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SOME REMARKS ON A RECENT CRITICAL POINT RESULT OF NONSMOOTH ANALYSIS

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The aim of this paper is to investigate some consequences of a nonsmooth version, established in [13], of Ghoussoub's general min-max principle [8, Theorem 1]. An application to a class of elliptic variational-hemivariational inequalities is also pointed out.

1. Introduction

In a recent paper [13], a general min-max principle for C^1-functions obtained by Ghoussoub [8, Theorem 1] has been extended to functionals f, on an infinite dimensional Banach space X, fulfilling the structural hypothesis

(H'_f) $f(x) := \Phi(x) + \psi(x)$ *for all* $x \in X$*, where* $\Phi : X \to \mathbb{R}$ *is locally Lipschitz continuous, while* $\psi : X \to \mathbb{R} \cup \{+\infty\}$ *turns out convex, proper, and lower semicontinuous. Moreover,* ψ *is continuous on any nonempty compact set* $A \subseteq X$ *such that* $\sup_{x\in A} \psi(x) < +\infty$.

Likewise the C^1-setting, this result leads to a nonsmooth version [12, Theorem 3.1] of the famous critical point theorem in presence of splitting established by Brézis and Nirenberg [2, Theorem 4].

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The main purpose of this paper is to present some consequences of both Theorem 3.1 in [13] and Theorem 3.1 of [12]. Section 2 is devoted to basic definitions and preliminary results. In Section 3 we first point out an immediate but useful consequence of Theorem 3.1 in [13]; see Theorem 3.2 below. In the locally Lipschitz continuous case, this result has already been obtained by X. Wu [15] through a different and longer proof. The above-mentioned nonsmooth version of Brézis-Nirenberg's critical point theorem is then presented and discussed; vide Theorem 3.3 below. Finally, Section 4 contains an application of Theorem 3.3 to an elliptic variational-hemivariational inequality problem. More precisely, let Ω be a nonempty, bounded, open subset of the real Euclidean N-space $(\mathbb{R}^N, |\cdot|)$, $N \geq 3$, having a smooth boundary $\partial\Omega$, and let

$$\mathscr{G}(u) := \int_\Omega G(x, u(x))dx, \quad u \in H_0^1(\Omega),$$

where $G(x,\xi) := \int_0^\xi -g(x,t)dt$ for all $(x,\xi) \in \Omega \times \mathbb{R}$, and

(h$_1$) $g : \Omega \times \mathbb{R} \to \mathbb{R}$ *is a locally bounded measurable function such that*

$$-\infty < \liminf_{|t|\to\infty} \frac{g(x,t)}{t} \leq \limsup_{|t|\to\infty} \frac{g(x,t)}{t} < \lambda_1,$$

uniformly in $x \in \Omega$.

Here, as usual, λ_1 denotes the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. The function $\mathscr{G}$ is well defined and locally Lipschitz continuous. Hence, we can consider its generalized directional derivative $\mathscr{G}^0$ in the sense of Clarke [4].

Let K be a suitable nonempty, convex, closed subset of $H_0^1(\Omega)$ and let (P_K) denote the following elliptic variational-hemivariational inequality problem:
Find $u \in K$ *such that*

$$-\int_\Omega \nabla u(x) \cdot \nabla(v-u)(x)dx \leq \mathscr{G}^0(u; v-u)$$

for all $v \in K$.
We shall prove that if g satisfies appropriate growth conditions, then (P_K) possesses at least two nontrivial solutions; see Theorem 4.1. Moreover, when

$$g^-(x,t) := \lim_{\delta\to 0} \inf_{|\xi-t|<\delta} g(x,\xi), \ \ g^+(x,t) := \lim_{\delta\to 0} \sup_{|\xi-t|<\delta} g(x,\xi),$$

turn out to be superposition measurable and $K := H_0^1(\Omega)$ one actually has two nontrivial solutions of the following multivalued Dirichlet problem:

Find $u \in H_0^1(\Omega)$ *such that*

$$\begin{cases} -\Delta u \in [g^-(x,u), g^+(x,u)] & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{W}$$

Let us finally note that Problem (W) has previously been investigated in [15] under different assumptions on the datum g; vide Remark 4.4.

