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STABILITY OF CONSTANT EQUILIBRIA IN A KELLER–SEGEL SYSTEM WITH GRADIENT DEPENDENT CHEMOTACTIC SENSITIVITY

S. KOHATSU - T. YOKOTA

This paper deals with the Keller–Segel system with gradient dependent chemotactic sensitivity,

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u|\nabla v|^{p-2}\nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ is a bounded domain with smooth boundary, and $\chi > 0$, $p \in (1,\infty)$ are constants. The purpose of this paper is to establish stability of constant equilibria under some smallness conditions for the initial data.

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1. Introduction

In this paper we consider the following Keller–Segel system with gradient dependent chemotactic sensitivity:

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u|\nabla v|^{p-2}\nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \\ \nabla u \cdot \nu = \nabla v \cdot \nu = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ is a bounded domain with smooth boundary $\partial\Omega$, $\chi > 0$, $p \in (1,\infty)$ are constants, ν is the outward normal vector to $\partial\Omega$, u_0 and v_0 satisfy

$$\begin{cases} u_0 \in C^0(\overline{\Omega}), \quad u_0 \geq 0 \text{ in } \overline{\Omega} \quad \text{and} \quad u_0 \neq 0, \\ v_0 \in W^{1,\infty}(\Omega) \quad \text{and} \quad v_0 \geq 0 \text{ in } \overline{\Omega}. \end{cases} \tag{1.2}$$

In recent years, chemotaxis systems with the term $-\chi\nabla \cdot (u|\nabla v|^{p-2}\nabla v)$ have been studied, where the unknown functions u and v describe the density of biological species and the concentration of chemical substances, respectively. When $p = 2$, the problem (1.1) is reduced to the classical Keller–Segel system proposed in [6], and there are a lot of work on large-time behavior of solutions. In the case $n = 1$, Osaki and Yagi [12] investigated asymptotic behavior of solutions. When $n \geq 2$, Winkler [15] and Cao [2] established large-time behavior of solutions, that is,

$$\|u(\cdot,t) - \overline{u_0}\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty, \tag{1.3}$$

under smallness conditions for $\|u_0\|_{L^{\frac{n}{2}}(\Omega)}$ and $\|\nabla v_0\|_{L^n(\Omega)}$, where $\overline{u_0} := \frac{1}{|\Omega|}\int_\Omega u_0$. For the case that $\Omega = \mathbb{R}^n$, see [10] and [3]. Considering these results, our focus will here be on how do the solutions of (1.1) behave in the case $p \neq 2$, especially whether (1.3) holds true or not.

We first review previous works for some related systems with the chemotactic term $-\chi\nabla \cdot (u|\nabla v|^{p-2}\nabla v)$. To the best of our knowledge, such systems were initially studied by Negreanu and Tello [11], where they considered the simplified parabolic–elliptic system

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u|\nabla v|^{p-2}\nabla v), & x \in \Omega,\ t > 0, \\ 0 = \Delta v - \overline{u_0} + u, & x \in \Omega,\ t > 0. \end{cases} \tag{1.4}$$

They proved uniform boundedness of $u(\cdot,t)$ in $L^\infty(\Omega)$ when $p \in (1,\infty)$ $(n = 1)$, and $p \in (1, \frac{n}{n-1})$ $(n \geq 2)$. They also showed existence of infinitely many steady states in the case that $p \in (1,2)$ $(n=1)$. On the other hand, Tello [13] proved that a solution to (1.4) blows up in finite time when $p \in (\frac{n}{n-1}, 2)$ $(n \geq 3)$. Wang and Li [14] studied the parabolic–parabolic system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u|\nabla v|^{p-2}\nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - uv, & x \in \Omega,\ t > 0, \end{cases}$$

where $p \in (1,2)$. They showed global existence of weak solutions in the case that $2 \leq n < \frac{8-2(p-1)}{p-1}$, and established global existence of renormalized solutions in the complementary case $n \geq \frac{8-2(p-1)}{p-1}$.

We next focus on the problem (1.1) and state known results and our purpose. The problem (1.1) was studied by Yan and Li [17]. They obtained global existence and boundedness of weak solutions in the case $p \in (1, \frac{n}{n-1})$ $(n \geq 2)$. We note that, following their proofs in [17], one can establish the same result in the case $p \in (1,2)$ $(n=1)$. However, large-time behavior of weak solutions to (1.1) has not been investigated yet. The purpose of this paper is to reveal behavior of solutions to the problem (1.1) for general $p \in (1,\infty)$, especially we focus on the asymptotic stability (1.3). Inspired by [8, 9], since boundedness was already obtained, we will impose some smallness conditions for $\|u_0\|_{L^1(\Omega)}$.

Before we state main results, we give a definition of weak solutions to (1.1) introduced by Yan and Li [17, Definition 2.1].

Definition 1.1. Let u_0 and v_0 satisfy (1.2). Let $T > 0$. A pair (u,v) of functions is called a *weak solution* of (1.1) in $\Omega \times (0,T)$ if

(i) $u \in L^\infty(\overline{\Omega} \times [0,T))$, $v \in L^\infty(\overline{\Omega} \times [0,T)) \cap L^2([0,T); W^{1,2}(\Omega))$,

(ii) $u \geq 0$ a.e. on $\Omega \times (0,T)$, $v \geq 0$ a.e. on $\Omega \times (0,T)$,

(iii) $|\nabla v|^{p-2}\nabla v \in L^2(\overline{\Omega} \times [0,T))$,

(iv) u has the mass conservation property

$$\int_\Omega u(\cdot,t) = \int_\Omega u_0 \quad \text{for a.a. } t > 0,$$

(v) for any nonnegative $\varphi \in C_c^\infty(\overline{\Omega} \times [0,T))$,

$$-\int_0^T \int_\Omega u\varphi_t - \int_\Omega u_0 \varphi(\cdot,0) = \int_0^T \int_\Omega u \cdot \Delta\varphi + \chi \int_0^T \int_\Omega u|\nabla v|^{p-2}\nabla v \cdot \nabla\varphi$$

and

$$-\int_0^T \int_\Omega v\varphi_t - \int_\Omega v_0 \varphi(\cdot,0) = -\int_0^T \int_\Omega \nabla v \cdot \nabla\varphi - \int_0^T \int_\Omega v\varphi + \int_0^T \int_\Omega u\varphi$$

hold true.

If $(u,v):\Omega\times(0,\infty)\to\mathbb{R}^2$ is a weak solution of (1.1) in $\Omega\times(0,T)$ for all $T>0$, then (u,v) is called a *global weak solution* of (1.1).

It would be possible to choose $u_0\in L^\infty(\Omega)$ in Definition 1.1, however, for sake of simplicity, we here assume that u_0 is continuous.

We now state the main theorems. The first theorem is concerned with stability of constant equlibria $\overline{u_0}$.

Theorem 1.2. *Let $n\in\mathbb{N}$. Assume that u_0 and v_0 satisfy* (1.2)*. Let $\overline{u_0}:=\frac{1}{|\Omega|}\int_\Omega u_0$ and $m:=\|u_0\|_{L^1(\Omega)}$. Suppose that*

$$\begin{cases} p\in(1,2) & \text{if } n=1,\\ p\in\left(1,\dfrac{n}{n-1}\right) & \text{if } n\geq 2. \end{cases} \tag{1.5}$$

Then there exist a global weak solution (u,v) of (1.1) *and $t_1>0$ such that*

$$\|u(\cdot,t)-\overline{u_0}\|_{L^\infty(\Omega)}\leq Cm^p(1+m^\alpha+m^\beta)\quad \text{for all } t\geq t_1, \tag{1.6}$$

where $C>0$, $\alpha>0$ and $\beta>0$ are constants. In particular, one can find $\eta_0>0$ such that for all $\eta\in(0,\eta_0)$, whenever u_0 fulfills

$$\|u_0\|_{L^1(\Omega)}\leq\eta,$$

u satisfies

$$\|u(\cdot,t)-\overline{u_0}\|_{L^\infty(\Omega)}\leq\eta\quad \text{for all } t\geq t_1.$$

The second theorem gives asymptotic stability of $\overline{u_0}$.

