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TRANSCENDENCE OF SOME INFINITE SERIES

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In the present paper and as an application of J. Hančl criterion for transcendental sequences which gave a sufficient conditions that will assure us that a series of positive rational terms is a transcendental number. With the same conditions, we establish a transcendental measure of $\sum_{n=1}^{\infty} 1/a_n$.

1. Introduction

The theory of transcendental numbers has a long history. We know since J. Liouville in 1844 that the very rapidly converging sequences of rational numbers provide examples of transcendental numbers. So, in his famous paper [7], Liouville showed that a real number admitting very good rational approximation can not be algebraic, then he explicitly constructed the first examples of transcendental numbers.

There are a number of sufficient conditions known within the literature for an infinite series, $\sum_{n=1}^{\infty} 1/a_n$, of positive rational numbers to converge to an irrational number, see [2, 9, 11]. These conditions, which are quite varied in form, share one common feature, namely, they all require rapid growth of the sequence (a_n) to deduce irrationality of the series. As an illustration consider the following results of J. Sándor which have been taken from [11] and [12].

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From this direction, the transcendence of some infinite series has been studied by several authors such as M.A. Nyblom [8], J. Hančl and J. Štepnicka [4]. we also note that the transcendence of some power series with rational coefficients is given by some authors such as J. P. Allouche [1] and G. K. Gözer [3]. The following Theorem gives Roth's Criterion for transcendental numbers, see [10].

Theorem 1.1. *Let α be a real number, δ a real number > 2, if there exists an infinity rational numbers $\frac{p}{q}$ with $gcd(p,q) = 1$ such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{\delta}},$$

then α is a transcendental number.

2. Transcendence

We recall that the concept of a transcendental sequence is defined by J. Hančl in [5] where he gave a criterion for transcendental series which depends only on the speed of convergence. This criterion is expressed in the following Theorem.

Theorem 2.1. *Let ε, γ and c be three positive real numbers satisfying*

$$\gamma > 2\varepsilon > \frac{\log_2(3+2\varepsilon)}{\log_2(3+2\gamma)}.$$

Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two sequences of positive integers, with $(a_n)_{n=1}^{\infty}$ nondeacresing, such that

$$\limsup a_n^{\frac{1}{(3+\gamma)^n}} > 1, \tag{1}$$

$$a_n > n^{1+\varepsilon}, \tag{2}$$

$$b_n < a_n^{\frac{\varepsilon}{1+\varepsilon}}.2^{-(\log_2 a_n)^c} \tag{3}$$

hold for every sufficiently large positive integer n. Then the sequence $(\frac{a_n}{b_n})_{n=1}^{\infty}$ is transcendental.

Proof. See Theorem 2.1 of [5]. □

Our first result is given in the following Theorem.

Theorem 2.2. *Let $(a_n)_{n\geq 1}$ be a sequence of non-zero natural integers and α be a positive real > 2 such that*

$$a_{n+1} > a_n^{\alpha+1}, \text{ for all } n \geq 1. \tag{4}$$

Then, the series

$$\sum_{n=1}^{+\infty} 1/a_n$$

converges to a transcendental number.

Proof. It is an immediate consequence of the previous Theorem 2.1 of J. Hančl. □

We will now give a corollary as an application of the previous result.

Corollary 2.3. *Any subseries of the series $\sum_{n=1}^{+\infty} 1/a_n$, where the terms $a_n \in \mathbb{N}^*$ satisfy (4) will have a transcendental sum.*

Proof. Consider an arbitrary subseries $\sum_{n\geq 1} 1/c_n$ then by definition there must exist a strictly monotone increasing function $g : \mathbb{N} \to \mathbb{N}$ such that $c_n = a_{g(n)}$. Clearly as $g(n+1) \geq g(n)+1$, one has

$$\frac{c_{n+1}}{c_n^{\alpha+1}} = \frac{a_{g(n+1)}}{a_{g(n)}^{\alpha+1}} \geq \frac{a_{g(n)+1}}{a_{g(n)}^{\alpha+1}} > 1,$$

and by Theorem 2.2 the subseries has a transcendental sum. □

Example 2.4. We consider the following sequence

$$\begin{cases} a_n = 2^{n!+1}, & n \geq 1 \\ a_{n+1} > a_n^3, & n \geq 3. \end{cases}$$

By applying Theorem 2.2, the series

$$\sum_n \frac{1}{2^{n!+1}}$$

converges to a transcendental number.

