2\} \subseteq V_1$ and let $\{R\xi_1, R\xi_2\} \subseteq V_2$. Observe that $R\pi_1 - R\xi_1 - R\pi_2 - R\xi_2 - R\pi_1$ is a cycle of length 4 in $\mathbb{CGI}(R)$ and hence, it is a cycle of length 4 in $\mathbb{G}(R)$. Therefore, we get that $gr(\mathbb{G}(R)) = 4$.

As $gr(\mathbb{G}(R)) \neq 3$, it follows that no two members of V_i are adjacent in $\mathbb{G}(R)$ for each $i \in \{1,2\}$ and so, we obtain that $\mathbb{CGI}(R) = \mathbb{G}(R)$ is a complete bipartite graph but not a star graph. □

The proof of Corollary 3.9 follows immediately from the results proved so far in this section. Yet for the sake of completeness, we include a proof of Corollary 3.9.

Corollary 3.9. *Let R be an atomic domain with $|Max(R)| \geq 2$. Then $gr(\mathbb{G}(R)) \in \{3,4,\infty\}$. And $gr(\mathbb{G}(R)) = \infty$ if and only if $|Max(R)| = 2$ with at least one of the maximal ideals of R is principal and in the case $gr(\mathbb{G}(R)) = \infty$, $\mathbb{G}(R)$ is a star graph.*

Proof. Suppose that $|Max(R)| \geq 3$. Then we obtain from Corollary 3.2 that $gr(\mathbb{G}(R)) = 3$. Suppose that $|Max(R)| = 2$. Assume that $\mathbb{G}(R)$ contains a cycle and $gr(\mathbb{G}(R)) \neq 3$. Then it follows from Proposition 3.8 that $gr(\mathbb{G}(R)) = 4$ and moreover, in this case, $\mathbb{CGI}(R) = \mathbb{G}(R)$ is a complete bipartite graph but not a star graph. Assume that $\mathbb{G}(R)$ does not contain any cycle. Then $gr(\mathbb{G}(R)) = \infty$ and by [2, Theorem 1.5.10], we get that $\mathbb{G}(R)$ is a bipartite graph and hence, $\mathbb{CGI}(R) = \mathbb{G}(R)$ is a complete bipartite graph by (3) $\Rightarrow$ (4) of Proposition 3.5. As $\mathbb{G}(R)$ does not contain any cycle by assumption, it follows that $\mathbb{CGI}(R) = \mathbb{G}(R)$ is a star graph. It follows from (3) $\Rightarrow$ (2) of Proposition 3.7 that at least one maximal ideal of R is principal. □

We provide Examples 3.11 and 3.12 to illustrate the results proved in this section. We use Lemma 3.10 in the verification of Example 3.11(3) and Example 3.12. We are thankful to the referee for the following version of Lemma 3.10 and its proof.

Lemma 3.10. *Let R be an atomic domain with $|Max(R)| \geq 2$ and $Max(R) = t - Max(R)$. Then $\mathbb{CGI}(R) = \mathbb{G}(R)$. In particular, if $dimR = 1$, then $\mathbb{CGI}(R) = \mathbb{G}(R)$.*

Proof. Assume that R is an atomic domain, $|Max(R)| \geq 2$, and $Max(R) = t - Max(R)$. For any atomic domain T with $|Max(T)| \geq 2$, $\mathbb{CGI}(T)$ is a spanning subgraph of $\mathbb{G}(T)$ and so, $\mathbb{CGI}(R)$ is a spanning subgraph of $\mathbb{G}(R)$. Let

$R\pi, R\pi' \in \mathcal{I}(R)$ be such that $R\pi$ and $R\pi'$ are adjacent in $\mathbb{G}(R)$. Then $(R\pi + R\pi')_v = R$ by Proposition 2.20. As $t = v_f$, it follows that $(R\pi + R\pi')_t = R$. Since $Max(R) = t - Max(R)$ by hypothesis, we obtain that $R\pi + R\pi' = R$. Hence, $R\pi$ and $R\pi'$ are adjacent in $\mathbb{CGI}(R)$. This shows that $\mathbb{G}(R)$ is a spanning subgraph of $\mathbb{CGI}(R)$ and so, $\mathbb{CGI}(R) = \mathbb{G}(R)$.