Theorem 1.3. *Suppose that*

$$n=1\quad \text{and}\quad p\in[2,\infty).$$

Assume that u_0 and v_0 satisfy (1.2)*, and $u_0\in\bigcup_{\theta\in(0,1)}C^\theta(\overline{\Omega})$. Then there exist a global classical solution*

$$(u,v)\in\left(C^0(\overline{\Omega}\times[0,\infty))\cap C^{2,1}(\overline{\Omega}\times(0,\infty))\right)^2$$

of (1.1) *and $t_2>0$ such that the estimate* (1.6) *holds for all $t\geq t_2$. Moreover, one can find $\eta_0>0$ such that for all $\eta\in(0,\eta_0)$, whenever u_0 fulfills*

$$\|u_0\|_{L^1(\Omega)}\leq\eta,$$

u satisfies

$$\|u(\cdot,t)-\overline{u_0}\|_{L^\infty(\Omega)}\leq\eta e^{-h(t-t_2)}\quad \text{for all } t\geq t_2, \tag{1.7}$$

where $h>0$ is a constant. In particular,

$$\|u(\cdot,t)-\overline{u_0}\|_{L^\infty(\Omega)}\to 0\quad \text{as } t\to\infty.$$

Remark 1.4. As we consider $u-M$ and $v-M$ instead of u and v in Theorem 1.3, where $M := \min_{x\in\overline{\Omega}} u_0(x)$, the stabilization (1.7) can be established even when u_0 fulfills

$$\|u_0 - M\|_{L^1(\Omega)} \le \eta$$

which means that the variation of u_0 in $\overline{\Omega}$ is sufficiently small.

Remark 1.5. It is unknown whether asymptotic stability of $\overline{u_0}$ under the condition (1.5) holds or not.

The proofs of the main theorems are based on [9]. As to the proof of Theorem 1.2, we consider a regularized problem of which global classical solvability is known, and construct a weak solution by taking the limit of solutions of the regularized problem. In order to prove (1.6), we first obtain

$$\|u_\varepsilon(\cdot,t) - \overline{u_0}\|_{L^\infty(\Omega)} \le Cm^p(1+m^\alpha+m^\beta) \quad \text{for all } t \ge t_1$$

with some $t_1 > 0$ which is independent of ε, where u_ε is the first component of solutions to a regularized problem. Then we let $\varepsilon \to 0$ and construct a weak solution which satisfies (1.6). However, unlike in [9], we cannot obtain the estimate for $\|\nabla v_\varepsilon\|_{L^q(\Omega)}$ with large q (Lemma 3.2), so we need to modify their proofs. With regard to the proof of Theorem 1.3, we first obtain the estimate (1.6). Next, to prove (1.7), we put

$$\mathcal{S} := \left\{T \ge t_2 \mid \|u(\cdot,t) - \overline{u_0}\|_{L^\infty(\Omega)} \le \eta e^{-h(t-t_2)} \quad \forall t \in [t_2,T]\right\}$$

and define $T^* := \sup\mathcal{S} \in (t_2,\infty]$. Then, since the power of m^p in (1.6) is greater than 1, we can obtain the sharper estimate $\|u(\cdot,t) - \overline{u_0}\|_{L^\infty(\Omega)} \le \frac{1}{2}\eta e^{-h(t-t_2)}$ for $t \in (t_2,T^*)$, and hence we have $T^* = \infty$, which shows exponential decay of $u(\cdot,t) - \overline{u_0}$.

This paper is organized as follows. In Section 2, we give some useful inequalities. Section 3 is devoted to the proofs of stability of $\overline{u_0}$ (Theorem 1.2). In Section 4, we show asymptotic stability of $\overline{u_0}$ (Theorem 1.3).

Throughout this paper, we put $m := \|u_0\|_{L^1(\Omega)}$, and we denote by c_i generic positive constants.

2. Preliminaries

In this section we collect some inequalities which will be used later. The following lemma provides an estimate for certain integral, which is established in [15, Lemma 1.2].

Lemma 2.1. *Let $\kappa < 1$, $\delta < 1$, $\gamma > 0$, $\mu > 0$, and $\gamma \neq \mu$. Then there exists a constant $C = C(\kappa, \delta, \gamma, \mu) > 0$ such that*

$$\int_0^t (1 + (t-s)^{-\kappa}) e^{-\gamma(t-s)} (1 + s^{-\delta}) e^{-\mu s}\, ds \le C(1 + t^{\min\{0, 1-\kappa-\delta\}}) e^{-\min\{\gamma,\mu\} t}$$

for all $t > 0$.

We next recall L^p-L^q estimates for the Neumann heat semigroup on bounded domains. We refer to [15, Lemma 1.3] for the proofs (see also [2, Lemma 2.1]).

Lemma 2.2. *Let $(e^{t\Delta})_{t\ge 0}$ be the Neumann heat semigroup in Ω, and denote by $\lambda_1 > 0$ the first nonzero eigenvalue of $-\Delta$ in Ω under the Neumann boundary condition. Then there exist constants $C_1, \dots, C_4 > 0$ which depend only on Ω such that the following hold:*

(i) If $1 \le q \le r \le \infty$, then

$$\|e^{t\Delta} w\|_{L^r(\Omega)} \le C_1 (1 + t^{-\frac{n}{2}(\frac{1}{q} - \frac{1}{r})}) e^{-\lambda_1 t} \|w\|_{L^q(\Omega)} \quad \text{for all } t > 0$$

holds for all $w \in L^q(\Omega)$ with $\int_\Omega w = 0$.

(ii) If $1 \le q \le r \le \infty$, then

$$\|\nabla e^{t\Delta} w\|_{L^r(\Omega)} \le C_2 (1 + t^{-\frac{1}{2} - \frac{n}{2}(\frac{1}{q} - \frac{1}{r})}) e^{-\lambda_1 t} \|w\|_{L^q(\Omega)} \quad \text{for all } t > 0$$

is valid for all $w \in L^q(\Omega)$.

(iii) If $2 \le q < \infty$, then

$$\|\nabla e^{t\Delta} w\|_{L^q(\Omega)} \le C_3 e^{-\lambda_1 t} \|\nabla w\|_{L^q(\Omega)} \quad \text{for all } t > 0$$

is true for all $w \in W^{1,q}(\Omega)$.

(iv) If $1 < q \le r \le \infty$, then

$$\|e^{t\Delta} \nabla \cdot w\|_{L^r(\Omega)} \le C_4 (1 + t^{-\frac{1}{2} - \frac{n}{2}(\frac{1}{q} - \frac{1}{r})}) e^{-\lambda_1 t} \|w\|_{L^q(\Omega)} \quad \text{for all } t > 0$$

holds for all $w \in (L^q(\Omega))^n$, where $e^{t\Delta}\nabla\cdot$ is the extension of the operator $e^{t\Delta}\nabla\cdot$ on $(C_c^\infty(\Omega))^n$ to $(L^q(\Omega))^n$.

We also recall the Gagliardo–Nirenberg inequality in the following lemma, which is the special case of [4, Proposition A.1].