2. Basic definitions and auxiliary results

Let $(X, \|\cdot\|)$ be a real Banach space. If V is a subset of X, we write $\overline{V}$ for the closure of V and ∂V for its boundary. When V is nonempty, $x \in X$, and $\delta > 0$, we define $B(x,\delta) := \{z \in X : \|z-x\| < \delta\}$ and

$$d(x,V) := \inf_{z \in V} \|x - z\|.$$

Given $x, z \in X$, the symbol $[x,z]$ indicates the line segment joining x to z, namely

$$[x,z] := \{(1-t)x + tz : t \in [0,1]\}.$$

We denote by X^* the dual space of X, while $\langle \cdot, \cdot \rangle$ stands for the duality pairing between X^* and X. A function $\Phi : X \to \mathbb{R}$ is called locally Lipschitz continuous when to every $x \in X$ there correspond a neighbourhood V_x of x and a constant $L_x \geq 0$ such that

$$|\Phi(z) - \Phi(w)| \leq L_x \|z - w\| \quad \forall z, w \in V_x.$$

If $x, z \in X$, we write $\Phi^0(x;z)$ for the generalized directional derivative of Φ at the point x along the direction z, i.e.,

$$\Phi^0(x;z) := \limsup_{w \to x,\, t \to 0^+} \Phi(w + tz) - \frac{\Phi(w)}{t}.$$

It is known [4, Proposition 2.1.1] that Φ^0 is upper semicontinuous on $X \times X$. The generalized gradient of the function Φ in x, denoted by $\partial\Phi(x)$, is the set

$$\partial\Phi(x) := \left\{ x^* \in X^* : \langle x^*, z \rangle \leq \Phi^0(x;z) \, \forall z \in X \right\}.$$

Proposition 2.1.2 of [4] ensures that $\partial\Phi(x)$ turns out nonempty, convex, in addition to weak* compact.
Let f be a function on X satisfying the structural hypothesis (H'_f). Put $D_\psi := \{x \in X : \psi(x) < +\infty\}$. Since ψ is continuous on $\mathrm{int}(D_\psi)$ (see for instance [5,

Exercise 1, p. 296]), the same holds regarding f. We say that $x \in D_\psi$ is a critical point of f when

$$\Phi^0(x;z-x)+\psi(z)-\psi(x)\geq 0 \quad \forall z\in X\,.$$

If $\psi \equiv 0$, it clearly signifies $0 \in \partial\Phi(x)$, namely x is a critical point of Φ according to [3, Definition 2.1].

Let S be a nonempty closed subset of X. The function f is said to fulfil the Palais-Smale condition at the level c and around the set S provided

$(\mathrm{PS})_{S,c}$ *Every sequence* $\{x_n\}\subseteq X$ *such that* $d(x_n,S)\to 0$, $f(x_n)\to c$, *and*

$$\Phi^0(x_n;x-x_n)+\psi(x)-\psi(x_n)\geq -\varepsilon_n\|x-x_n\| \tag{1}$$

for all $n\in\mathbb{N}$ *and* $x\in X$, *where* $\varepsilon_n\to 0^+$, *possesses a convergent subsequence.*

When $S = X$ we simply write $(\mathrm{PS})_c$ in place of $(\mathrm{PS})_{S,c}$. Moreover f satisfies $(\mathrm{PS})_f$ means that $(\mathrm{PS})_c$ hold true at any level c.

3. Some remarks on a recent critical point theorem

Let B be a nonempty closed subset of X and let $\mathscr{F}$ be a class of nonempty compact sets in X. We say that $\mathscr{F}$ is a homotopy-stable family with extended boundary B when for every $A\in\mathscr{F}$ and every $\eta\in C^0([0,1]\times X,X)$ such that $\eta(t,x)=x$ in $(\{0\}\times X)\cup([0,1]\times B)$ one has $\eta(\{1\}\times A)\in\mathscr{F}$. The following assumptions will be posited in the sequel.

(a_1) $\mathscr{F}$ *is a homotopy-stable family with extended boundary B, the function f fulfills condition* (H'_f), *and*

$$c=\inf_{A\in\mathscr{F}}\sup_{x\in A} f(x)<+\infty.$$

(a_2) *There exists a closed subset F of X such that*

$$(A\cap F)\setminus B\neq\emptyset \quad \forall A\in\mathscr{F}, \tag{2}$$

and, moreover,

$$\sup_{x\in B} f(x)\leq \inf_{x\in F} f(x). \tag{3}$$

Gathering (a_1) and (a_2) together one has

$$\inf_{x\in F} f(x)\leq c. \tag{4}$$

The next result [13, Theorem 3.1] holds.