3. Transcendental measure

Definition 3.1. Let $P \in \mathbb{Z}[X]/\{0\}$ be a polynomial of degree d. The height of polynomial P is maximum of the absolute value of its coefficients.

The second main result of this paper is to give a transcendental measure of $\theta = \sum_{n=1}^{\infty} \dfrac{1}{a_n}$. In this section, we keep the same notations as in the second section.

Theorem 3.2. *Let $P \in \mathbb{Z}[X]/\{0\}$ be a polynomial of degree $d \geq 2$ and height H. Let $\alpha > d$ and $k > 1$ be two real numbers such that*

$$a_n^{\alpha+1} \leq a_{n+1} < a_n^{k\alpha}, \text{ for all } n \geq 1.$$

Then, we have

$$|P(\theta)| > \frac{1}{(Hd\,(d+1))^{\frac{kd(\alpha+1)}{\alpha-d}}}.$$

In order to prove this Theorem, we need the following Lemmas.

Lemma 3.3. *Let*

$$\frac{p_m}{q_m} = \sum_{k=1}^{m} \frac{1}{a_k}$$

such that $(p_m, q_m) = 1$. Then, we have

$$q_m \leq a_1 a_2 \ldots a_m. \tag{5}$$

Proof. Since $(p_m, q_m) = 1$, the lowest common denominator of the fraction $\dfrac{1}{a_1} \cdots \dfrac{1}{a_m}$ must be greater than or equal to q_m. So we deduce $q_m \leq a_1 a_2 \cdots a_m$. □

Lemma 3.4. *Let $(a_n)_{n\geq 1}$ be a sequence of natural integers ¿ 0, and α be a given real > 2. The hypothesis $a_{n+1} > a_n^{\alpha+1}$ implies that*

$$(i)\ \lim_{n\to\infty} \frac{(a_1 a_2 \ldots a_n)^{\alpha}}{a_{n+1}} = 0. \tag{6}$$

$$(ii)\ \left|\theta - \frac{p_m}{q_m}\right| < \frac{1}{q_m^{\alpha}}.$$

Proof. (i) We set

$$b_n = \frac{(a_1 a_2 \ldots a_n)^{\alpha}}{a_{n+1}},$$

and we show that $\lim_{n\to\infty} b_n = 0$. We have

$$\begin{aligned}\ln\left(\frac{1}{b_n}\right) &= \ln(a_{n+1}) - \alpha\sum_{k=1}^{n}\ln(a_k)\\ &= \sum_{k=1}^{n}\ln\left(\frac{a_{k+1}}{a_k}\right) + \ln(a_1) - \alpha\sum_{k=1}^{n}\ln(a_k)\\ &= \sum_{k=1}^{n}\ln\left(\frac{a_{k+1}}{a_k^{\alpha+1}}\right) + \ln(a_1)\\ &\geq \sum_{k=1}^{n}\ln\left(\frac{a_{k+1}}{a_k^{\alpha+1}}\right).\end{aligned}$$

Since $\frac{a_{n+1}}{a_n^{\alpha+1}} > 1$, then there exists $\delta > 0$ such that $\frac{a_{n+1}}{a_n^{\alpha+1}} > 1+\delta$. Therefore, we get

$$\ln\left(\frac{1}{b_n}\right) \geq n\ln(1+\delta).$$

From this, we deduce that, $\lim_{n\to+\infty}\ln\left(\frac{1}{b_n}\right) = +\infty$, then $\lim_{n\to+\infty} b_n = 0$.

(ii) According to the hypothesis, the series $\sum_n \frac{1}{a_n}$ is convergent.

Set $\theta = \sum_{n=1}^{\infty}\frac{1}{a_n}$ and $\frac{p_m}{q_m} = \sum_{n=1}^{m}\frac{1}{a_n}$. From the equality,

$$\left|\theta - \frac{p_m}{q_m}\right| = \sum_{n=m+1}^{\infty}\frac{1}{a_n},$$

we obtain

$$q_m^{\alpha}\left|\theta - \frac{p_m}{q_m}\right| = \sum_{n=m+1}^{\infty}\frac{q_m^{\alpha}}{a_n}.$$

The relationship (5) implies that

$$q_m^{\alpha}\left|\theta - \frac{p_m}{q_m}\right| \leq (a_1a_2...a_m)^{\alpha}$$

$$\sum_{n=m+1}^{\infty}\frac{1}{a_n} \leq b_m\sum_{n=m+1}^{\infty}\frac{a_{m+1}}{a_n},$$

with $b_m = \frac{(a_1a_2...a_m)^{\alpha}}{a_{m+1}}$.

