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SOME PROPERTIES FOR ν-ZEROS OF PARABOLIC CYLINDER FUNCTIONS

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Abstract. Let $D_\nu(z)$ be the Parabolic Cylinder function. We study the ν-zeros of the function $\nu \mapsto D_\nu(z)$ with respect to the real variable z. We establish a formula for the derivative of a zero and deduce some monotonicity results. Then we also give an asymptotic expansion for ν-zeros for large positive z.

1. Introduction

Since the mid-twentieth century, real and complex zeros of special functions such as Bessel functions, Parabolic Cylinder functions, Hankel functions etc. have been intensively studied for various applications in physics, applied mathematics and engineering.

Studies on zeros for a special function of order ν and argument z have been performed by several authors. For example, Olver finds the z-zeros of Parabolic Cylinder functions [14] for large values of ν. The case of Bessel functions has been frequently studied (see for example Olver [13], Watson [17], Laforgia and Natalini [9]. In [5], the author presents a selection of results on the zeros of Bessel functions. Other authors have been interested in the z-zeros of Hermite functions or Confluent Hypergeometric functions (see for example [6], [7]). In [6], Elbert and Muldoon study the variation of the z-zeros of the Hermite

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function and establish a formula for the derivative of a zero with respect to the parameter ν.

Fewer studies have been published on the ν-zeros. In [11] or [8], the authors study the behavior of the ν-zeros of the Hankel function of the first kind. Later on, these results were improved by Cochran [3]. Conde and Kalla [4] compute the ν-zeros of the Bessel function. Slater [15] gives an asymptotic formula for large ν-zeros of the Parabolic Cylinder function when z is fixed. Besides that, little is known, about the ν-zeros of the Parabolic Cylinder function. However, these zeros appear in the first passage time law of an Ornstein Uhlenbeck process and other associated laws ([1], [2], [10]).

In this paper we study the ν-zeros of the Parabolic Cylinder function, the solution of the differential equation

$$\begin{cases} y''(z) + \left(\nu + \frac{1}{2} - \frac{1}{4}z^2\right) y(z) = 0, \\ y(z) \underset{z\to+\infty}{\sim} z^{\nu} e^{-z^2/4}, \end{cases}$$

where the Parabolic Cylinder function, denoted $D_\nu(z)$, is to be considered as function of its order ν.

The aim of this paper is to complete Slater's study and to propose a formula for ν-zeros for large values of z. We also establish a formula for the derivative of a ν-zero and deduce some monotonicity results. Since the z-zeros of Hermite functions are linked to those of Parabolic Cylinder functions, our analysis is based on the results of [6]. Asymptotic expressions for the ν-zeros are derived from the expansion of Olver [14]. Our analysis is similar to that of [3] for Hankel functions. Only real parameters are considered in this paper.

The paper is organized as follows : in Section 2 we present some properties for the ν-zeros of the Parabolic Cylinder function $D_\nu(z)$. Section 3 focusses on the behavior of the ν-zeros for large z. Moreover, numerical verifications of the asymptotic expansion are displayed.

2. Variation of zeros

In this section we present some properties for the ν-zeros of the Parabolic Cylinder function $D_\nu(z)$ with respect to the real variable z. Since the function is holomorphic (see [12], ch. 10) in the complex plane, the set of ν-zeros has no accumulation points and there is a countably infinite number of zeros. Moreover, in the real case they are strictly positive [2]. In the following, we denote by $(\nu_n(z))_{n\geq 1}$ the ordered sequence of zeros of the function $\nu \mapsto D_\nu(z)$.

The following proposition gives some monotonicity properties of the zeros.

Proposition 2.1. *For all $n \in \mathbb{N}^*$:*

1. *The first derivative of the $n-th$ ν-zero is given by :*

$$\partial_z \nu_n(z) = \frac{2}{\sqrt{\pi}\int_0^\infty e^{-(2\nu_n(z)+1)u+\frac{z^2}{2}\tanh(u)} erfc\left(z\sqrt{\frac{\tanh(u)}{2}}\right)\frac{du}{\sqrt{\sinh(u)\cosh(u)}}} \tag{1}$$

