LE MATEMATICHE
Vol. LXIV (2009) – Fasc. I, pp. 93–96

ESSENTIALLY HYPONORMAL OPERATORS WITH ESSENTIAL SPECTRUM CONTAINED IN A CIRCLE

SHQIPE I. LOHAJ - MUHIB R. LOHAJ

In this paper two results are given . It is proved that if the essential spectrum $\sigma(\pi(T))$ of the bounded hyponormal operator T is contained in a circle, then T is essentially normal operator. Based on this result it is proved that if $T \in L(H)$ with ind $T = 0$ then $T = \lambda U + K$ (where $\lambda \in \mathbf{R}^+$, U is a unitary operator and K is a compact operator) if and only if TT^* is quasi-diagonal with respect to any sequence $\{P_n\}$ in $PF(H)$ such that $P_n \to I$, strongly.

1. Introduction

Let $L(H)$ be the algebra of all bounded linear operators acting on a separable Hilbert space H and let $K(H)$ be the ideal of all compact operators. Denote by $A(H) = L(H)/K(H)$ the cosset and let $\pi : L(H) \to A(H)$ be the canonical projection. $A(H)$ is a Banach algebra with respect to the norm $\|\pi(T)\| = inf\{\|T - K\| : K \in K(H)\}$. π is a continuous linear map and $A(H)$ is a C^* -algebra with respect to the involution * : $\pi(T) \to [\pi(T)]^* = \pi(T^*)$, called Calkin Algebra.

We say that an operator $T \in L(H)$ is Fredholm if $\pi(T)$ is invertible element in Calkin algebra $A(H)$. Denote by $F_0(H)$ the set of all Fredholm operators and $\sigma(\pi(T)) = \{\lambda \in \mathbf{C} : T - \lambda I \notin F(H)\}$ the spectrum of element $\pi(T)$ in

Entrato in redazione: 15 gennaio 2009

AMS 2000 Subject Classification: 47Bxx.
Keywords: essential spectrum, quasidiagonal operator.

$A(H)$ which is called the essential spectrum of operator T . Further, we say that an operator T is essentially unitary operator, if $\pi(T)$ is unitary element in $A(H)$. An operator is said to be hyponormal (essentially hyponormal) if $TT^* \leq T^*T\left(\pi(TT^*) \leq \pi(T^*T)\right)$.

2. Main Results

Theorem 2.1. *Let $T \in L(H)$ be an essentially hyponormal operator. If the essential spectrum $\sigma\left(\pi(T)\right)$ of the operator T is contained in a circle then T is an essentially normal operator.*

Before proving Theorem 2.1 we need to prove the following result:

Proposition 2.2. *Let A be C^* -algebra and $a \in A$ an invertible element, then*

$$\|a\| \cdot \left\|a^{-1}\right\| = 1 \Leftrightarrow a = \lambda u, \quad \lambda \in \mathbf{R}^+$$

where u is a unitary element in A.

Proof. Suppose that $\|a\| \cdot \left\|a^{-1}\right\| = 1$. Since aa^* is a positive self-adjoint element in algebra A, its spectrum $\sigma\left(aa^*\right) \subset [0,\infty)$. Further, $\|a\|^2 = \left\|aa^*\right\|$ and the spectral radius $r\left(aa^*\right) = \left\|aa^*\right\| = \|a\|^2$. Now

$$\sigma\left(aa^*\right) \subseteq \left\{\lambda \in \mathbf{R} : 0 \leq \lambda \leq \|a\|^2\right\} \tag{1}$$

On the other hand, $\left(aa^*\right)^{-1}$ is also positive self-adjoint element and because $\|a\| \cdot \left\|a^{-1}\right\| = 1$ we will have

$$\sigma\left(\left(aa^*\right)^{-1}\right) \subseteq \left\{\lambda \in \mathbf{R} : 0 \leq \lambda \leq \frac{1}{\|a\|^2}\right\} \tag{2}$$

(since $r\left(\left(aa^*\right)^{-1}\right) = \frac{1}{\|a\|^2}$). From the above inclusions (1) and (2) we obtain $\sigma\left(aa^*\right) = \left\{\|a\|^2\right\}$. Since the spectrum of element aa^* contains only one point, we conclude that $aa^* - \|a\|^2 e = q$ is a quasi-nilpotent element in A. Moreover, q is a self adjoint element in A , which implies $q = 0$. Therefore, $aa^* = \|a\|^2 e$ and $a/\|a\| = u$ is a unitary element in C^* -algebra A by which we have proved the necessary condition of the proposition.

The proof of the sufficient condition of the proposition is trivial. □

Remark 2.3. Since $L(H)$ is a C^* -algebra from the Proposition 2.2 we observe that every invertible operator $T \in L(H)$ that satisfies the property $\|T\| \cdot \left\|T^{-1}\right\| = 1$ is a scalar multiple of some unitary operator $U \in L(H)$.