Lemma 2.3. *Let $q > 0$, $r \in (0, q)$, and $s \in [1, \infty]$ such that $\frac{1}{s} - \frac{1}{n} < \frac{1}{q}$. Then there exists a constant $C = C(\Omega, s, r, n) > 0$ such that for all $u \in L^r(\Omega)$ with $\nabla u \in L^s(\Omega)$,*

$$\|u\|_{L^q(\Omega)} \le C \|\nabla u\|_{L^s(\Omega)}^a \|u\|_{L^r(\Omega)}^{1-a} + C \|u\|_{L^r(\Omega)},$$

where $a := \frac{\frac{1}{r} - \frac{1}{q}}{\frac{1}{r} + \frac{1}{n} - \frac{1}{s}}$.

3. Stability of $\overline{u_0}$ when $p \in (1,2)$ if $n = 1$, and $p \in (1, \frac{n}{n-1})$ if $n \geq 2$

In this section we will show Theorem 1.2. In the following, we let $\lambda_1 > 0$ be the first nonzero eigenvalue of $-\Delta$ in Ω under the Neumann boundary condition. Also, we suppose that u_0 and v_0 satisfy (1.2), and p satisfies (1.5).

3.1. Regularized problem of (1.1)

According to the idea from [17], we consider the regularized problem

$$\begin{cases} (u_\varepsilon)_t = \Delta u_\varepsilon - \chi \nabla \cdot \left(u_\varepsilon (|\nabla v_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla v_\varepsilon \right), & x \in \Omega,\ t > 0, \\ (v_\varepsilon)_t = \Delta v_\varepsilon - v_\varepsilon + u_\varepsilon, & x \in \Omega,\ t > 0, \\ \nabla u_\varepsilon \cdot \nu = \nabla v_\varepsilon \cdot \nu = 0, & x \in \partial\Omega,\ t > 0, \\ u_\varepsilon(x,0) = u_0(x),\ v_\varepsilon(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{3.1}$$

where $\varepsilon \in (0,1)$. Global existence of classical solutions to (3.1) has already been proved in [17, Lemma 2.2 and Lemma 4.1]. Hereinafter, we let $(u_\varepsilon, v_\varepsilon)$ be the global classical solution of (3.1). We first note that u_ε satisfies

$$\|u_\varepsilon(\cdot,t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)} = m \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1). \tag{3.2}$$

The next lemma asserts that the solution of (3.1) is uniformly bounded with respect to time and ε. For the proof, see [17, Lemma 4.1].

Lemma 3.1. *Suppose that p satisfies* (1.5). *Then there exists a constant $C > 0$ such that*

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq C \tag{3.3}$$

for all $t > 0$ and $\varepsilon \in (0,1)$.

3.2. L^r-estimate for u_ε in terms of mass

We give an L^r-estimate for u_ε in terms of mass m which will be used later (Lemma 3.4). To this end, we first employ semigroup techniques to estimate ∇v_ε in $L^q(\Omega)$ for some q.

Lemma 3.2. *Let q satisfy*

$$\begin{cases} q \in [1,\infty) & \text{if } n = 1, \\ q \in \left[1, \dfrac{n}{n-1}\right) & \text{if } n \geq 2. \end{cases} \tag{3.4}$$

Then there exists a constant $C > 0$ such that

$$\|\nabla v_\varepsilon(\cdot,t)\|_{L^q(\Omega)} \leq C(1 + t^{-\frac{1}{2}}) e^{-(1+\lambda_1)t} + Cm$$

for all $t > 0$ and $\varepsilon \in (0,1)$.

Proof. Since v_ε solves the second equation of (3.1), it follows that

$$v_\varepsilon(\cdot,t) = e^{t(\Delta-1)}v_0 + \int_0^t e^{(t-\sigma)(\Delta-1)}u_\varepsilon(\cdot,\sigma)\,d\sigma$$

for all $t>0$ and $\varepsilon\in(0,1)$. Thus,

$$\|\nabla v_\varepsilon(\cdot,t)\|_{L^q(\Omega)} \le e^{-t}\|\nabla e^{t\Delta}v_0\|_{L^q(\Omega)} + \int_0^t e^{-(t-\sigma)}\|\nabla e^{(t-\sigma)\Delta}u_\varepsilon(\cdot,\sigma)\|_{L^q(\Omega)}\,d\sigma \tag{3.5}$$

for all $t>0$ and $\varepsilon\in(0,1)$. Here, according to Lemma 2.2 (ii), we have

$$e^{-t}\|\nabla e^{t\Delta}v_0\|_{L^q(\Omega)} \le c_1(1+t^{-\frac{1}{2}})e^{-(1+\lambda_1)t}\|v_0\|_{L^q(\Omega)} \tag{3.6}$$

for all $t>0$. Moreover, invoking Lemma 2.2 (ii) and (3.2), we can see that

$$\begin{aligned}
&\int_0^t e^{-(t-\sigma)}\|\nabla e^{(t-\sigma)\Delta}u_\varepsilon(\cdot,\sigma)\|_{L^q(\Omega)}\,d\sigma\\
&\quad\le c_2\int_0^t \left(1+(t-\sigma)^{-\frac{1}{2}-\frac{n}{2}\left(1-\frac{1}{q}\right)}\right)e^{-(1+\lambda_1)(t-\sigma)}\|u_\varepsilon(\cdot,\sigma)\|_{L^1(\Omega)}\,d\sigma\\
&\quad= c_2m\int_0^t \left(1+\sigma^{-\frac{1}{2}-\frac{n}{2}\left(1-\frac{1}{q}\right)}\right)e^{-(1+\lambda_1)\sigma}\,d\sigma\\
&\quad\le c_2m\int_0^\infty \left(1+\sigma^{-\frac{1}{2}-\frac{n}{2}\left(1-\frac{1}{q}\right)}\right)e^{-(1+\lambda_1)\sigma}\,d\sigma
\end{aligned} \tag{3.7}$$

for all $t>0$ and $\varepsilon\in(0,1)$. Since the integral $\int_0^\infty(1+\sigma^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})})e^{-(1+\lambda_1)\sigma}\,d\sigma$ is finite according to (3.4), the claim follows from (3.5), (3.6) and (3.7). □

From Lemma 3.2 we prove an L^r-estimate for u_ε in terms of mass m.

Lemma 3.3. *Suppose that p satisfies* (1.5). *Let $r\in(1,\infty)$. Then there exists a constant $C>0$ such that*

$$\|u_\varepsilon(\cdot,t)\|_{L^r(\Omega)} \le Ce^{-\frac{t-1}{r}} + Cm\left(1+m^{\frac{2(p-1)}{r}}+m^{\frac{2(p-1)}{r(1-a)}}\right)$$

for all $t>1$ and $\varepsilon\in(0,1)$, with some $a\in(0,1)$.

Proof. Let $r\in(1,\infty)$. Multiplying the first equation of (3.1) by u_ε^{r-1}, integrating by parts yield, and noting that $p<2$, we obtain

$$\begin{aligned}
\frac{1}{r}\frac{d}{dt}\int_\Omega u_\varepsilon^r + \frac{4(r-1)}{r^2}\int_\Omega |\nabla u_\varepsilon^{\frac{r}{2}}|^2 &= (r-1)\chi\int_\Omega (|\nabla v_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}(\nabla v_\varepsilon\cdot\nabla u_\varepsilon)u_\varepsilon^{r-1}\\
&\le (r-1)\chi\int_\Omega |\nabla v_\varepsilon|^{p-1}|\nabla u_\varepsilon|u_\varepsilon^{r-1}\\
&= \frac{2(r-1)\chi}{r}\int_\Omega |\nabla v_\varepsilon|^{p-1}|\nabla u_\varepsilon^{\frac{r}{2}}|u_\varepsilon^{\frac{r}{2}}
\end{aligned} \tag{3.8}$$

for all $t > 0$ and $\varepsilon \in (0,1)$. Here thanks to the Young inequality, we can see that

$$\begin{aligned}\frac{2(r-1)\chi}{r}\int_\Omega |\nabla v_\varepsilon|^{p-1}\big|\nabla u_\varepsilon^{\frac{r}{2}}\big|u_\varepsilon^{\frac{r}{2}}\\ \le \frac{r-1}{r^2}\int_\Omega \big|\nabla u_\varepsilon^{\frac{r}{2}}\big|^2 + (r-1)\chi^2\int_\Omega |\nabla v_\varepsilon|^{2(p-1)}u_\varepsilon^r.\end{aligned} \tag{3.9}$$