2. *The function $z \mapsto \nu_n(z)$ is strictly increasing and convex.*

Proof. 1. Let $z(\nu,\alpha)$ a zero of the function $z \mapsto \cos(\alpha)H_\nu(z)+\sin(\alpha)G_\nu(z)$ where α is fixed and $H_\nu(z)$ and $G_\nu(z)$ are linear independent solutions of $y''-2zy'+2\nu y=0$ with $H_\nu(z) \sim (2z)^\nu$ and $G_\nu(z) \sim \frac{1}{\sqrt{\pi}}\Gamma(1+\nu)z^{-\nu-1}e^{z^2}$ when $z \to +\infty$. In [6], the authors compute the derivative with respect to ν and obtain

$$\partial_\nu z(\nu,\alpha) = \frac{\sqrt{\pi}}{2}\int_0^\infty e^{-(2\nu+1)u+z(\nu,\alpha)^2\tanh(u)} erfc\left(z(\nu,\alpha)\sqrt{\tanh(u)}\right)\frac{du}{\sqrt{\sinh(u)\cosh(u)}}.$$

Since $D_\nu(z) = 2^{-\frac{\nu}{2}}e^{-\frac{z^2}{4}}H_\nu\left(\frac{z}{\sqrt{2}}\right)$, then choosing $\alpha = 0$ the result is a direct. consequence of the local inversion theorem.

2. Elbert and Muldoon [6] (Corollary 7.2) prove that $\nu \mapsto z(\nu,\alpha)$ is completely monotonic : $\partial_\nu z(\nu,\alpha) > 0$, $(-1)^k\partial_\nu^{k+1}z(\nu,\alpha) \geq 0$, $k = 1,2,\ldots,n$. The conclusion follows from the local inversion theorem.

□

If $z = 0$, Formula (1) can be simplified. Indeed, the zeros $(\nu_n(0))_{n\geq 1}$ of $\nu \mapsto D_\nu(0)$ are the positive odd integers, $\nu_n(0) = 2n-1$. In this particular case, (1) becomes:

$$\partial_z \nu_n(z)\,|_{z=0} = \frac{2}{\sqrt{\pi}\int_0^\infty \frac{e^{-(4n-1)u}du}{\sqrt{\sinh(u)\cosh(u)}}} = \begin{cases} \frac{2}{\sqrt{\pi}} & \text{if} \quad n=1, \\ \frac{2}{\sqrt{\pi}(n-1)B\left(n-1,\frac{3}{2}\right)} & \text{if} \quad n \geq 2. \end{cases}$$

Remark 2.1. We can prove that the function $z \mapsto \nu_n(z)$ is strictly increasing without using the form (1) of the derivative $\partial_z \nu_n(z)$. Indeed, on the one hand, thanks to [2] (Proposition 3.14), we have :

$$\int_z^\infty D^2_{\nu_n(z)}(x)\,dx = -\nu_n(z)\,D_{\nu_n(z)-1}(z)\,\partial_\nu D_{\nu_n(z)}(z).$$

On the other hand, by differentiating $D_{\nu_n(z)}(z) = 0$ with respect to z, we get

$$\partial_\nu D_{\nu_n(z)}(z)\,\partial_z \nu_n(z) + \nu_n(z)\,D_{\nu_n(z)-1}(z) = 0.$$

Therefore

$$\partial_\nu D_{\nu_n(z)}(z) = -\frac{\nu_n(z) D_{\nu_n(z)-1}(z)}{\partial_z \nu_n(z)}.$$

So that we finally get

$$\partial_z \nu_n(z) = \frac{\nu_n(z)^2 D^2_{\nu_n(z)-1}(z)}{\int_z^\infty D^2_{\nu_n(z)}(x)\,dx} > 0.$$

As a consequence of (1), we obtain some bounds on the derivative of a ν-zero with respect to z.

Corollary 2.1.1. The following inequalities hold

1. If $z > 0$, then $\nu_n(z) > 1$ for all $n \in \mathbb{N}^*$ and

$$\frac{4}{\sqrt{\pi}(\nu_n(z)-1)B\left(\frac{\nu_n(z)-1}{2},\frac{3}{2}\right)} \leq \partial_z \nu_n(z) \leq \frac{4e^{-\frac{z^2}{2}}}{\sqrt{\pi}(\nu_n(z)-1)\,erfc\left(\frac{z}{\sqrt{2}}\right)B\left(\frac{\nu_n(z)-1}{2},\frac{3}{2}\right)}.$$