Assume that $dimR = 1$. Let $\mathfrak{p} \in t - Max(R)$. Then $\mathfrak{p} \in Spec(R)\backslash\{(0)\} = Max(R)$. Hence, $t - Max(R) \subseteq Max(R)$. Let $\mathfrak{m} \in Max(R)$. Let $m \in \mathfrak{m}\backslash\{0\}$. Then Rm is a t-ideal of R and $\mathfrak{m}$ is minimal over Rm. Therefore, $\mathfrak{m}$ is a t-ideal of R and so, $\mathfrak{m} \in t - Max(R)$. Therefore, $Max(R) \subseteq t - Max(R)$ and hence, $Max(R) = t - Max(R)$. It follows from the proof given in the previous paragraph that $\mathbb{CGI}(R) = \mathbb{G}(R)$. □

Let R be a ring and $\mathfrak{p} \in Spec(R)$. Recall that the *height* of $\mathfrak{p}$ is defined to be the supermum of the length of chains of prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{p}$ which end at $\mathfrak{p}$ [1, page 120]. Let R be a Noetherian domain. If $\mathfrak{p} \in Spec(R)$ is such that the height of $\mathfrak{p}$ is greater than equal to 2, then $\mathfrak{p}$ cannot be principal by Krull's principal ideal theorem [1, Corollary 11.17].

Example 3.11. (1) Consider $T = K[X]$, the polynomial ring in one variable X over a field K. Let $R = K + X^2K[X]$. Then $gr(\mathbb{G}(R)) = 3$.
(2) Let $K = \mathbb{R}$ and let R be as in (1). Let $A = R[Y]$ be the polynomial ring in one variable Y over R. Let $\mathfrak{m}_1 = A(1+X^2) + AY$ and $\mathfrak{m}_2 = A(X^2\mathbb{R}[X]) + A(Y-1)$. Let $B = S^{-1}A$, where $S = A\backslash(\mathfrak{m}_1 \cup \mathfrak{m}_2)$. Then $gr(\mathbb{G}(B)) = 3$, whereas $\mathbb{CGI}(B)$ is a complete bipartite graph but not a star graph.
(3) Let $K = \mathbb{R}$ and let R be as in (1). Let $\mathfrak{m}_1 = (1+X^2)\mathbb{R}[X] \cap R$ and $\mathfrak{m}_2 = X^2\mathbb{R}[X]$. Let $R_1 = S^{-1}R$, where $S = R\backslash(\mathfrak{m}_1 \cup \mathfrak{m}_2)$. Then $gr(\mathbb{G}(R_1)) = \infty$.

Proof. (1) It is already noted in the proof of Lemma 2.16 that R is Noetherian, $dimR = 1$, and $J(R) = (0)$. Hence, R has an infinite number of maximal ideals. Therefore, $gr(\mathbb{G}(R)) = 3$ by Corollary 3.2.
(2) It is already noted in the proof of Example 2.17(1) that $1+X^2$ is a prime element of R. Therefore, $1+X^2$ is a prime element of $A = R[Y]$. Observe that $X^2\mathbb{R}[X] \in Max(R)$. It is clear that $\{\mathfrak{m}_1 = A(1+X^2) + AY, \mathfrak{m}_2 = A(X^2\mathbb{R}[X]) + A(Y-1)\} \subseteq Max(A)$. Note that $S = A\backslash(\mathfrak{m}_1 \cup \mathfrak{m}_2)$ is a multiplicatively closed subset of A. Since R is Noetherian, it follows from Hilbert's Basis Theorem [1, Theorem 7.5] that $A = R[Y]$ is Noetherian and so, we obtain from [1, Proposition 7.3] that $B = S^{-1}A$ is Noetherian. Hence, B is an atomic domain. Now, as $B = S^{-1}A$, $\{\mathfrak{m}_i \mid i \in \{1,2\}\}$ is the set of all prime ideals of A maximal with respect to the property of not meeting S, it follows from [3, Corollary 4.6] that $Max(B) = \{\mathfrak{m}_1B, \mathfrak{m}_2B\}$. Note that $U(A) = U(R)$ by [1, Exercise 2(i), page 11] and so, $U(A) = \mathbb{R}\backslash\{0\}$. Observe that $1+X^2, Y, Y-1$ are non-associate prime elements of A. As $A(1+X^2) \cap S = AY \cap S = A(Y-1) \cap S = \emptyset$, we obtain from [1,