Furthermore, for all $n \geq 1$, we have

$$\frac{a_n}{a_{n+1}} < \frac{1}{a_n^{\alpha}} < \frac{1}{a_n}. \tag{7}$$

Then, we obtain

$$\begin{aligned} q_m^{\alpha}\left|\theta-\frac{p_m}{q_m}\right| &< b_m\left(1+\sum_{k=1}^{\infty}\frac{a_{m+1}}{a_{m+k+1}}\right) \\ &< b_m\left(1+\sum_{k=1}^{\infty}\frac{a_{m+k}}{a_{m+k+1}}\right) \\ &< b_m\left(1+\sum_{k=1}^{\infty}\frac{1}{a_{m+k}}\right) \\ &< b_m\left(1+\theta\right). \end{aligned}$$

According to the relationship (6), and for m sufficiently large, we get $b_m < (1+\theta)^{-1}$.

Therefore for m sufficiently large, we have

$$q_m^{\alpha}\left|\theta-\frac{p_m}{q_m}\right| < 1.$$

Finally we find

$$\left|\theta-\frac{p_m}{q_m}\right| < \frac{1}{q_m^{\alpha}}. \tag{8}$$

□

Lemma 3.5. *(i) The hypothesis* $a_n^{\alpha+1} \leq a_{n+1}$ *implies that for all* $n \geq 1$, *we have*

$$q_n \leq a_n^{\frac{\alpha+1}{\alpha}}. \tag{9}$$

(ii) The hypothesis $a_{n+1} < a_n^{k\alpha}$ *implies that*

$$q_{n+1} < q_n^{k(\alpha+1)}, \quad \textit{for all} \quad n \geq 1. \tag{10}$$

Proof. (i) The hypothesis of (i) implies that

$$a_n \leq a_{n+1}^{\frac{1}{\alpha+1}}.$$

Then for all $1 \leq j \leq n-1$, we obtain

$$a_j \leq a_n^{\left(\frac{1}{\alpha+1}\right)^{n-j}}.$$

On the other hand, according to the relationship (5), one has

$$q_n \leq a_1 \ldots a_{n-1}.a_n,$$

this implies

$$q_n \leq a_n^{1+\frac{1}{\alpha+1}+\frac{1}{(\alpha+1)^2}+\ldots+\frac{1}{(\alpha+1)^{n-1}}}.$$

Which gives

$$q_n \leq a_n^{\frac{1}{1-\frac{1}{\alpha+1}}}.$$

Finally we obtain

$$q_n \leq a_n^{\frac{\alpha+1}{\alpha}}, \quad for\ all\ n \geq 1.$$

(ii) According to the relationship (9), we have

$$q_n \leq a_n^{\frac{\alpha+1}{\alpha}} < a_{n-1}^{\frac{\alpha+1}{\alpha}k\alpha} = a_{n-1}^{k(\alpha+1)}.$$

Since $a_n < q_n$ for all $n \geq 1$, we obtain

$$q_n < q_{n-1}^{k(\alpha+1)}.$$

□

Proof of Theorem 3.2. Set

$$\theta_n = \frac{p_n}{q_n} = \sum_{k=1}^{n} \frac{1}{a_k}.$$

From the equality,

$$P(\theta_n) = P(\theta_n) - P(\theta) + P(\theta),$$

we get

$$|P(\theta_n)| \leq |P(\theta_n) - P(\theta)| + |P(\theta)|.$$

Therefore,

$$|P(\theta)| \geq |P(\theta_n)| - |P(\theta) - P(\theta_n)|. \tag{11}$$

Set $P = \sum_{k=1}^{d} e_k X^k$, then

$$P(\theta_n) = P\left(\frac{p_n}{q_n}\right) = \sum_{k=1}^{d} e_k \frac{p_n^k}{q_n^k} = \frac{1}{q_n^d} \sum_{k=1}^{d} e_k p_n^k q_n^{d-k}. \tag{12}$$

