ESSENTIALLY HYPONORMAL OPERATORS
WITH ESSENTIAL SPECTRUM CONTAINED IN A CIRCLE

SHQIPE I. LOHAJ - MUHIB R. LOHAJ

In this paper two results are given. It is proved that if the essential spectrum \( \sigma(\pi(T)) \) of the bounded hyponormal operator \( T \) is contained in a circle, then \( T \) is essentially normal operator. Based on this result it is proved that if \( T \in L(H) \) with ind \( T = 0 \) then \( T = \lambda U + K \) (where \( \lambda \in \mathbb{R}^+ \), \( U \) is a unitary operator and \( K \) is a compact operator) if and only if \( TT^* \) is quasi-diagonal with respect to any sequence \( \{P_n\} \) in \( PF(H) \) such that \( P_n \to I \), strongly.

1. Introduction

Let \( L(H) \) be the algebra of all bounded linear operators acting on a separable Hilbert space \( H \) and let \( K(H) \) be the ideal of all compact operators. Denote by \( A(H) = L(H)/K(H) \) the coset and let \( \pi : L(H) \to A(H) \) be the canonical projection. \( A(H) \) is a Banach algebra with respect to the norm \( \|\pi(T)\| = \inf\{\|T - K\| : K \in K(H)\} \). \( \pi \) is a continuous linear map and \( A(H) \) is a \( C^* \)-algebra with respect to the involution \( * : \pi(T) \to [\pi(T)]^* = \pi(T^*) \), called Calkin Algebra.

We say that an operator \( T \in L(H) \) is Fredholm if \( \pi(T) \) is invertible element in Calkin algebra \( A(H) \). Denote by \( F_0(H) \) the set of all Fredholm operators and \( \sigma(\pi(T)) = \{\lambda \in \mathbb{C} : T - \lambda I \notin F(H)\} \) the spectrum of element \( \pi(T) \) in the

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A(H) which is called the essential spectrum of operator T. Further, we say that an operator T is essentially unitary operator, if \( \pi(T) \) is unitary element in \( A(H) \). An operator is said to be hyponormal (essentially hyponormal) if \( TT^* \leq T^*T \left( \pi(TT^*) \leq \pi(T^*T) \right) \).

### 2. Main Results

**Theorem 2.1.** Let \( T \in L(H) \) be an essentially hyponormal operator. If the essential spectrum \( \sigma(\pi(T)) \) of the operator \( T \) is contained in a circle then \( T \) is an essentially normal operator.

Before proving Theorem 2.1 we need to prove the following result:

**Proposition 2.2.** Let \( A \) be \( C^* \)-algebra and \( a \in A \) an invertible element, then

\[
\|a\| \cdot \|a^{-1}\| = 1 \iff a = \lambda u, \quad \lambda \in \mathbb{R}^+
\]

where \( u \) is a unitary element in \( A \).

**Proof.** Suppose that \( \|a\| \cdot \|a^{-1}\| = 1 \). Since \( aa^* \) is a positive self-adjoint element in algebra \( A \), its spectrum \( \sigma(aa^*) \subset [0, \infty) \). Further, \( \|a\|^2 = \|aa^*\| \) and the spectral radius \( r(aa^*) = \|aa^*\| = \|a\|^2 \). Now

\[
\sigma(aa^*) \subseteq \left\{ \lambda \in \mathbb{R} : 0 \leq \lambda \leq \|a\|^2 \right\} \quad (1)
\]

On the other hand, \( (aa^*)^{-1} \) is also positive self-adjoint element and because \( \|a\| \cdot \|a^{-1}\| = 1 \) we will have

\[
\sigma((aa^*)^{-1}) \subseteq \left\{ \lambda \in \mathbb{R} : 0 \leq \lambda \leq \frac{1}{\|a\|^2} \right\} \quad (2)
\]

(since \( r((aa^*)^{-1}) = \frac{1}{\|a\|^2} \)). From the above inclusions (1) and (2) we obtain \( \sigma(aa^*) = \{\|a\|^2\} \). Since the spectrum of element \( aa^* \) contains only one point, we conclude that \( aa^* - \|a\|^2e = q \) is a quasi-nilpotent element in \( A \). Moreover, \( q \) is a self adjoint element in \( A \), which implies \( q = 0 \). Therefore, \( aa^* = \|a\|^2e \) and \( a/\|a\| = u \) is a unitary element in \( C^* \)-algebra \( A \) by which we have proved the necessary condition of the proposition.