Proposition 3.11(iv)] that $(1+X^2), Y, Y-1$ are non-associate prime elements of B. It is clear that $B(1+X^2), BY, B(Y-1) \in \mathcal{I}(B)$ are pairwise distinct and we obtain from Lemma 2.5 that $B(1+X^2) - BY - B(Y-1) - B(1+X^2)$ is a cycle of length 3 in $\mathbb{G}(B)$. This proves that $gr(\mathbb{G}(B)) = 3$.

Now, $|Max(B)| = 2$. We know from Lemma 3.4 that $\mathbb{CGI}(B)$ is a complete bipartite graph with vertex partition $\mathcal{I}(B) = \bigcup_{i=1}^{2} V_i$, where $V_1 = \{B\pi \in \mathcal{I}(B) \mid \pi \in \mathfrak{m}_1 B \backslash \mathfrak{m}_2 B\}$ and $V_2 = \{B\pi' \in \mathcal{I}(B) \mid \pi' \in \mathfrak{m}_2 B \backslash \mathfrak{m}_1 B\}$. Since R is a Noetherian domain with $dimR = 1$ and $A = R[Y]$, it follows from [3, Theorem 30.5] that $dimA = 2$. Note that height of $\mathfrak{m}_i$ is equal to 2 for each $i \in \{1,2\}$. Hence, height of $\mathfrak{m}_i B$ is equal to 2 for each $i \in \{1,2\}$ and as B is Noetherian, it follows that $\mathfrak{m}_i B$ cannot be principal for each $i \in \{1,2\}$. Hence, we obtain from Lemma 3.6 that $|V_i| \geq 2$ for each $i \in \{1,2\}$ and so, $gr(\mathbb{CGI}(B)) = 4$. Therefore, $\mathbb{CGI}(B)$ is a complete bipartite graph but not a star graph.

(3) It is already noted in the proof of (1) that R is Noetherian and $dimR = 1$. It is noted in the proof of Example 2.17(1) that $\mathfrak{m}_1 = (1+X^2)\mathbb{R}[X] \cap R \in Max(R)$ and $\mathfrak{m}_1 = (1+X^2)R$. Since $\frac{R}{X^2\mathbb{R}[X]} \cong \mathbb{R}$, the field of real numbers, it follows that $\mathfrak{m}_2 = X^2\mathbb{R}[X] \in Max(R)$. It is clear that $\mathfrak{m}_1 \neq \mathfrak{m}_2$. Note that $S = R \backslash (\mathfrak{m}_1 \cup \mathfrak{m}_2)$ is a multiplicatively closed subset of R. As R is Noetherian, we obtain from [1, Proposition 7.3] that $R_1 = S^{-1}R$ is Noetherian. Hence, R_1 is an atomic domain . Since $\{\mathfrak{m}_i \mid i \in \{1,2\}\}$ is the set of all prime ideals of R maximal with respect to the property of not meeting S, it follows from [3, Corollary 4.6] that $Max(R_1) = \{\mathfrak{m}_1 R_1, \mathfrak{m}_2 R_1\}$. It follows from [1, Proposition 3.11(iv)] that $dimS^{-1}R \leq dimR = 1$. Therefore, we obtain that $dimS^{-1}R = 1$ and moreover, $Spec(S^{-1}R) \backslash \{(0)\} = \{\mathfrak{m}_1 R_1, \mathfrak{m}_2 R_1\} = Max(R_1)$. Note that $\mathbb{CGI}(R_1) = \mathbb{G}(R_1)$ by Lemma 3.10. As $|Max(R_1)| = 2$, it follows from Lemma 3.4 that $\mathbb{CGI}(R_1)$ is a complete bipartite graph and so, $\mathbb{G}(R_1)$ is a complete bipartite graph. Observe that $\mathfrak{m}_1 R_1 = (1+X^2)R_1$ is a principal ideal of R_1. Hence, we obtain from (2) $\Rightarrow$ (3) of Proposition 3.7 that $\mathbb{CGI}(R_1) = \mathbb{G}(R_1)$ is a star graph and so, $gr(\mathbb{G}(R_1)) = \infty$. □