Besides, by the Hölder inequality, we infer that

$$\int_\Omega |\nabla v_\varepsilon|^{2(p-1)}u_\varepsilon^r \le \left[\int_\Omega |\nabla v_\varepsilon|^s\right]^{\frac{2(p-1)}{s}}\left[\int_\Omega u_\varepsilon^{\frac{sr}{s-2(p-1)}}\right]^{\frac{s-2(p-1)}{s}}, \tag{3.10}$$

where

$$s \in (2(p-1),\infty). \tag{3.11}$$

From (3.8), (3.9), and (3.10) we have

$$\begin{aligned}\frac{1}{r}\frac{d}{dt}\int_\Omega u_\varepsilon^r + \frac{3(r-1)}{r^2}\int_\Omega \big|\nabla u_\varepsilon^{\frac{r}{2}}\big|^2\\ \le (r-1)\chi^2\left[\int_\Omega |\nabla v_\varepsilon|^s\right]^{\frac{2(p-1)}{s}}\left[\int_\Omega u_\varepsilon^{\frac{sr}{s-2(p-1)}}\right]^{\frac{s-2(p-1)}{s}}\end{aligned}$$

and hence,

$$\begin{aligned}\frac{d}{dt}\int_\Omega u_\varepsilon^r + \int_\Omega u_\varepsilon^r + \frac{3(r-1)}{r}\int_\Omega \big|\nabla u_\varepsilon^{\frac{r}{2}}\big|^2\\ \le r(r-1)\chi^2\left[\int_\Omega |\nabla v_\varepsilon|^s\right]^{\frac{2(p-1)}{s}}\left[\int_\Omega u_\varepsilon^{\frac{sr}{s-2(p-1)}}\right]^{\frac{s-2(p-1)}{s}} + \int_\Omega u_\varepsilon^r\end{aligned} \tag{3.12}$$

for all $t > 0$ and $\varepsilon \in (0,1)$, with s satisfying (3.11). We now estimate the first term on the right-hand side of (3.12). According to Lemma 3.2, we know that

$$\begin{aligned}\left[\int_\Omega |\nabla v_\varepsilon|^s\right]^{\frac{2(p-1)}{s}} &= \|\nabla v_\varepsilon(\cdot,t)\|_{L^s(\Omega)}^{2(p-1)}\\ &\le c_1\big((1+t^{-\frac{1}{2}})e^{-(1+\lambda_1)t}+m\big)^{2(p-1)}\\ &\le c_2(1+m^{2(p-1)})\end{aligned} \tag{3.13}$$

for all $t > 1$ and $\varepsilon \in (0,1)$, where s satisfies (3.11) and

$$s \in [1,\infty) \quad \text{if } n = 1, \quad \text{and} \quad s \in \left[1, \frac{n}{n-1}\right) \quad \text{if } n \ge 2. \tag{3.14}$$

Moreover, the Gagliardo–Nirenberg inequality (Lemma 2.3) shows that

$$\begin{aligned}\left[\int_\Omega u_\varepsilon^{\frac{sr}{s-2(p-1)}}\right]^{\frac{s-2(p-1)}{s}} &= \|u_\varepsilon^{\frac{r}{2}}\|^2_{L^{\frac{2s}{s-2(p-1)}}(\Omega)}\\ &\le c_3\|\nabla u_\varepsilon^{\frac{r}{2}}\|^{2a}_{L^2(\Omega)}\|u_\varepsilon^{\frac{r}{2}}\|^{2(1-a)}_{L^{\frac{2}{r}}(\Omega)} + c_3\|u_\varepsilon^{\frac{r}{2}}\|^2_{L^{\frac{2}{r}}(\Omega)}\\ &= c_3\|\nabla u_\varepsilon^{\frac{r}{2}}\|^{2a}_{L^2(\Omega)} m^{r(1-a)} + c_3 m^r, \end{aligned}\tag{3.15}$$

where s fulfills (3.11), (3.14) and

$$s \in (n(p-1), \infty),\tag{3.16}$$

and $a := \frac{\frac{r}{2}-\frac{s-2(p-1)}{2s}}{\frac{r}{2}+\frac{1}{n}-\frac{1}{2}} \in (0,1)$. Here we can choose s satisfying (3.11), (3.14) and (3.16), because the condition (1.5) implies

$$n(p-1) < \frac{n}{n-1} \quad \text{for } n \ge 2.\tag{3.17}$$

Now, invoking (3.13), (3.15) and the Young inequality, we obtain

$$\begin{aligned} r(r-1)\chi^2 &\left[\int_\Omega |\nabla v_\varepsilon|^s\right]^{\frac{2(p-1)}{s}} \left[\int_\Omega u_\varepsilon^{\frac{sr}{s-2(p-1)}}\right]^{\frac{s-2(p-1)}{s}}\\ &\le c_4(1+m^{2(p-1)})\|\nabla u_\varepsilon^{\frac{r}{2}}\|^{2a}_{L^2(\Omega)} m^{r(1-a)} + c_4(1+m^{2(p-1)})m^r\\ &\le \frac{2(r-1)}{r}\int_\Omega |\nabla u_\varepsilon^{\frac{r}{2}}|^2 + c_5(1+m^{2(p-1)})^{\frac{1}{1-a}} m^r + c_4(1+m^{2(p-1)})m^r\\ &\le \frac{2(r-1)}{r}\int_\Omega |\nabla u_\varepsilon^{\frac{r}{2}}|^2 + c_6 m^r\left(1+m^{2(p-1)}+m^{\frac{2(p-1)}{1-a}}\right) \end{aligned}\tag{3.18}$$

for all $t > 1$ and $\varepsilon \in (0,1)$, with s satisfying (3.11), (3.14) and (3.16). Next, again by using Lemma 2.3 and the Young inequality, we have

$$\begin{aligned}\int_\Omega u_\varepsilon^r &= \|u_\varepsilon^{\frac{r}{2}}\|^2_{L^2(\Omega)}\\ &\le c_7\|\nabla u_\varepsilon^{\frac{r}{2}}\|^{2b}_{L^2(\Omega)}\|u_\varepsilon^{\frac{r}{2}}\|^{2(1-b)}_{L^{\frac{2}{r}}(\Omega)} + c_7\|u_\varepsilon^{\frac{r}{2}}\|^2_{L^{\frac{2}{r}}(\Omega)}\\ &= c_7\|\nabla u_\varepsilon^{\frac{r}{2}}\|^{2b}_{L^2(\Omega)} m^{r(1-b)} + c_7 m^r\\ &\le \frac{r-1}{r}\int_\Omega |\nabla u_\varepsilon^{\frac{r}{2}}|^2 + c_8 m^r \end{aligned}\tag{3.19}$$

for all $t > 0$ and $\varepsilon \in (0,1)$, where $b := \frac{\frac{r}{2}-\frac{1}{2}}{\frac{r}{2}+\frac{1}{n}-\frac{1}{2}} \in (0,1)$. Plugging (3.18) and (3.19) into (3.12), we finally derive the differential inequality

$$\frac{d}{dt}\int_\Omega u_\varepsilon^r + \int_\Omega u_\varepsilon^r \le c_9 m^r\left(1+m^{2(p-1)}+m^{\frac{2(p-1)}{1-a}}\right)\tag{3.20}$$

for all $t > 1$ and $\varepsilon \in (0,1)$. Integrating (3.20) over $(1,t)$ and applying (3.3), we have