Theorem 3.1. *Let* (a_1) *and* (a_2) *be satisfied. Then to every sequence* $\{A_n\} \subseteq \mathscr{F}$ *such that* $\lim_{n\to+\infty} \sup_{x\in A_n} f(x) = c$ *there corresponds a sequence* $\{x_n\} \subseteq X \setminus B$ *having the following properties:*

- (i_1) $\lim_{n\to+\infty} f(x_n) = c$.
- (i_2) $\Phi^0(x_n; z - x_n) + \psi(z) - \psi(x_n) \geq -\varepsilon_n \|z - x_n\|$, $\forall n \in \mathbb{N}$, $z \in X$, *where* $\varepsilon_n \to 0^+$.
- (i_3) $\lim_{n\to+\infty} d(x_n, F) = 0$ *provided* $\inf_{x\in F} f(x) = c$.
- (i_4) $\lim_{n\to+\infty} d(x_n, A_n) = 0$.

Now, let X be reflexive, let K be a compact metric space, and let K^* be a nonempty closed subset of K. Define $\mathscr{A} = \{p \in C^0(K,X) : p|_{K^*} = p^*\}$, where $p^* : K^* \to X$ is a fixed continuous function. If

$$c := \inf_{p\in\mathscr{A}} \sup_{x\in K} f(p(x)),$$

then $c \geq \sup_{x\in K^*} f(p^*(x))$.

An immediate consequence of Theorem 3.1 is the following.

Theorem 3.2. *Let the function* $f : X \to \mathbb{R} \cup \{+\infty\}$ *fulfill the following assumptions in addition to* (H'_f).

- (j_1) $\sup_{x\in K} f(p(x)) < +\infty$ *for some* $p \in \mathscr{A}$.
- (j_2) *There exists a closed subset* D *of* X *such that* $(p(K) \cap D) \setminus p^*(K^*) \neq \emptyset$ *for every* $p \in \mathscr{A}$ *and, moreover,* $\sup_{x\in K^*} f(p^*(x)) \leq \inf_{x\in D} f(x)$.

Then there is a sequence $\{u_n\} \subseteq X$ *having properties* (i_1)–(i_3), *with* $F := D$. *If, in addition,* f *satisfies condition* $(\mathrm{PS})_c$, *then it has a critical point* $u \in D$ *such that* $f(u) = c$.

Proof. Define $B := p^*(K^*)$. Obviously, setting

$$\mathscr{F} := \{p(K) : p \in \mathscr{A}\}$$

we obtain a homotopy-stable family with extended boundary B. Moreover, thanks (j_1), $c = \inf_{p\in\mathscr{A}} \sup_{x\in K} f(p(x)) < +\infty$. Hence, (a_1) holds true. Bearing in mind (j_2) yields (a_2). Now, the conclusion is an immediate consequence of Theorem 3.1. □

Remark 3.3. In the locally Lipschitz continuous case, Theorem 3.2 has been established by X. Wu in [15] using different methods and a longer proof. We also point out that weaker Palais-Smale's type compactness conditions might be adopted once one exploits Theorem 3.1 of [11].

Now, let X be reflexive and let f be a function from X into $\mathbb{R} \cup \{+\infty\}$. The following hypothesis will be posited in the sequel:

(f_1) *f is bounded below and fulfils $(\mathrm{PS})_f$ besides (H'_f),*

(f_2) *x_0 is a global minimum point of the function f.*

Since under (f_1) each minimizing sequence for f possesses a convergent subsequence (see [12]), the function f must attain its minimum at some point $x_0 \in X$. So, (f_2) is quite natural. Suppose further

$$X := X_1 \oplus X_2\,,$$

where $\dim(X_1) > 0$, while $0 < \dim(X_2) < \infty$.

The following nonsmooth version of the famous Brézis-Nirenberg critical point theorem in presence of splitting is proved in [12].

Theorem 3.4. *If* (f_1)–(f_2) *are satisfied,* $\inf_{x\in X} f(x) < f(0)$, $f(0) = 0$, *and, moreover,*

(f_3) *the set $\{x \in X : f(x) < a\}$ is open for some constant $a > 0$,*

(f_4) *there exists an $r \in]0, \frac{\|x_0\|}{2}[$ such that $f|_{\overline{B}(0,r)\cap X_1} \geq 0$, $f|_{\overline{B}(0,r)\cap X_2} \leq 0$, and $f|_{\partial B(0,r)\cap X_2} < 0$,*

then the function f possesses at least two nontrivial critical points.