We provide in Example 3.12, a Noetherian domain R with $|Max(R)| = 2$ and $dimR = 1$ such that $\mathbb{CGI}(R) = \mathbb{G}(R)$ is a complete bipartite graph but not a star graph. We are thankful to the referee for suggesting Example 3.12 and its proof.

Example 3.12. Let $T = \mathbb{C}[X]$ be the polynomial ring in one variable X over $\mathbb{C}$. Let $D_1 = \mathbb{R} + X\mathbb{C}[X]$ and $D_2 = \mathbb{R} + (X-1)\mathbb{C}[X]$. Let $D = D_1 \cap D_2$. Let $\mathfrak{p}_1 = X\mathbb{C}[X] \cap D$ and $\mathfrak{p}_2 = (X-1)\mathbb{C}[X] \cap D$. Let $R = S^{-1}D$, where $S = D \backslash (\mathfrak{p}_1 \cup \mathfrak{p}_2)$. Then R is a Noetherian domain, $dimR = 1$, and $|Max(R)| = 2$ such that $\mathbb{CGI}(R) = \mathbb{G}(R)$ is a complete bipartite graph but not a star graph.

Proof. Note that $\mathbb{R}[X]$ is a subring of $D = D_1 \cap D_2$ and D is a subring of $T = \mathbb{C}[X]$. As the Noetherian domain $T = \mathbb{C}[X]$ is a finitely generated $\mathbb{R}[X]$-module, it follows that D is a finitely generated $\mathbb{R}[X]$-module and so, D is Noetherian. Note that D is integral over $\mathbb{R}[X]$ and so, $dimD = dim\mathbb{R}[X] = 1$. It is clear that $\mathfrak{p}_1 = X\mathbb{C}[X] \cap D, \mathfrak{p}_2 = (X-1)\mathbb{C}[X] \cap D \in Spec(D)\backslash\{(0)\} = Max(D)$. Observe that $X \in \mathfrak{p}_1\backslash\mathfrak{p}_2$ and $X-1 \in \mathfrak{p}_2\backslash\mathfrak{p}_1$ and so, $\mathfrak{p}_1 \neq \mathfrak{p}_2$. Note that $S = D\backslash(\mathfrak{p}_1 \cup \mathfrak{p}_2)$ is a multiplicatively closed subset of D. Let $R = S^{-1}D$. Since $\{\mathfrak{p}_1, \mathfrak{p}_2\}$ is the set of all prime ideals of D maximal with respect to the property of not meeting S, it follows that $Max(R) = \{S^{-1}\mathfrak{p}_1, S^{-1}\mathfrak{p}_2\}$. It is convenient to denote $S^{-1}\mathfrak{p}_i$ by $\mathfrak{m}_i$ for each $i \in \{1,2\}$. As D is Noetherian and $dimD = 1$, it follows that R is Noetherian and $dimR = 1$. Thus R is an atomic domain and $|Max(R)| = 2$. We know from Lemma 3.4 that $\mathbb{CGI}(R)$ is a complete bipartite graph with vertex partition $\mathcal{I}(R) = V_1 \cup V_2$, where $V_1 = \{R\pi \in \mathcal{I}(R) \mid \pi \in \mathfrak{m}_1\backslash\mathfrak{m}_2\}$ and $V_2 = \{R\pi' \in \mathcal{I}(R) \mid \pi' \in \mathfrak{m}_2\backslash\mathfrak{m}_1\}$. It follows from Lemma 3.10 that $\mathbb{CGI}(R) = \mathbb{G}(R)$. We claim that $|V_i| \geq 2$ for each $i \in \{1,2\}$. To prove this claim, in view of Lemma 3.6, it is enough to show that $\mathfrak{m}_i$ is not principal for each $i \in \{1,2\}$.