$$\begin{aligned}\int_\Omega u_\varepsilon^r(\cdot,t) &\le e^{-(t-1)}\int_\Omega u_\varepsilon^r(\cdot,1) + c_9 m^r\big(1+m^{2(p-1)}+m^{\frac{2(p-1)}{1-a}}\big)\\ &\le c_{10}|\Omega|e^{-(t-1)} + c_9 m^r\big(1+m^{2(p-1)}+m^{\frac{2(p-1)}{1-a}}\big)\end{aligned}$$

for all $t > 1$ and $\varepsilon \in (0,1)$, which leads to the conclusion. □

3.3. Boundedness of $u_\varepsilon - \overline{u_0}$ in the large-time limit

We now derive an L^∞-estimate for $u_\varepsilon - \overline{u_0}$, which is crucial to obtain (1.6). In order to compute more directly, we introduce

$$\begin{cases} U_\varepsilon(x,t) := u_\varepsilon(x,t) - \overline{u_0},\\ V_\varepsilon(x,t) := v_\varepsilon(x,t) - \overline{u_0}\end{cases}$$

for $\varepsilon \in (0,1)$, $x \in \Omega$ and $t > 0$. Then $(U_\varepsilon, V_\varepsilon)$ satisfies the following problem:

$$\begin{cases} (U_\varepsilon)_t = \Delta U_\varepsilon - \chi\nabla\cdot\Big(u_\varepsilon(|\nabla V_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla V_\varepsilon\Big), & x\in\Omega,\ t>0,\\ (V_\varepsilon)_t = \Delta V_\varepsilon - V_\varepsilon + U_\varepsilon, & x\in\Omega,\ t>0,\\ \nabla U_\varepsilon\cdot\nu = \nabla V_\varepsilon\cdot\nu = 0, & x\in\partial\Omega,\ t>0,\\ U_\varepsilon(x,0) = u_0(x)-\overline{u_0},\ V_\varepsilon(x,0) = v_0(x)-\overline{u_0}, & x\in\Omega.\end{cases} \tag{3.21}$$

Lemma 3.4. *Suppose that p satisfies* (1.5). *Then there exist $C > 0$ and $t_1 > 0$ such that*

$$\|U_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le Cm^p(1+m^\alpha+m^\beta) \quad \text{for all } t \ge t_1 \text{ and } \varepsilon\in(0,1)$$

with some $\alpha > 0$ and $\beta > 0$.

Proof. Rewriting the first equation in (3.21) as

$$\begin{aligned}U_\varepsilon(\cdot,t) &= e^{(t-1)\Delta}U_\varepsilon(\cdot,1)\\ &\quad -\chi\int_1^t e^{(t-\sigma)\Delta}\nabla\cdot\big(u_\varepsilon(\cdot,\sigma)(|\nabla V_\varepsilon(\cdot,\sigma)|^2+\varepsilon)^{\frac{p-2}{2}}\nabla V_\varepsilon(\cdot,\sigma)\big)\,d\sigma,\end{aligned}$$

we have

$$\begin{aligned}&\|U_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}\\ &\quad\le \|e^{(t-1)\Delta}U_\varepsilon(\cdot,1)\|_{L^\infty(\Omega)}\\ &\qquad +\chi\int_1^t \|e^{(t-\sigma)\Delta}\nabla\cdot\big(u_\varepsilon(\cdot,\sigma)(|\nabla V_\varepsilon(\cdot,\sigma)|^2+\varepsilon)^{\frac{p-2}{2}}\nabla V_\varepsilon(\cdot,\sigma)\big)\|_{L^\infty(\Omega)}\,d\sigma\end{aligned} \tag{3.22}$$

for all $t > 1$ and $\varepsilon \in (0,1)$. Here, in view of (3.2) we employ Lemma 2.2 (i) and (3.3) to derive

$$\begin{aligned}\|e^{(t-1)\Delta}U_\varepsilon(\cdot,1)\|_{L^\infty(\Omega)} &\le c_1 e^{-\lambda_1(t-1)}\|U_\varepsilon(\cdot,1)\|_{L^\infty(\Omega)}\\ &\le c_2 e^{-\lambda_1(t-1)}\end{aligned} \tag{3.23}$$

for all $t > 1$ and $\varepsilon \in (0,1)$. From now on, we estimate the second term on the right-hand side of (3.22). We can apply Lemma 2.2 (iv) and the Hölder inequality to see that

$$\begin{aligned}&\chi\int_1^t \|e^{(t-\sigma)\Delta}\nabla\cdot\big(u_\varepsilon(\cdot,\sigma)(|\nabla V_\varepsilon(\cdot,\sigma)|^2+\varepsilon)^{\frac{p-2}{2}}\nabla V_\varepsilon(\cdot,\sigma)\big)\|_{L^\infty(\Omega)}\,d\sigma\\ &\le c_3\int_1^t \big(1+(t-\sigma)^{-\frac12-\frac{n}{2k}}\big)e^{-\lambda_1(t-\sigma)}\big\|u_\varepsilon(\cdot,\sigma)|\nabla V_\varepsilon(\cdot,\sigma)|^{p-1}\big\|_{L^k(\Omega)}\,d\sigma\\ &\le c_3\int_1^t \big(1+(t-\sigma)^{-\frac12-\frac{n}{2k}}\big)e^{-\lambda_1(t-\sigma)}\|\nabla V_\varepsilon(\cdot,\sigma)\|^{p-1}_{L^{k_1(p-1)}(\Omega)}\|u_\varepsilon(\cdot,\sigma)\|_{L^{k_2}(\Omega)}\end{aligned} \tag{3.24}$$

for all $t > 1$ and $\varepsilon \in (0,1)$, with $k_1 > n$ and $k_2 > n$ to be fixed later, and $k > n$ satisfying $\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$. In view of the obvious identity $\nabla V_\varepsilon = \nabla v_\varepsilon$, Lemma 3.2 implies that

$$\|\nabla V_\varepsilon(\cdot,t)\|^{p-1}_{L^{k_1(p-1)}(\Omega)} \le c_4(e^{-\lambda_1(p-1)(t-1)} + m^{p-1}) \tag{3.25}$$

for all $t > 1$ and $\varepsilon \in (0,1)$, where

$$k_1 \in \left[\frac{1}{p-1},\infty\right) \quad \text{if } n=1, \quad \text{and} \quad k_1 \in \left[\frac{1}{p-1},\frac{n}{(p-1)(n-1)}\right) \quad \text{if } n \ge 2. \tag{3.26}$$

We can actually choose $k_1 > n$ as in (3.26), since the relation (3.17) ensures that

$$n < \frac{n}{(p-1)(n-1)} \quad \text{for } n \ge 2.$$

Recalling Lemma 3.3, we have

$$\|u_\varepsilon(\cdot,t)\|_{L^{k_2}(\Omega)} \le c_5\left[e^{-\frac{t-1}{k_2}} + m\left(1 + m^{\frac{2(p-1)}{k_2}} + m^{\frac{2(p-1)}{k_2(1-a)}}\right)\right] \tag{3.27}$$

for all $t > 1$, $\varepsilon \in (0,1)$, with some $a \in (0,1)$. Using (3.25) and (3.27) in (3.24), we derive that