Proof of Theorem 2.1. Let T be an essentially hyponormal operator. It is easy to see that the translation and inverse of essentially hyponormal operator is essentially hyponormal. For this reason without loss of generality suppose that

$$\sigma\big(\pi(T)\big) \subseteq \{\lambda \in \mathbf{C} : |\lambda| = l\}$$

Then, $r\big(\pi(T)\big) = l$ and $r\big(\pi(T)^{-1}\big) = \frac{1}{l}$. Further, since the element $\pi(T)$ is normal in the Calkin algebra, we will have $r\big(\pi(T)\big) = \|\pi(T)\|$ (see [3]) and therefore

$$\|\pi(T)\| \cdot \left\|\pi(T)^{-1}\right\| = 1$$

Now, by Proposition 1 and the latter relation we conclude that the operator T is essentially normal. By which we have completed the proof of Theorem 2.1.

Further on, let $(QD)\{P_n\}$ be a class of quasidiagonal operators such that the quasidiagonality is assumed with a common sequence of orthogonal projections $\{P_n\}$ (see[6] , [7]). Then we can give the following result:

Proposition 2.4. *Let $T \in L(H)$ with $indT = dimKerT - dimKerT^* = 0$ then*

$$TT^* \in \bigcap_{\substack{\{P_n\} \subset PF(H) \\ P_n \xrightarrow{s} I}} (QD)\{P_n\} \Leftrightarrow T = \lambda U + K$$

where $\lambda \in \mathbf{R}^+$, U is a unitary operator and K is a compact operator.

Proof. Let

$$TT^* \in \bigcap_{\substack{\{P_n\} \subset PF(H) \\ P_n \xrightarrow{s} I}} (QD)\{P_n\}.$$

It means that $TT^* \in (QD)\{P_n\}$ for every sequence $\{P_n\}$ in $PF(H)$ such that $P_n \to I$, strongly. Hence, operator TT^* is uniformly quasidiagonal and thus TT^* is thin(see [5]). Therefore, $TT^* = \alpha I + K$, where $\alpha \geq 0$ and $K \in K(H)$. Further,

$$\pi(TT^*) = \pi(T)\pi(T^*) = \alpha\pi(I)$$

Hence $\pi(T)/\sqrt{\alpha}$ is a unitary element in Calkin Algebra $L(H)/K(H)$. Since ind $T = 0$, there exists a unitary operator $U \in L(H)$, and a compact operator $K \in L(H)$ such that $T/\sqrt{\alpha} = U + K$ (see Theorem 3.1 in [2]).

Conversely. If $T = \lambda U + K$ then

$$TT^* = \big(\lambda U + K\big)\big(\bar{\lambda} U^* + K^*\big) = |\lambda|^2 I + \lambda U K^* + \bar{\lambda} K U^* + K K^* = |\lambda|^2 I + K_1,$$

where $K_1 \in K(H)$, which completes the proof of Proposition 2.4, because the operators I, $K_1 \in (QD)\{P_n\}$, for every sequence $\{P_n\}$ in $PF(H)$, such that $P_n \to I$, strongly. □

REFERENCES

[1] W. B. Arveson, *An invitation to C^* -algebras*, Springer-Verlag, New York, 1976.

[2] L. G. Brown - R. G. Douglas - P. A. Fillmore, *Unitary equivalence modulo the compact operators and extensions of C^* - algebras*, L. N. M. n. 345, 58–128, Springer–Verlag, New York, 1973.

[3] J. Dixmier, *Les C^* - algèbres et leurs représentations*, Paris, Gauthier-Villars, 1969.

[4] P. A. Fillmore - J. G. Stampfli - J.P. Williams, *On the essential numerical range, the essential spectrum, and a problem of Halmos*, Acta Sci. Math. (Szeged) 33 (1972), 179–192.

[5] G. R. Luecke, *A note on quasidiagonal and quasitriangular operators*, Pacific. J. Math. 56 (1975), 179–185.

[6] M. R. Lohaj, *Necessary conditions for quasidiagonality of some special nilpotent operators*, Rad. Mat. 10 (2001), 209–217.

[7] M. R. Lohaj, *Necessary conditions for quasidiagonality of certain bounded operators defined in the direct sum $H \bigoplus H \bigoplus H$* , Rad. Mat. 11 n. 1 (2002), 113-117.

SHQIPE I. LOHAJ
Electronic Faculty
University of Prishtina
e-mail: shqipe_lohaj@yahoo.com

MUHIB R. LOHAJ
Faculty of Natural Science
University of Prishtina
e-mail: muhib_lohaj@yahoo.com