Remark 3.5. Hypothesis (f_4) is obviously fulfilled in the meaningful special case when

(f'_4) *for some $r > 0$ one has $f|_{\overline{B}(0,r)\cap X_1} \geq 0$ as well as $f|_{\overline{B}(0,r)\cap X_2\setminus\{0\}} < 0$,*

namely 0 turns out a local minimum of $f|_{X_1}$ and a proper local maximum for $f|_{X_2}$.

Assuming that f is a locally Lipschitz continuous, i.e. $\psi \equiv 0$, and substituting hypothesis (f_4) with

(f^*_4) *there exists a positive constant r such that $f|_{\overline{B}(0,r)\cap X_1} \geq 0$, $f|_{\overline{B}(0,r)\cap X_2} \leq 0$,*

one can get Theorem 2.3 of [15].

4. Application

Let Ω and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be as in the Introduction. The main purpose of this section is to investigate a variational-hemivariational inequality version of Problem (W). The symbol $H_0^1(\Omega)$ indicates the closure of $C_0^\infty(\Omega)$ in $W^{1,2}(\Omega)$ with respect to the norm

$$\|u\| := \left(\int_\Omega |\nabla u(x)|^2 dx \right)^{\frac{1}{2}} .$$

Denote by 2^* the critical exponent for the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$. Recall that $2^* = \frac{2N}{N-2}$, if $p \in [1, 2^*]$ then there exists a positive constant c_p such that

$$\|u\|_{L^p(\Omega)} \le c_p \|u\|, \quad u \in H_0^1(\Omega), \tag{5}$$

and, in particular, the embedding is compact whenever $p \in [1, 2^*[$; see e.g.[14, Proposition B.7].
Consider the following eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6}$$

It is well known [7, Section 8.12] that (6) possesses a sequence $\{\lambda_n\}$ of eigenvalues fulfilling $0 < \lambda_1 < \lambda_2 \le \cdots \le \lambda_n \le \cdots$. The number of times an eigenvalue appears in the sequence equals its multiplicity.
Let $\{\varphi_n\}$ be a corresponding sequence of eigenfunctions normalized as follows:

$$\int_\Omega |\nabla \varphi_n(x)|^2 dx = \lambda_n \int_\Omega \varphi_n(x)^2 dx = \lambda_n \quad \forall n \in \mathbb{N}; \tag{7}$$

$$\int_\Omega \nabla \varphi_m(x) \cdot \nabla \varphi_n(x) dx = \int_\Omega \varphi_m(x) \varphi_n(x) dx = 0, \tag{8}$$

provided $m, n \in \mathbb{N}$ and $m \neq n$.

By (h_1) there are constants $\varepsilon \in]0, \lambda_1[$ and $r > 0$ such that

$$g(x,t) < (\lambda_1 - \varepsilon) t \tag{9}$$

for all $|t| \ge r$ and $x \in \Omega$. Since g is locally bounded, we also have

$$M := \sup_{(x,t) \in \Omega \times [-r,r]} |g(x,t)| < +\infty. \tag{10}$$

Now, let $\kappa > 0$. Define

$$r_\kappa := \sqrt{\frac{\kappa + Mr\mu(\Omega)}{\varepsilon} 2\lambda_1}, \tag{11}$$

where $\mu(\Omega)$ is the Lebesgue measure of Ω. A set $K \subseteq H_0^1(\Omega)$ is called of type (K) provided

(K) *K turns out to be nonempty, convex, closed in $H_0^1(\Omega)$. Moreover, there exists a $\kappa > 0$ such that $\overline{B}(0, r_\kappa) \subset K$.*

The following result provides an application of Theorem 3.3.

Theorem 4.1. *Let g satisfy condition (h_1) and let K be of type (K). Assume that there exists an integer $k \geq 1$ such that*

$$\lambda_k < \lambda_{k+1}. \tag{12}$$

If, moreover,

$$\liminf_{\xi \to 0} \frac{\int_0^\xi g(x,t)dt}{|\xi|^2} > \frac{\lambda_k}{2}, \tag{13}$$

and

$$\limsup_{|\xi| \to 0} \frac{g(x,\xi)}{\xi} < \lambda_{k+1} \tag{14}$$

uniformly in $x \in \Omega$, then Problem (P_K) possesses at least two nontrivial solutions.