$$
\begin{aligned}
&\chi \int_1^t \|e^{(t-\sigma)\Delta}\nabla \cdot \big(u_\varepsilon(\cdot,\sigma)(|\nabla V_\varepsilon(\cdot,\sigma)|^2+\varepsilon)^{\frac{p-2}{2}}\nabla V_\varepsilon(\cdot,\sigma)\big)\|_{L^\infty(\Omega)}\,d\sigma \\
&\le c_6 \int_1^t \big(1+(t-\sigma)^{-\frac{1}{2}-\frac{n}{2k}}\big)e^{-\lambda_1(t-\sigma)}e^{-\left(\lambda_1(p-1)+\frac{1}{k_2}\right)(\sigma-1)}\,d\sigma \\
&\quad + c_6 m\left(1+m^{\frac{2(p-1)}{k_2}}+m^{\frac{2(p-1)}{k_2(1-a)}}\right) \\
&\qquad \times \int_1^t \big(1+(t-\sigma)^{-\frac{1}{2}-\frac{n}{2k}}\big)e^{-\lambda_1(t-\sigma)}e^{-\lambda_1(p-1)(\sigma-1)}\,d\sigma \\
&\quad + c_6 m^{p-1}\int_1^t \big(1+(t-\sigma)^{-\frac{1}{2}-\frac{n}{2k}}\big)e^{-\lambda_1(t-\sigma)}e^{-\frac{\sigma-1}{k_2}}\,d\sigma \\
&\quad + c_6 m^p\left(1+m^{\frac{2(p-1)}{k_2}}+m^{\frac{2(p-1)}{k_2(1-a)}}\right)\int_1^t \big(1+(t-\sigma)^{-\frac{1}{2}-\frac{n}{2k}}\big)e^{-\lambda_1(t-\sigma)}\,d\sigma \\
&=: c_6 I_1(\cdot,t)+c_6 m\left(1+m^{\frac{2(p-1)}{k_2}}+m^{\frac{2(p-1)}{k_2(1-a)}}\right)I_2(\cdot,t) \\
&\quad + c_6 m^{p-1}I_3(\cdot,t)+c_6 m^p\left(1+m^{\frac{2(p-1)}{k_2}}+m^{\frac{2(p-1)}{k_2(1-a)}}\right)I_4(\cdot,t)
\end{aligned} \tag{3.28}
$$

for all $t>1$ and $\varepsilon\in(0,1)$, with some $a\in(0,1)$. Here, by virtue of Lemma 2.1, we deduce

$$
\begin{aligned}
I_1(\cdot,t) &= \int_0^{t-1} \big(1+(t-1-\tau)^{-\frac{1}{2}-\frac{n}{2k}}\big)e^{-\lambda_1(t-1-\tau)}e^{-\left(\lambda_1(p-1)+\frac{1}{k_2}\right)\tau}\,d\tau \\
&\le c_7\big(1+(t-1)^{\min\{0,1-\frac{1}{2}-\frac{n}{2k}\}}\big)e^{-\min\left\{\lambda_1,\lambda_1(p-1)+\frac{1}{k_2}\right\}(t-1)} \\
&= 2c_7 e^{-\min\left\{\lambda_1,\lambda_1(p-1)+\frac{1}{k_2}\right\}(t-1)}
\end{aligned} \tag{3.29}
$$

for all $t>1$, with k_2 satisfying

$$
\lambda_1 \ne \lambda_1(p-1)+\frac{1}{k_2}. \tag{3.30}
$$

Since $\lambda_1(p-1)<\lambda_1$ according to (1.5), we can also estimate $I_2(\cdot,t)$ by using Lemma 2.1 as

$$
\begin{aligned}
I_2(\cdot,t) &= \int_0^{t-1} \big(1+(t-1-\tau)^{-\frac{1}{2}-\frac{n}{2k}}\big)e^{-\lambda_1(t-1-\tau)}e^{-\lambda_1(p-1)\tau}\,d\tau \\
&\le c_8\big(1+(t-1)^{\min\{0,1-\frac{1}{2}-\frac{n}{2k}\}}\big)e^{-\min\{\lambda_1,\lambda_1(p-1)\}(t-1)} \\
&= 2c_8 e^{-\lambda_1(p-1)(t-1)}
\end{aligned} \tag{3.31}
$$

for all $t > 1$. Similarly, we utilize Lemma 2.1 again to obtain

$$\begin{aligned} I_3(\cdot,t) &= \int_0^{t-1} \left(1+(t-1-\tau)^{-\frac{1}{2}-\frac{n}{2k}}\right) e^{-\lambda_1(t-1-\tau)} e^{-\frac{1}{k_2}\tau}\, d\tau \\ &\le c_9\left(1+(t-1)^{\min\{0,1-\frac{1}{2}-\frac{n}{2k}\}}\right) e^{-\min\left\{\lambda_1,\frac{1}{k_2}\right\}(t-1)} \\ &= 2c_9 e^{-\min\left\{\lambda_1,\frac{1}{k_2}\right\}(t-1)} \end{aligned} \tag{3.32}$$

for all $t > 1$, provided that

$$\lambda_1 \neq \frac{1}{k_2}. \tag{3.33}$$

On the other hand, recalling that $k > n$, we see that

$$I_4(\cdot,t) \le \int_0^{\infty} (1+\sigma^{-\frac{1}{2}-\frac{n}{2k}}) e^{-\lambda_1\sigma}\, d\sigma < \infty \tag{3.34}$$

for all $t > 1$. Now, let $k_1 > n$ be as in (3.26), and take $k_2 > n$ large enough so that k_2 satisfies (3.30), (3.33) and $\frac{1}{k_1}+\frac{1}{k_2} < \frac{1}{n}$. Then, by plugging (3.29), (3.31), (3.32) and (3.34) into (3.28) we can derive that

$$\begin{aligned} &\chi \int_1^t \|e^{(t-\sigma)\Delta}\nabla\cdot\left(u_\varepsilon(\cdot,\sigma)(|\nabla V_\varepsilon(\cdot,\sigma)|^2+\varepsilon)^{\frac{p-2}{2}}\nabla V_\varepsilon(\cdot,\sigma)\right)\|_{L^\infty(\Omega)}\, d\sigma \\ &\quad \le c_{10} e^{-\min\left\{\lambda_1,\lambda_1(p-1)+\frac{1}{k_2}\right\}(t-1)} \\ &\qquad + c_{10} m \left(1+m^{\frac{2(p-1)}{k_2}}+m^{\frac{2(p-1)}{k_2(1-a)}}\right) e^{-\lambda_1(p-1)(t-1)} \\ &\qquad + c_{10} m^{p-1} e^{-\min\left\{\lambda_1,\frac{1}{k_2}\right\}(t-1)} + c_{10} m^p \left(1+m^{\frac{2(p-1)}{k_2}}+m^{\frac{2(p-1)}{k_2(1-a)}}\right) \end{aligned} \tag{3.35}$$

for all $t > 1$ and $\varepsilon \in (0,1)$, with some $a \in (0,1)$. Combining (3.23) and (3.35) with (3.22) yields

$$\begin{aligned} \|U_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} &\le c_2 e^{-\lambda_1(t-1)} + c_{10} e^{-\min\left\{\lambda_1,\lambda_1(p-1)+\frac{1}{k_2}\right\}(t-1)} \\ &\quad + c_{10} m \left(1+m^{\frac{2(p-1)}{k_2}}+m^{\frac{2(p-1)}{k_2(1-a)}}\right) e^{-\lambda_1(p-1)(t-1)} \\ &\quad + c_{10} m^{p-1} e^{-\min\left\{\lambda_1,\frac{1}{k_2}\right\}(t-1)} + c_{10} m^p \left(1+m^{\frac{2(p-1)}{k_2}}+m^{\frac{2(p-1)}{k_2(1-a)}}\right) \end{aligned}$$

for all $t > 1$ and $\varepsilon \in (0,1)$, with some $a \in (0,1)$, and hence we arrive at the conclusion. □

3.4. Proof of Theorem 1.2

The following lemma provides global existence of weak solutions to (1.1) and some convergence results, which were already shown in [17, Theorem 1.1 and Lemma 4.3].