2. If $z < 0$, then $0 < \nu_1(z) < 1$ and $\nu_n(z) > 1$ for $n \geq 2$. We also have

$$\frac{2e^{-\frac{z^2}{2}}}{c_{\nu_1(z)}\sqrt{\pi}erfc\left(\frac{z}{\sqrt{2}}\right)} \leq \partial_z \nu_1(z) \leq \frac{2}{\sqrt{\pi}c_{\nu_1(z)}},$$

$$\frac{4e^{-\frac{z^2}{2}}}{\sqrt{\pi}(\nu_n(z)-1)\,erfc\left(\frac{z}{\sqrt{2}}\right)B\left(\frac{\nu_n(z)-1}{2},\frac{3}{2}\right)} \leq \partial_z \nu_n(z) \leq \frac{4}{\sqrt{\pi}(\nu_n(z)-1)B\left(\frac{\nu_n(z)-1}{2},\frac{3}{2}\right)}, \quad n \geq 2,$$

where $c_{\nu_1(z)} \in \left[1\ \frac{\pi}{2}\right)$ is a constant depending on $\nu_1(z)$.

Proof. 1. If $z = 0$, the zeros $(\nu_n(0))_{n\geq 1}$ are the positive odd integers. Since the $z \mapsto \nu_n(z)$ is strictly increasing (see Proposition 2.1), then for $z > 0$, we get

$$\nu_n(z) > \nu_n(0) \geq \nu_1(0) = 1, \ \ n \in \mathbb{N}^*.$$

Since u is positive, then $\tanh(u) \in [0,1]$ and

$$e^{\frac{z^2}{2}} erfc\left(\frac{z}{\sqrt{2}}\right) \leq e^{\frac{z^2}{2}\tanh(u)} erfc\left(z\sqrt{\frac{\tanh(u)}{2}}\right) \leq 1.$$

Moreover, $\nu_n(z) > 1$ for all $n \in \mathbb{N}^*$, then

$$\int_0^\infty \frac{e^{-(2\nu_n(z)+1)u}}{\sqrt{\sinh(u)\cosh(u)}} = \frac{\nu_n(z)-1}{2} B\left(\frac{\nu_n(z)-1}{2},\frac{3}{2}\right).$$

2. On the one hand, in the case of negative z, the strictly monotonicity property of $z \mapsto \nu_n(z)$ gives $\nu_n(z) < \nu_n(0) = 2n-1$ for all $n \in \mathbb{N}^*$.

   On the other hand, the behavior of $D_\nu(z)$ for large negative z is ([12]) :

$$D_\nu(z) = z^\nu e^{-\frac{z^2}{4}}\left(1+O\left(|z|^{-2}\right)\right) - \frac{\sqrt{2\pi}e^{-\nu\pi i}}{\Gamma(-\nu)} z^{-\nu-1} e^{\frac{z^2}{4}}\left(1+O\left(|z|^{-2}\right)\right). \tag{2}$$

   If $\nu \in \mathbb{N}$, the dominant part (second term) in (2) vanishes and $D_\nu(z) \underset{z\to-\infty}{\longrightarrow} 0$. Therefore $\nu_n(z) \underset{z\to-\infty}{\longrightarrow} n-1$ for all $n \in \mathbb{N}^*$. We deduce that for $n \in \mathbb{N}^*$, we have

$$n-1 < \nu_n(z) < 2n-1.$$

   If $z<0$, then $1 \le e^{\frac{z^2}{2}\tanh(u)} erfc\left(z\sqrt{\frac{\tanh(u)}{2}}\right) \le e^{\frac{z^2}{2}} erfc\left(\frac{z}{\sqrt{2}}\right)$.

   Moreover

$$\int_0^\infty \frac{e^{-(2\nu+1)u}}{\sqrt{\sinh(u)\cosh(u)}} = \begin{cases} c_\nu \in \left[1\ \frac{\pi}{2}\right) & \text{if} \quad 0<\nu\le 1, \\ \frac{\nu-1}{2}B\left(\frac{\nu-1}{2},\frac{3}{2}\right) & \text{if} \quad \nu>1, \end{cases}$$

   where c_ν is a constant depending on ν.

□

Remark 2.2. In the case $z<0$, by using the inequality $n-1 < \nu_n(z) < 2n-1$, we obtain less accurate bounds depending only on n.

3. Asymptotic expansions of ν-zeros for large z

We are now interested in the behavior of ν-zeros for large positive values of z. Since the ν-zeros are positive, we restrict ourselves to the case of real positive ν.