Proof. Pick $X := H_0^1(\Omega)$, $p \in]2, 2^*[$ and define, whenever $u \in X$,

$$\Phi(u) := \frac{1}{2} \int_\Omega |\nabla u(x)|^2 dx + \mathscr{G}(u),$$

as well as

$$\psi(u) := \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{otherwise,} \end{cases} \qquad f(u) := \Phi(u) + \psi(u).$$

Owing to (h_1) the function $\Phi : X \to \mathbb{R}$ turns out to be locally Lipschitz continuous. Consequently, f satisfies condition (H'_f).
We shall prove that f is bounded from below and coercive. By (9) and (10) one has

$$\int_0^\xi g(x,t)dt \leq Mr + \frac{1}{2}(\lambda_1 - \varepsilon)\xi^2 \quad \forall \xi \in \mathbb{R}. \tag{15}$$

Hence,

$$f(u) \geq \Phi(u) =$$
$$\frac{1}{2}\|u\|^2 - \int_\Omega dx \int_0^{u(x)} g(x,t)dt$$
$$\geq \frac{1}{2}\|u\|^2 - \int_\Omega \left[Mr + \frac{1}{2}(\lambda_1 - \varepsilon)|u(x)|^2\right] dx =$$
$$= \frac{1}{2}\|u\|^2 - \frac{1}{2}(\lambda_1 - \varepsilon)\int_\Omega |u(x)|^2 dx - Mr\mu(\Omega).$$

From $\|u\|_{L^2(\Omega)} \leq \frac{1}{\sqrt{\lambda_1}}\|u\|$ (see for instance [7, p. 213]) it follows that

$$f(u) \geq \frac{1}{2}\|u\|^2 - \frac{1}{2}\left(1 - \frac{\varepsilon}{\lambda_1}\right)\|u\|^2 - Mr\mu(\Omega).$$

Thus,

$$f(u) \geq \frac{\varepsilon}{2\lambda_1}\|u\|^2 - Mr\mu(\Omega) \;\; \forall u \in X, \tag{16}$$

which shows the claim.

Let us next show that the function f satisfies condition $(\mathrm{PS})_f$. So, pick a sequence $\{u_n\} \subseteq X$ such that $\{f(u_n)\}$ is bounded and

$$\Phi^0(u_n; v - u_n) + \psi(v) - \psi(u_n) \geq -\varepsilon_n\|v - u_n\| \tag{17}$$

for all $n \in \mathbb{N}$, $v \in X$, where $\varepsilon_n \to 0^+$. One evidently has $\{u_n\} \subseteq K$. Since f is coercive, the sequence $\{u_n\}$ turns out bounded. Thus, passing to a subsequence if necessary, we may suppose both $u_n \rightharpoonup u$ in X and $u_n \to u$ in $L^2(\Omega)$. The point u belongs to K because this set is weakly closed. Exploiting (17) with $v := u$ we then get

$$\int_\Omega \nabla u_n(x) \cdot \nabla(u - u_n)(x)dx + \mathscr{G}^0(u_n; u - u_n) \geq -\varepsilon_n\|u - u_n\| \quad \forall n \in \mathbb{N}. \tag{18}$$

The upper semicontinuity of $\mathscr{G}^0$ on $L^2(\Omega) \times L^2(\Omega)$ forces

$$\limsup_{n\to+\infty} \mathscr{G}^0(u_n; u - u_n) \leq \mathscr{G}^0(u; 0) = 0. \tag{19}$$

By (19), besides the weak convergence of $\{u_n\}$ to u, inequality (18) yields, as $n \to +\infty$,

$$\limsup_{n\to\infty} \int_\Omega |\nabla u_n(x)|^2 dx \leq \int_\Omega |\nabla u(x)|^2 dx.$$

Hence, thanks to [1, Proposition III.30], $u_n \to u$ in X, i.e., hypothesis (f_1) of Theorem 3.3 is fulfilled.

The next step is to verify (f_3). Since K is of type (K), there exists a $\kappa > 0$ such that the set $\{u \in X : f(u) < \kappa\}$ is open. Indeed, through (16) we obtain

$$\{u \in X : f(u) < \kappa\} \subseteq B(0, r_\kappa) \subseteq K.$$

Consequently,

$$\{u \in X : f(u) < \kappa\} = \{u \in K : \Phi(u) < \kappa\} = \{u \in X : \Phi(u) < \kappa\},$$

which is open.