Lemma 3.5. *Suppose that p satisfies* (1.5)*. Then there exist a global weak solution (u,v) of* (1.1) *as well as a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ such that*

$$u_\varepsilon \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(\Omega\times(0,\infty)) \tag{3.36}$$

and

$$u_\varepsilon \to u \quad \text{in } C^0_{\mathrm{loc}}\Big([0,\infty);(W^{2,2}_0(\Omega))^*\Big) \tag{3.37}$$

as $\varepsilon = \varepsilon_k \searrow 0$.

We are in a position to complete the proof of Theorem 1.2.

Proof of Theorem 1.2. The first half of the proof is similar to that of [16, Lemma 4.2]. By Lemma 3.4 and (3.36), there exists a null set $N\subset[t_1,\infty)$ such that

$$\|u(\cdot,t)-\overline{u_0}\|_{L^\infty(\Omega)} \le c_1 m^p(1+m^\alpha+m^\beta) \quad \text{for all } t\in[t_1,\infty)\setminus N \tag{3.38}$$

for some $\alpha>0$ and $\beta>0$. Indeed, from (3.36) it follows that

$$u_\varepsilon-\overline{u_0} \overset{*}{\rightharpoonup} u-\overline{u_0} \quad \text{in } L^\infty(\Omega\times[t_1,\infty))$$

as $\varepsilon=\varepsilon_k\searrow 0$, and then due to the weak lower semicontinuity of the norm (see e.g., [1, Proposition 3.13 (iii)]), we infer from Lemma 3.4 that

$$\begin{aligned}\|u-\overline{u_0}\|_{L^\infty(\Omega\times[t_1,\infty))} &\le \liminf_{\varepsilon=\varepsilon_k\searrow 0}\|u_\varepsilon-\overline{u_0}\|_{L^\infty(\Omega\times[t_1,\infty))}\\ &\le c_1 m^p(1+m^\alpha+m^\beta)\end{aligned}$$

for some $\alpha>0$ and $\beta>0$, and moreover the measure theory ensures the existence of a null set $N\subset[t_1,\infty)$ such that

$$u(\cdot,t)-\overline{u_0}\in L^\infty(\Omega) \quad \text{and} \quad \|u(\cdot,t)-\overline{u_0}\|_{L^\infty(\Omega)} \le \|u-\overline{u_0}\|_{L^\infty(\Omega\times[t_1,\infty))}$$

for all $t\in[t_1,\infty)\setminus N$. We claim that the inequality (3.38) actually holds for every $t\in[t_1,\infty)$. To see this, first, for each $t\in[t_1,\infty)$ we can find $(\widetilde{t_k})_{k\in\mathbb{N}}\subset[t_1,\infty)\setminus N$ such that $\widetilde{t_k}\to t$ as $k\to\infty$, and extracting a subsequence if necessary we also have

$$u(\cdot,\widetilde{t_k}) \overset{*}{\rightharpoonup} \widetilde{u} \quad \text{in } L^\infty(\Omega) \text{ as } k\to\infty$$

with some $\widetilde{u} \in L^\infty(\Omega)$ (see [1, Theorem 3.18]). On the other hand, (3.37) implies

$$u(\cdot,\widetilde{t_k}) \to u(\cdot,t) \quad \text{in } (W_0^{2,2}(\Omega))^* \text{ as } k \to \infty.$$

We thus get $\widetilde{u} = u(\cdot,t)$, and due to the weak lower semicontinuity of the norm, we arrive at

$$\begin{aligned} \|u(\cdot,t) - \overline{u_0}\|_{L^\infty(\Omega)} &\le \liminf_{k\to\infty} \|u(\cdot,\widetilde{t_k}) - \overline{u_0}\|_{L^\infty(\Omega)} \\ &\le c_1 m^p (1 + m^\alpha + m^\beta) \quad \text{for all } t \in [t_1,\infty), \end{aligned}$$

which proves the claim, and hence establishes (1.6). For the latter part of the theorem, let η_0 be such that

$$c_1 \eta_0^{p-1}(1 + \eta_0^\alpha + \eta_0^\beta) \le 1,$$

and for each $\eta \in (0,\eta_0)$ fix $m = \|u_0\|_{L^1(\Omega)}$ such that $m \le \eta$. Then we have

$$\begin{aligned} \|u(\cdot,t) - \overline{u_0}\|_{L^\infty(\Omega)} &\le \eta \cdot c_1 \eta_0^{p-1}(1 + \eta_0^\alpha + \eta_0^\beta) \\ &\le \eta \end{aligned}$$

for all $t \ge t_1$, and the proof is complete. □

4. Asymptotic stability of $\overline{u_0}$ when $p \in [2,\infty)$ and $n = 1$

In this section we will prove Theorem 1.3. Throughout this section, we let $n = 1$, and denote by $\lambda_1 > 0$ the first nonzero eigenvalue of $-\Delta$ in Ω. Also, we suppose that u_0 and v_0 satisfy (1.2), $u_0 \in \bigcup_{\theta\in(0,1)} C^\theta(\overline{\Omega})$ and $p \in [2,\infty)$.

We first give a result on global existence and boundedness of classical solutions to (1.1) without a proof. Thanks to the regularity for u_0, local existence can be proved by standard arguments based on Schauder's fixed point theorem (see e.g., [5, 7]); boundedness, and hence globality, can be shown similarly as in [11, Lemma 2.5] and [17, Lemma 4.1].

Lemma 4.1. *Suppose that $n = 1$ and $p \in [2,\infty)$. Then there exists a global classical solution*

$$(u,v) \in \left(C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty))\right)^2$$

of (1.1) *which is bounded in the sense that there exists $C > 0$ such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C \quad \textit{for all } t > 0. \tag{4.1}$$

In the following, we denote by (u,v) the classical solution of (1.1) given in Lemma 4.1. We next establish an L^r-estimate for u, which will be used to show (1.6).

Lemma 4.2. *Suppose that $n=1$. Let $q \in [1,\infty)$. Then there exists a constant $C>0$ such that*

$$\limsup_{t\to\infty} \|\nabla v(\cdot,t)\|_{L^q(\Omega)} \le Cm. \tag{4.2}$$

Proof. According to the variation-of-constants formula associated with v, we see that

$$v(\cdot,t) = e^{t(\Delta-1)}v_0 + \int_0^t e^{(t-\sigma)(\Delta-1)}u(\cdot,\sigma)\,d\sigma \quad \text{for all } t>0.$$

Repeating the proof of Lemma 3.2 with u_ε and v_ε replaced by u and v, we can obtain

$$\begin{aligned} \|\nabla v(\cdot,t)\|_{L^q(\Omega)} &\le c_1(1+t^{-\frac{1}{2}})e^{-(1+\lambda_1)t}\|v_0\|_{L^q(\Omega)} + c_1 m \\ &\le c_1|\Omega|^{\frac{1}{q}}(1+t^{-\frac{1}{2}})e^{-(1+\lambda_1)t}\|v_0\|_{W^{1,\infty}(\Omega)} + c_1 m \quad \text{for all } t>0. \end{aligned}$$

The claim therefore results by the fact that $(1+t^{-\frac{1}{2}})e^{-(1+\lambda_1)t} \to 0$ as $t\to\infty$. □

Lemma 4.3. *Suppose that $n=1$ and $p\in[2,\infty)$. Let $r\in(1,\infty)$. Then there exists a constant $C>0$ such that*

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^r(\Omega)} \le Cm\left(1+m^{\frac{2(p-1)}{r}} + m^{\frac{2(p-1)}{r(1-a)}}\right) \tag{4.3}$$

with some $a\in(0,1)$.