Asymptotic expansion of Parabolic cylinder function

Recall that the Parabolic cylinder function $D_\nu(z)$ is solution of the differential equation :

$$y''(z) + \left(\nu + \frac{1}{2} - \frac{1}{4}z^2\right) y(z) = 0, \quad z \in \mathbb{C}. \tag{3}$$

The behavior of $D_\nu(z)$ for large positive z and $z >> |\nu|$ is ([12]) :

$$D_\nu(z) = e^{-\frac{z^2}{4}} z^\nu \left[1 + O\left(z^{-2}\right)\right]. \tag{4}$$

Equation (3) has two turning points at $\sqrt{4\nu+2}$ and $-\sqrt{4\nu+2}$ The asymptotic behavior of $D_\nu(z)$ changes significantly depending on the relative position of z with respect to the turning points. The asymptotic behavior (4) is still not valid if z runs through an interval containing one of the turning points. In this case, an Airy type expansion is needed to obtain those of the Parabolic cylinder function. Its expression is ([14], [16]) :

$$D_{\frac{1}{2}\mu^2-\frac{1}{2}}\left(\mu t\sqrt{2}\right) \underset{\mu\to+\infty}{=} 2\sqrt{\pi}\mu^{\frac{1}{3}}g(\mu)\left(\frac{\xi(t)}{t^2-1}\right)^{\frac{1}{4}}\left[Ai\left(\mu^{\frac{4}{3}}\xi(t)\right)A_\mu(\xi)+\frac{Ai'\left(\mu^{\frac{4}{3}}\xi(t)\right)}{\mu^{\frac{8}{3}}}B_\mu(\xi)\right], \tag{5}$$

where Ai is the Airy function of the first kind,

$$\frac{2}{3}(-\xi(t))^{\frac{3}{2}} = \int_t^1 \sqrt{1-u^2}du, \ \ -1<t\leq 1\,(\xi\leq 0),$$

$$\frac{2}{3}(\xi(t))^{\frac{3}{2}} = \int_1^t \sqrt{u^2-1}du, \ \ t\geq 1\,(\xi\geq 0),$$

$$g(\mu) \underset{\mu\to+\infty}{\sim} 2^{-\frac{1}{4}\mu^2-\frac{1}{4}}e^{-\frac{1}{4}\mu^2}\mu^{\frac{1}{2}\mu^2-\frac{1}{2}}\left(1+\frac{1}{2}\sum_{s\geq 1}\frac{2^s\gamma_s}{\mu^{2s}}\right)$$ and the coefficients γ_s are defined by

$$\Gamma\left(\frac{1}{2}+z\right) \underset{z\to+\infty}{\sim} \sqrt{2\pi}e^{-z}z^z\sum_{s\geq 0}\frac{\gamma_s}{z^s}.$$

More details on these coefficients γ_s and their computation can be found in [14], pages 134-135. For example, Olver finds for $s\leq 4$:

$$\gamma_0 = 1,\ \gamma_1 = -\frac{1}{24},\ \gamma_2 = \frac{1}{1152},$$
$$\gamma_3 = \frac{1003}{414720},\ \gamma_4 = \frac{4027}{39813120}.$$

The functions A_μ and B_μ satisfy $A_\mu(\xi) \underset{\mu\to+\infty}{\sim} \sum_{s\geq 0}\frac{A_s(\xi(t))}{\mu^{4s}}$, $B_\mu(\xi) \underset{\mu\to+\infty}{\sim} \sum_{s\geq 0}\frac{B_s(\xi(t))}{\mu^{4s}}$, where the coefficients $A_s(\xi(t))$ and $B_s(\xi(t))$ are given by

$$A_s(\xi(t)) = \xi(t)^{-3s}\sum_{m=0}^{2s}\beta_m\left(\frac{\xi(t)}{t^2-1}\right)^{\frac{3}{2}}(2s-m)u_{2s-m}(t),$$

$$B_s(\xi(t)) = -\xi(t)^{-3s}\sum_{m=0}^{2s+1}\alpha_m\left(\frac{\xi(t)}{t^2-1}\right)^{\frac{3}{2}}(2s-m+1)u_{2s-m+1}(t),$$

$$\alpha_0 = 1 \ \text{ and } \alpha_m = \frac{(2m+1)(2m+3)\dots(6m-1)}{m!(144)^m}, \beta_m = -\frac{6m+1}{6m-1}\alpha_m$$

and $u_s(t)$ are polynomials in t of degrees $3s$ (s odd), $3s-2$ (s even, $s\geq 2$) and they satisfy the recurrence relation

$$(t^2-1)u'(t)-3stu_s(t) = r_{s-1}(t),$$

where

$$8r_s(t) = (3t^2+2)u_s(t) - 12(s+1)tr_{s-1}(t) + 4(t^2-1)r'_{s-1}(t).$$

Formula (5) gives the asymptotic behavior of $D_\nu(z)$ if z runs through an interval containing the turning point $\sqrt{4\nu+2}$. Near the other turning point $-\sqrt{4\nu+2}$ (so when $z<0$), the asymptotic behavior of $D_\nu(z)$ is given by another formula (see [14], (9.7.)). As in this section we study the ν-zeros only in the case of large positive z, this second formula will not be useful here.