Let $X_2 := \mathrm{span}\{\varphi_1, \dots, \varphi_k\}$ and let $X_1 := X_2^\perp$, where the orthogonal complement is taken in $H_0^1(\Omega)$. One clearly has $X = X_1 \oplus X_2$, $\dim(X_1) > 0$, and $0 < \dim(X_2) < +\infty$. Due to (13) there exists a $\delta > 0$ such that

$$\int_0^\xi g(x,t)dt > \frac{\lambda_k}{2}|\xi|^2,$$

provided $0 < |\xi| < \delta$. Since X_2 is finite dimensional, we can find a positive constant $\rho_1 < r_\kappa$ such that if $u \in X_2$ and $\|u\| \le \rho_1$, then $\|u\|_{L^\infty(\Omega)} < \delta$. So, $0 < \|u\|_{L^\infty(\Omega)} < \delta$ for all $u \in \overline{B}(0,\rho_1) \cap X_2 \setminus \{0\}$, which forces $0 \le |u(x)| < \delta$ almost everywhere in Ω as well as $0 < |u(x)|$ in $\Omega_0 \subseteq \Omega$ with $\mu(\Omega_0) > 0$. Consequently,

$$\int_0^{u(x)} g(x,t)dt \ge \frac{\lambda_k}{2}|u(x)|^2$$

for almost all $x \in \Omega$ and with a strict inequality in Ω_0. Now if $u \in X_2$, $u = \sum_{i=1}^k \alpha_i \varphi_i$, for suitable $\alpha_i \in \mathbb{R}$, $i = 1, ..., k$. Owing (7) and (8) one has

$$\|u\|^2 = \sum_{i=1}^k \alpha_i^2 \int_\Omega |\nabla \varphi_i(x)|^2 dx = \sum_{i=1}^k \alpha_i^2 \lambda_i \int_\Omega \varphi_i(x)^2 dx \le \lambda_k \int_\Omega |u(x)|^2 dx.$$

This implies

$$\begin{aligned} f(u) = \frac{1}{2}\|u\|^2 + \int_\Omega G(x,u(x))dx & \\ &\le \frac{\lambda_k}{2}\int_\Omega |u(x)|^2 dx - \int_\Omega \Big[\int_0^{u(x)} g(x,t)dt\Big]dx \\ &= \int_\Omega \Big[\frac{\lambda_k}{2}|u(x)|^2 - \int_0^{u(x)} g(x,t)dt\Big]dx < 0, \end{aligned}$$

which clearly means

$$f(u) < 0 \quad \forall u \in \overline{B}(0,\rho_1) \cap X_2 \setminus \{0\}. \tag{20}$$

By (14) there exist $\lambda \in]0,\lambda_{k+1}[$ and $\sigma \in]0,r[$ such that $\frac{g(x,t)}{t} < \lambda$ for every $|t| \in]0,\sigma[$ and $x \in \Omega$. Hence

$$\int_{|u(x)|<\sigma} \Big[\int_0^{u(x)} g(x,t)dt\Big]dx \leq \frac{\lambda}{2}\int_\Omega |u(x)|^2 dx. \tag{21}$$

Due to (15), one has

$$G(x,\xi) \geq -Mr - \frac{1}{2}(\lambda_1 - \varepsilon)\xi^2 \geq -\Big(\frac{Mr}{\sigma^p} + \frac{\lambda_1-\varepsilon}{\sigma^{p-2}}\Big)|\xi|^p, \tag{22}$$

provided $|\xi| \geq \sigma$. The Sobolev embedding theorem gives

$$\int_{|u(x)|\geq\sigma} G(x,u(x))dx \geq -\Big(\frac{Mr}{\sigma^p} + \frac{\lambda_1-\varepsilon}{\sigma^{p-2}}\Big)\|u\|_{L^p(\Omega)} \geq -c^*\|u\|^p, \tag{23}$$

where $c^* := \Big(\frac{Mr}{\sigma^p} + \frac{\lambda_1-\varepsilon}{\sigma^{p-2}}\Big)c_p^p$. Now if $u \in X_1$, $u = \sum_{j=k+1}^{+\infty}\beta_j\varphi_j$, for suitable $\beta_j \in \mathbb{R}$, $j = k+1,\ldots$ Owing (7) and (8), one has