Notice that

$$\sum_{k=1}^{d} e_k p_n^k q_n^{d-k} \neq 0,$$

because, if we assume that

$$\sum_{k=1}^{d} e_k p_n^k q_n^{d-k} = 0,$$

then we would have

$$q_n^d.P(\theta_n) = 0,$$

which implies that

$$P(\theta_n) = 0.$$

We also have

$$P(\theta) = P\left(\lim_{n\to+\infty} \theta_n\right) = \lim_{n\to+\infty} P(\theta_n) = 0,$$

which contradicts the fact that θ is a transcendental number. Therefore,

$$\left|\sum_{k=1}^{d} e_k p_n^k q_n^{d-k}\right| \geq 1.$$

According to equality (12), we get

$$|P(\theta_n)| \geq \frac{1}{q_n^d}. \tag{13}$$

On the other hand, according to the mean value theorem applied to P, there exists a real number $F \in]\theta_n, \theta[$ or $]\theta, \theta_n[$ such that

$$P(\theta) - P(\theta_n) = P'(F)(\theta - \theta_n).$$

From this, we obtain

$$|P(\theta) - P(\theta_n)| = \left|P'(F)\right| |\theta - \theta_n|. \tag{14}$$

Furthermore, as

$$P'(F) = \sum_{k=1}^{d} kF^{k-1} e_k,$$

which implies that

$$\begin{aligned} \left|P'(F)\right| &\leq \sum_{k=1}^{d} k|F|^{k-1}|e_k| \\ &\leq \sum_{k=1}^{d} k|e_k| \leq H\sum_{k=1}^{d} k \\ &\leq H\frac{d(d+1)}{2}. \end{aligned}$$

Therefore, equality (14) becomes

$$|P(\theta)-P(\theta_n)|<H\frac{d(d+1)}{2}|\theta-\theta_n|. \tag{15}$$

According to relationship (8), we have

$$|\theta-\theta_n|<\frac{1}{q_n^{\alpha}},$$

then

$$|P(\theta)-P(\theta_n)|<\frac{Hd(d+1)}{2q_n^{\alpha}}. \tag{16}$$

By combining (13) and (14), the relationship (11) becomes

$$|P(\theta)|>\frac{1}{q_n^d}-\frac{Hd(d+1)}{2q_n^{\alpha}}, \text{ for n sufficiently large.}$$

In order to have $|P(\theta)|>\frac{1}{2q_n^d}$, it suffices to have

$$\frac{1}{q_n^d}-\frac{Hd(d+1)}{2q_n^{\alpha}}>\frac{1}{2q_n^d},$$

which is equivalent to

$$\frac{1}{2q_n^d}>\frac{Hd(d+1)}{2q_n^{\alpha}}\Longleftrightarrow q_n^{\alpha-d}>Hd(d+1).$$

So that, we take n_1 the smallest integer such that

$$q_{n_1-1}^{\alpha-d}<Hd(d+1)<q_{n_1}^{\alpha-d}. \tag{17}$$

Remark 3.6. The natural number n_1 exists because $\lim\limits_{n\to\infty} q_n^{\alpha-d}=+\infty$, then, we obtain

$$q_{n_1}^{\alpha-d}>Hd(d+1).$$

Therefore, we get

$$|p(\theta)|>\frac{1}{2q_{n_1}^d}. \tag{18}$$

Using (ii) from Lemma 3, the relationship (18) becomes

$$|P(\theta)|>\frac{1}{2q_{n_1}^d}>\frac{1}{2q_{n_1-1}^{kd(\alpha+1)}}. \tag{19}$$

According to the relationship (17), we have

$$\frac{1}{q_{n_1-1}^{\alpha-d}} > \frac{1}{Hd(d+1)},$$

then

$$\frac{1}{q_{n_1-1}^{kd(\alpha+1)}} > \frac{1}{(Hd\,(d+1))^{\frac{kd(\alpha+1)}{\alpha-d}}}.$$

So, relationship (19) becomes

$$|P(\theta)| > \frac{1}{(Hd\,(d+1))^{\frac{kd(\alpha+1)}{\alpha-d}}},$$

which completes the proof of Theorem 3.

Example 3.7. Let

$$\begin{cases} a_0 = 0, a_1 = 2, \\ a_{n+1} = a_n^4, \quad n \geq 1, \\ \alpha = 4, k = 2. \end{cases}$$

Let $P \in \mathbb{Z}[X]n\{0\}$ be a quadratic polynomial of height H. By applying Theorem 3, a transcendental measure of $\theta = \sum_{n=1}^{\infty} 1/a_n$ is given by

$$|P(\theta)| > \frac{1}{(6H)^{10}}.$$

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