Remark 3.1. If z belongs to an interval containing the other turning point $-\sqrt{4\nu+2}$, the study of the asymptotic behavior of the zeros is easier. The zeros tend to positive integers.

Indeed, in this case the asymptotic behavior of the Parabolic cylinder function is given by ([14], [16]) :

$$D_{\frac{1}{2}\mu^2-\frac{1}{2}}\left(-\mu t\sqrt{2}\right) \underset{\mu\to+\infty}{=} 2\sqrt{\pi}\mu^{\frac{1}{3}}g(\mu)\left(\frac{\xi(t)}{t^2-1}\right)^{\frac{1}{4}}\left[\sin\left(\frac{1}{2}\pi\mu^2\right)\left(Ai\left(\mu^{\frac{4}{3}}\xi(t)\right)A_\mu(\xi)+\frac{Ai'\left(\mu^{\frac{4}{3}}\xi(t)\right)}{\mu^{\frac{8}{3}}}B_\mu(\xi)\right)\right.$$
$$\left.+\cos\left(\frac{1}{2}\pi\mu^2\right)\left(Bi\left(\mu^{\frac{4}{3}}\xi(t)\right)A_\mu(\xi)+\frac{Bi'\left(\mu^{\frac{4}{3}}\xi(t)\right)}{\mu^{\frac{8}{3}}}B_\mu(\xi)\right)\right], \quad (6)$$

where Bi is the Airy function of the second kind. Recall that ([13])

$$Ai(z) \underset{z\to+\infty}{\sim} \frac{1}{2\sqrt{\pi}}z^{-\frac{1}{4}}e^{-\frac{2}{3}z^{\frac{3}{2}}} \quad Ai'(z) \underset{z\to+\infty}{\sim} -\frac{1}{2\sqrt{\pi}}z^{\frac{1}{4}}e^{-\frac{2}{3}z^{\frac{3}{2}}},$$
$$Bi(z) \underset{z\to+\infty}{\sim} \frac{1}{\sqrt{\pi}}z^{-\frac{1}{4}}e^{\frac{2}{3}z^{\frac{3}{2}}} \quad Bi'(z) \underset{z\to+\infty}{\sim} \frac{1}{\sqrt{\pi}}z^{\frac{1}{4}}e^{\frac{2}{3}z^{\frac{3}{2}}}.$$

As the factor $Bi\left(\mu^{\frac{4}{3}}\xi(t)\right)A_\mu(\xi)+\frac{Bi'\left(\mu^{\frac{4}{3}}\xi(t)\right)}{\mu^{\frac{8}{3}}}B_\mu(\xi)$ goes to infinity when $\mu\to+\infty$, to obtain the zeros of $D_{\frac{1}{2}\mu^2-\frac{1}{2}}$ we must cancelled this terme. If $\frac{1}{2}\mu^2 = n+\frac{1}{2}$ in (6), the cosine vanishes and, hence, the dominant part vanishes.

Asymptotic expansion of ν-zeros

For large positive values of z, the ν-zeros of $D_\nu(z)$ are linked to a_n, $n = 1,2,\ldots$, the zeros of the Airy function ($Ai(a_n)=0$). The zeros of Ai have been studied ([13]). Olver shows that they are all real and negative. They may be expressed asymptotically as

$$a_n \underset{n\to+\infty}{\sim} -\left(\frac{3\pi}{2}\left(n-\frac{1}{4}\right)\right)^{\frac{2}{3}}.$$

The following proposition gives the behavior of $\nu_n(z)$ when $z\to\infty$.

Proposition 3.1. *For large positive z, we have :*

$$\nu_n(z) = \frac{z^2}{4} - \frac{1}{2} - z^{\frac{2}{3}} 2^{-\frac{2}{3}} a_n + O\left(z^{-\frac{2}{3}}\right), \tag{7}$$

where a_n, $n = 1, 2, \ldots$ are the zeros of the Airy function.