$$\|u\|^2_{L^2(\Omega)} = \sum_{j=k+1}^{+\infty}\beta_j^2\int_\Omega \varphi_j(x)^2 dx = \sum_{j=k+1}^{+\infty}\frac{\beta_j^2}{\lambda_j}\int_\Omega |\nabla\varphi_j(x)|^2 dx \leq \frac{1}{\lambda_{k+1}}\|u\|^2,$$

i.e.,

$$\|u\|_{L^2(\Omega)} \leq \frac{1}{\sqrt{\lambda_{k+1}}}\|u\| \tag{24}$$

for each $u \in X_1$. Then, by (21), (23) and (24) we get

$$\begin{aligned} f(u) &= \frac{1}{2}\|u\|^2 + \int_\Omega G(x,u(x))dx = \\ &= \frac{1}{2}\|u\|^2 - \int_{|u(x)|<\sigma}\Big[\int_0^{u(x)} g(x,t)dt\Big]dx + \int_{|u(x)|\geq\sigma} G(x,u(x))dx \geq \\ &\geq \frac{1}{2}\Big(1 - \frac{\lambda}{\lambda_{k+1}}\Big)\|u\|^2 - c^*\|u\|^p. \end{aligned} \tag{25}$$

Since $p > 2$, putting $\rho_2 := \Big[\frac{1}{2c^*}\Big(1 - \frac{\lambda}{\lambda_{k+1}}\Big)\Big]^{1/(p-2)}$, from (25) it follows that

$$f(u) \geq 0 \quad \forall u \in \overline{B}(0,\rho_2)\cap X_1. \tag{26}$$

Choose $\rho := \min\{\rho_1,\rho_2\}$. Then

$$f(u) < 0 \quad \forall u \in \overline{B}(0,\rho)\cap X_2,$$

and

$$f(u) \geq 0 \quad \forall u \in \overline{B}(0,\rho) \cap X_1.$$

Bearing in mind Remark 3.5, this immediately yields (f_4).
Finally, observe that by (20) one has $\inf_{u \in X} f(u) < 0$. We are in a position now to apply Theorem 3.3. Thus there exist at least two points $u_1, u_2 \in X \setminus \{0\}$ such that

$$\Phi^0(u_i; v - u_i) + \psi(v) - \psi(u_i) \geq 0$$

for all $v \in X$, $i = 1,2$. The choice of ψ gives both $u_i \in K$ and $\Phi^0(u_i; v - u_i) \geq 0$, $v \in K$, $i = 1,2$, namely u_1, u_2 turn out to be nontrivial solutions of Problem (P_K), which completes the proof. □

Example 4.2. The aim of this example is to exhibit a non-trivial case of set in $H_0^1(\Omega)$ of type (K). Let $h : H_0^1(\Omega) \to \mathbb{R}$ be a weakly continuous and convex function. For $\overline{\kappa} > 0$ fixed, put

$$r_{\overline{\kappa}} := \sqrt{\frac{\overline{\kappa} + Mr\mu(\Omega)}{\varepsilon} 2\lambda_1}, \tag{27}$$

with the same notation as before. The ball $\overline{B}(0, r_{\overline{\kappa}})$ is a weakly compact subset of $H_0^1(\Omega)$, hence $h|_{\overline{B}(0,r_{\overline{\kappa}})}$ admits a global maximum. Then the set

$$K := \{u \in H_0^1(\Omega) : h(u) \leq \alpha + 1\},$$

where $\alpha := \max_{u \in \overline{B}(0,r_{\overline{\kappa}})} h(u)$, is a subset of $H_0^1(\Omega)$ of type (K).

Remark 4.3. Recall that a function $q : \Omega \times \mathbb{R} \to \mathbb{R}$ is called superposition measurable when $x \mapsto q(x, u(x))$ is measurable for all measurable $u : \Omega \to \mathbb{R}$. Let $K := H_0^1(\Omega)$. Assume that:

(h_2) *The functions*

$$g^-(x,t) := \lim_{\delta \to 0} \inf_{|\xi - t| < \delta} g(x,\xi), \;\; g^+(x,t) := \lim_{\delta \to 0} \sup_{|\xi - t| < \delta} g(x,\xi)$$

are superposition measurable.