We first verify that $\mathfrak{m}_1$ is not principal. Suppose that $\mathfrak{m}_1$ is principal. Then $\mathfrak{m}_1 R_{\mathfrak{m}_1}$ is principal. Note that $Max(R_{\mathfrak{m}_1}) = \{\mathfrak{m}_1 R_{\mathfrak{m}_1}\}$ by [1, Example (1), page 38]. As R is a Noetherian domain with $dimR = 1$, it follows that $R_{\mathfrak{m}_1}$ is a local Noetherian domain and $dimR_{\mathfrak{m}_1} = 1$ and its unique maximal ideal $\mathfrak{m}_1 R_{\mathfrak{m}_1}$ is principal by assumption. Hence, $R_{\mathfrak{m}_1}$ is integrally closed by $(iii) \Rightarrow (ii)$ of [1, Proposition 9.2]. Since $R_{\mathfrak{m}_1} = (S^{-1}D)_{S^{-1}\mathfrak{p}_1} \cong D_{\mathfrak{p}_1}$ as rings by [9, Proposition 19, page 165], we get that $D_{\mathfrak{p}_1}$ is integrally closed. Observe that $X(X-1), \sqrt{-1}X(X-1) = iX(X-1) \in D_1 \cap D_2 = D$ and so, $i = \frac{iX(X-1)}{X(X-1)}$ belongs to the quotient field of D. Hence, i belongs to the quotient field of $D_{\mathfrak{p}_1}$. From $i^2 = -1 \in D_{\mathfrak{p}_1}$, we obtain that $i \in D_{\mathfrak{p}_1}$. Since $D_{\mathfrak{p}_1} \subseteq (D_1)_{X\mathbb{C}[X]}$, we get that $i \in (D_1)_{X\mathbb{C}[X]}$. Note that $X\mathbb{C}[X] \in Max(D_1)$ and $i \notin D_1$ and so, the conductor $(D_1 :_{D_1} i) = X\mathbb{C}[X]$. Therefore, $i \in (D_1)_{X)\mathbb{C}[X]}$ is impossible and hence, $\mathfrak{m}_1$ is not principal.

We next verify that $\mathfrak{m}_2$ is not principal. Suppose that $\mathfrak{m}_2$ is principal. Then $\mathfrak{m}_2 R_{\mathfrak{m}_2}$ is principal. Now, it follows as in the previous paragraph that $R_{\mathfrak{m}_2}$ is integrally closed. Since $R_{\mathfrak{m}_2} = (S^{-1}D)_{S^{-1}\mathfrak{p}_2} \cong D_{\mathfrak{p}_2}$ as rings, we get that $D_{\mathfrak{p}_2}$ is integrally closed. Hence, $i \in D_{\mathfrak{p}_2}$. As $D_{\mathfrak{p}_2} \subseteq (D_2)_{(X-1)\mathbb{C}[X]}$, we get that $i \in (D_2)_{(X-1)\mathbb{C}[X]}$. As $(X-1)\mathbb{C}[X] \in Max(D_2)$ and $i \notin D_2$, it follows that the conductor $(D_2 :_{D_2} i) = (X-1)\mathbb{C}[X]$. Therefore, $i \in (D_2)_{(X-1)\mathbb{C}[X]}$ is impossible and hence, $\mathfrak{m}_2$ is not principal.

Thus $\mathfrak{m}_i$ is not principal for each $i \in \{1,2\}$ and so, $|V_i| \geq 2$ for each $i \in \{1,2\}$. Therefore, $\mathbb{CGI}(R) = \mathbb{G}(R)$ is a complete bipartite graph but not a star graph. Hence, $gr(\mathbb{CGI}(R)) = gr(\mathbb{G}(R)) = 4$. $\square$

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