Proof. We apply the method given in [3] to compute the zeros of the Hankel function.

Taking $\mu = \sqrt{2\nu+1}$ in (5), the original argument z of $D_\nu(z)$ has temporarily been replaced by $t(\xi)\sqrt{2}\sqrt{2\nu+1}$. The ν-zeros of $D_\nu\left(t(\xi)\sqrt{2}\sqrt{2\nu+1}\right)$ are given asymptotically by the ν-zeros of the right hand side of (5). These zeros in turn appear to be given by the ν-solutions of $Ai\left((2\nu+1)^{\frac{2}{3}}\xi(t)\right) = 0$, from which we deduce that

$$(2\nu+1)^{\frac{2}{3}}\xi(t) \sim a_n,$$

as $\nu \to +\infty$. Since a_n is negative, then $\xi(t) < 0$. Hence we obtain the asymptotic relation between zeros of the Airy function and ν-zeros of the Parabolic cylinder function (we restore the original argument $z = t(\xi)\sqrt{2}\sqrt{2\nu+1}$):

$$\left(\frac{a_n}{\xi(t)}\right)^{\frac{3}{2}} \sim 2\nu_n + 1 = \frac{z^2}{2t^2(\xi)},$$

where ξ and t are related by the relation $\frac{2}{3}(-\xi(t))^{\frac{3}{2}} = \int_t^1 \sqrt{1-u^2}du$. We deduce that the limiting case that gives rise to large values of z (so large values of ν_n) is $\xi(t) \to 0$. As ξ is negative, so the case $\xi(t) \uparrow 0$ is associated with the behavior of the ν-zeros of $D_\nu(z)$ for large positive z. We easily deduce that if $\xi(t) \uparrow 0$, then $t \uparrow 1$ and $t(\xi) = 1 + 2^{-\frac{1}{3}}\xi - \frac{1}{10}2^{-\frac{2}{3}}\xi^2 + o\left(\xi^3\right)$.

Thus, for $z \to \infty$,

$$\begin{aligned}
2\nu_j + 1 &= \frac{z^2}{2t^2(\xi)} = \frac{z^2}{2}\frac{1}{\left(1 + 2^{-\frac{1}{3}}\xi - \frac{1}{10}2^{-\frac{2}{3}}\xi^2 + o(\xi^3)\right)^2} \\
&= \frac{z^2}{2}\left(1 - 2^{\frac{2}{3}}\xi + \frac{1}{5}2^{-\frac{2}{3}}\xi^2 + o\left(\xi^3\right)\right) \\
&= \frac{z^2}{2} - z^{\frac{2}{3}}2^{\frac{1}{3}}a_j + \frac{1}{10}2^{\frac{2}{3}}z^{-\frac{2}{3}}a_j^2 + z^{-2}O(1).
\end{aligned}$$

□

Remark 3.2. The expansions (7) are still valid for complex values of the parameters.

Numerical verification

The values of the function $\nu \mapsto D_\nu(z)$ exceed the computer capabilities, the zeros are no longer observable and computable. For large values of z, the function oscillates between $+\infty$ and $-\infty$, so numerical verifications will be performed for moderate values of z. To check the quality of our results, we compare graphically the ν-zeros given by the asymptotic expansion (7) with those of the function. The computations are done in Python.

n	1	2	3	4	5
Dichotomy	143.8036	153.0062	160.6533	167.4948	173.8159
Asymptotic expansion	143.6623	152.5775	159.8764	166.3272	172.2242
Relative error	0.0009	0,0028	0,0048	0,0069	0,0091

Figure 1: Comparison of ν-zeros of $D_\nu(z)$ for $z = 23$ with the asymptotic development (7)

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Figure 1 provides the graph of the Parabolic Cylinder functions $D_\nu(23)$ in blue. The small red crosses mark the ν-zeros calculated with the formula (7).

We observe that, although the asymptotic formula (7) is for large z, for $z = 23$ we already obtain acceptable estimates. We clearly see the loss of accuracy, but numerical right shift of the ν-zeros estimated with (7) can be explained as follows. Since simulations cannot be performed for very large z, as n increase, the zeros of Airy function become dominant on (7). The quantity $-z^{\frac{2}{3}}2^{-\frac{2}{3}}a_n > 0$ becomes too small, which involves a right shift on the simulation. This example shows that our formula allows to evaluate the first five ν-zeros even for moderates values of z.

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