The proof of the sufficient condition of the proposition is trivial. \( \square \)

**Remark 2.3.** Since \( L(H) \) is a \( C^* \)-algebra from the Proposition 2.2 we observe that every invertible operator \( T \in L(H) \) that satisfies the property \( \|T\| \cdot \|T^{-1}\| = 1 \) is a scalar multiple of some unitary operator \( U \in L(H) \).
Proof of Theorem 2.1. Let $T$ be an essentially hyponormal operator. It is easy to see that the translation and inverse of essentially hyponormal operator is essentially hyponormal. For this reason without loss of generality suppose that

$$\sigma(\pi(T)) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = l\}$$

Then, $r(\pi(T)) = l$ and $r(\pi(T)^{-1}) = \frac{1}{l}$. Further, since the element $\pi(T)$ is normal in the Calkin algebra, we will have $r(\pi(T)) = \|\pi(T)\|$ (see [3]) and therefore

$$\|\pi(T)\| \cdot \|\pi(T)^{-1}\| = 1$$

Now, by Proposition 1 and the latter relation we conclude that the operator $T$ is essentially normal. By which we have completed the proof of Theorem 2.1.

Further on, let $(QD)\{P_n\}$ be a class of quasidiagonal operators such that the quasidiagonality is assumed with a common sequence of orthogonal projections $\{P_n\}$ (see [6], [7]). Then we can give the following result:

**Proposition 2.4.** Let $T \in L(H)$ with $\text{ind}T = \dim \ker T - \dim \ker T^* = 0$ then

$$TT^* \in \bigcap_{\{P_n\} \subset PF(H)} (QD)\{P_n\} \iff T = \lambda U + K$$

where $\lambda \in \mathbb{R}^+$, $U$ is a unitary operator and $K$ is a compact operator.

**Proof.** Let

$$TT^* \in \bigcap_{\{P_n\} \subset PF(H)} (QD)\{P_n\}.$$

It means that $TT^* \in (QD)\{P_n\}$ for every sequence $\{P_n\}$ in $PF(H)$ such that $P_n \to I$, strongly. Hence, operator $TT^*$ is uniformly quasidiagonal and thus $TT^*$ is thin(see [5]). Therefore, $TT^* = \alpha I + K$, where $\alpha \geq 0$ and $K \in K(H)$. Further,

$$\pi(TT^*) = \pi(T)\pi(T^*) = \alpha \pi(I)$$

Hence $\pi(T)/\sqrt{\alpha}$ is a unitary element in Calkin Algebra $L(H)/K(H)$. Since $\text{ind} T = 0$, there exists a unitary operator $U \in L(H)$, and a compact operator $K \in L(H)$ such that $T/\sqrt{\alpha} = U + K$ (see Theorem 3.1 in [2]).

Conversely. If $T = \lambda U + K$ then

$$TT^* = (\lambda U + K)\left(\bar{\lambda}U^* + K^*\right) = |\lambda|^2 I + \lambda UK^* + \bar{\lambda}KU^* + KK^* = |\lambda|^2 I + K_1,$$

where $K_1 \in K(H)$, which completes the proof of Proposition 2.4, because the operators $I, K_1 \in (QD)\{P_n\}$, for every sequence $\{P_n\}$ in $PF(H)$, such that $P_n \to I$, strongly. □
REFERENCES


SHQIPE I. LOHAJ
Electronic Faculty
University of Prishtina
e-mail: shqipe_lohaj@yahoo.com

MUHIB R. LOHAJ
Faculty of Natural Science
University of Prishtina
e-mail: muhib_lohaj@yahoo.com