If $u \in H_0^1(\Omega)$ complies with

$$-\int_\Omega \nabla u(x) \cdot \nabla (v - u)(x) dx \leq \mathscr{G}^0(u; v - u) \;\; \forall v \in H_0^1(\Omega),$$

then

$$\int_{\Omega} -\Delta u(x) \cdot w(x) dx \leq (-\mathscr{G})^0(u; w), \quad w \in H_0^1(\Omega).$$

This implies

$$-\Delta u \in \partial(-\mathscr{G})(u) \subseteq \{w \in L^2(\Omega) : g^-(x, u(x)) \leq w(x) \leq g^+(x, u(x)) \text{ a.e. in } \Omega\},$$

i.e.,

$$-\Delta u(x) \in [g^-(x, u(x)), g^+(x, u(x))] \quad \text{for almost all } x \in \Omega.$$

Hence, Theorem 4.1 gives at least two nontrivial solutions of Problem (W).

Remark 4.4. Very recently, in [15], an existence result has been obtained by using hypotheses (h_1),(h_2),(14), and the following one:
There exists a $\delta > 0$ such that

$$\lambda_k \leq \frac{g(x,t)}{t}$$

for all $0 < |t| < \delta$ and almost all $x \in \Omega$; cf. [15, Theorem 3.1]. It is simple matter to see that this result and Theorem 4.1 above are mutually independent.

Remark 4.5. Let Ω and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be as in the Introduction and let A be a uniformly elliptic operator of the form

$$Au = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\Big(a^{ij}(x)\frac{\partial u}{\partial x_i}\Big) + c(x)u,$$

where $a^{ij} : \Omega \to \mathbb{R}$, $i, j = 1, ..., N$, satisfy the following conditions:

A_1) $a^{ij} = a^{ji} \in L^\infty(\Omega)$;

A_2) $c \in L^\infty(\Omega)$ and $c \geq 0$ almost everywhere in Ω;

A_3) There is a $\gamma > 0$ such that

$$\sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j \geq \gamma|\xi|^2 \qquad \forall x \in \Omega,\ \xi \in \mathbb{R}^N.$$

On $H_0^1(\Omega)$ we consider the norm (equivalent to the usual one)

$$\|u\| := \langle Au, u\rangle^{1/2}$$

induced by the inner product

$$\langle Au, v\rangle = \int_{\Omega} \Big[\sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_j} + c(x)u(x)v(x)\Big] dx,$$

see for instance [6, p. 650]. It is well known (vide Proposition 6.1.15, p. 652, and Theorem 6.1.21, p. 654, of [6]) that the eigenvalue Dirichlet problem

$$\begin{cases} Au = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{28}$$

possesses a sequence $\{\lambda_n\}$ of eigenvalues fulfilling

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$$

and a corresponding sequence $\{\varphi_n\}$ of eigenfunctions normalized as follows:

$$\langle A\varphi_n, \varphi_n \rangle = \lambda_n \int_\Omega \varphi_n(x)^2 dx = \lambda_n \quad \forall n \in \mathbb{N}; \tag{29}$$

$$\langle A\varphi_n, \varphi_m \rangle = \int_\Omega \varphi_m(x)\varphi_n(x) dx = 0, \tag{30}$$

provided $m, n \in \mathbb{N}$ and $m \neq n$; see [6, Proposition 6.1.19, p. 653].
Arguing as in the proof of Theorem 4.1, but with Au in place of $-\Delta u, u \in H_0^1(\Omega)$, it is possible to obtain two nontrivial solutions of the following variational-hemivariational inequality problem:

$$\begin{cases} u \in K, \\ \langle Au, v-u \rangle + \mathscr{G}^0(u; v-u) \geq 0 \quad \forall v \in K, \end{cases} \tag{$\mathrm{P}_{A,K}$}$$

where K is of type (K) in $H_0^1(\Omega)$. Recently, in [16], X. Wu and T. Leng studied the existence of two nontrivial solutions in $H_0^1(\Omega)$ to the problem

$$\begin{cases} Au = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{31}$$

where g is a locally bounded Carathéodory function, under assumption (h_1) and the following one:
There exists an integer $k \geq 1$ *such that*

$$\lambda_k \leq \liminf_{t \to 0} \frac{g(x,t)}{t} \leq \limsup_{t \to 0} \frac{g(x,t)}{t} < \lambda_{k+1}, \tag{32}$$

uniformly for a.e. $x \in \Omega$.
We point out that if the first inequality in (32) is strict, then Theorem 1.1 of [16] is a very special case of Theorem 4.1 written for $(\mathrm{P}_{A,K})$. For example, we require only that g be a locally bounded measurable function.

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