Proof. Let $r\in(1,\infty)$. Testing the first equation of (1.1) with u^{r-1} and integrating by parts give

$$\begin{aligned} &\frac{1}{r}\frac{d}{dt}\int_\Omega u^r + \frac{4(r-1)}{r^2}\int_\Omega |\nabla u^{\frac{r}{2}}|^2 \\ &\quad = (r-1)\chi\int_\Omega |\nabla v|^{p-2}(\nabla v\cdot\nabla u)u^{r-1} \\ &\quad \le (r-1)\chi\int_\Omega |\nabla v|^{p-1}|\nabla u|u^{r-1} \\ &\quad = \frac{2(r-1)\chi}{r}\int_\Omega |\nabla v|^{p-1}|\nabla u^{\frac{r}{2}}|u^{\frac{r}{2}} \quad \text{for all } t>0. \end{aligned}$$

We will modify the argument of Lemma 3.3. Indeed, instead of Lemma 3.2 we will use the fact that the estimate (4.2) guarantees existence of $t_0 > 0$ such that

$$\|\nabla v(\cdot,t)\|_{L^q(\Omega)} \le c_1 m \quad \text{for all } t > t_0 \text{ and } q \in [1,\infty),$$

and we can follow the proof as in Lemma 3.3 to observe that

$$\frac{d}{dt}\int_\Omega u^r + \int_\Omega u^r \le c_2 m^r \left(1 + m^{2(p-1)} + m^{\frac{2(p-1)}{1-a}}\right) \quad \text{for all } t > t_0$$

with some $a \in (0,1)$. This will lead to the conclusion. □

As just as in Section 3, we introduce

$$\begin{cases} U(x,t) := u(x,t) - \overline{u_0}, \\ V(x,t) := v(x,t) - \overline{u_0} \end{cases}$$

for $x \in \Omega$ and $t > 0$. Then (U,V) satisfies the following problem:

$$\begin{cases} U_t = \Delta U - \chi \nabla \cdot (u |\nabla V|^{p-2} \nabla V), & x \in \Omega,\ t > 0, \\ V_t = \Delta V - V + U, & x \in \Omega,\ t > 0, \\ \nabla U \cdot \nu = \nabla V \cdot \nu = 0, & x \in \partial\Omega,\ t > 0, \\ U(x,0) = u_0(x) - \overline{u_0},\ V(x,0) = v_0(x) - \overline{u_0}, & x \in \Omega. \end{cases} \tag{4.4}$$

We are now in a position to establish the estimate (1.6).

Lemma 4.4. *Suppose that $n = 1$ and $p \in [2,\infty)$. Then there exists a constant $C > 0$ such that*

$$\limsup_{t\to\infty} \|U(\cdot,t)\|_{L^\infty(\Omega)} \le C m^p (1 + m^\alpha + m^\beta) \tag{4.5}$$

with some $\alpha > 0$ and $\beta > 0$.

Proof. In light of the identity $\nabla V = \nabla v$, for all $q \in [1,\infty)$ and $r \in (1,\infty)$, the estimates (4.2) and (4.3) provide $t_1 = t_1(r) > 0$ such that

$$\|\nabla V(\cdot,t)\|_{L^q(\Omega)} \le c_1 m \quad \text{for all } t > t_1 \tag{4.6}$$

and

$$\|u(\cdot,t)\|_{L^r(\Omega)} \le c_1 m \left(1 + m^{\frac{2(p-1)}{r}} + m^{\frac{2(p-1)}{r(1-a)}}\right) \quad \text{for all } t > t_1 \tag{4.7}$$

with some $a \in (0,1)$. We now make use of the representation formula for U to see that

$$U(\cdot,t) = e^{(t-t_1)\Delta}U(\cdot,t_1) - \chi \int_{t_1}^{t} e^{(t-\sigma)\Delta}\nabla \cdot \big(u(\cdot,\sigma)|\nabla V(\cdot,\sigma)|^{p-2}\nabla V(\cdot,\sigma)\big)\, d\sigma,$$

and hence,

$$\|U(\cdot,t)\|_{L^\infty(\Omega)} \le \|e^{(t-t_1)\Delta}U(\cdot,t_1)\|_{L^\infty(\Omega)} + \chi \int_{t_1}^{t} \|e^{(t-\sigma)\Delta}\nabla \cdot \big(u(\cdot,\sigma)|\nabla V(\cdot,\sigma)|^{p-2}\nabla V(\cdot,\sigma)\big)\|_{L^\infty(\Omega)}\, d\sigma \tag{4.8}$$

for all $t > t_1$. Here, in view of the fact $\int_\Omega U = 0$, we employ Lemma 2.2 (i) and (4.1) to confirm that

$$\begin{aligned} \|e^{(t-t_1)\Delta}U(\cdot,t_1)\|_{L^\infty(\Omega)} &\le c_2 e^{-\lambda_1(t-t_1)}\|U(\cdot,t_1)\|_{L^\infty(\Omega)} \\ &\le c_3 e^{-\lambda_1(t-t_1)} \end{aligned} \tag{4.9}$$

for all $t > t_1$. Moreover, we see from Lemma 2.2 (iv), the Hölder inequality, and (4.6) in conjunction with (4.7) that

$$\begin{aligned} &\chi \int_{t_1}^{t} \|e^{(t-\sigma)\Delta}\nabla \cdot \big(u(\cdot,\sigma)|\nabla V(\cdot,\sigma)|^{p-2}\nabla V(\cdot,\sigma)\big)\|_{L^\infty(\Omega)}\, d\sigma \\ &\le c_4 \int_{t_1}^{t} \big(1+(t-\sigma)^{-\frac{1}{2}-\frac{1}{4}}\big)e^{-\lambda_1(t-\sigma)}\big\|u(\cdot,\sigma)|\nabla V(\cdot,\sigma)|^{p-1}\big\|_{L^2(\Omega)}\, d\sigma \\ &\le c_4 \int_{t_1}^{t} \big(1+(t-\sigma)^{-\frac{3}{4}}\big)e^{-\lambda_1(t-\sigma)}\|u(\cdot,\sigma)\|_{L^4(\Omega)}\|\nabla V(\cdot,\sigma)\|^{p-1}_{L^{4(p-1)}(\Omega)}\, d\sigma \\ &\le c_5 m^p \Big(1+m^{\frac{p-1}{2}}+m^{\frac{p-1}{2(1-b)}}\Big)\int_0^\infty (1+\sigma^{-\frac{3}{4}})e^{-\lambda_1\sigma}\, d\sigma \end{aligned} \tag{4.10}$$

for all $t > t_1$, with some $b \in (0,1)$. Therefore, a combination of (4.9) and (4.10) with (4.8) ensures that this lemma holds. □

In light of (4.2) and (4.5), we can pick $t_2 = t_2(u,v) > 0$ such that

$$\|\nabla V(\cdot,t)\|_{L^q(\Omega)} \le c_1 m \quad \text{for all } t \ge t_2 \text{ and } q \in [1,\infty), \tag{4.11}$$

and

$$\|U(\cdot,t)\|_{L^\infty(\Omega)} \le c_1 m^p(1+m^\alpha+m^\beta) \quad \text{for all } t \ge t_2 \tag{4.12}$$

with some $\alpha > 0$ and $\beta > 0$. We now select $\eta_0 > 0$ such that

$$2c_1\eta_0^{p-1}(1+\eta_0^\alpha+\eta_0^\beta) \le 1, \tag{4.13}$$

and for each $\eta \in (0,\eta_0)$ fix $m = \|u_0\|_{L^1(\Omega)}$ such that $m \le \eta$. Then we infer from (4.12) and (4.13) that

$$\|U(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{1}{2}\eta \cdot 2c_1\eta_0^{p-1}(1+\eta_0^\alpha+\eta_0^\beta) \le \frac{1}{2}\eta \tag{4.14}$$

for all $t \ge t_2$. Consequently, we have that

$$\mathcal{S} := \left\{T \ge t_2 \mid \|U(\cdot,t)\|_{L^\infty(\Omega)} \le \eta e^{-h(t-t_2)} \quad \forall t \in [t_2,T]\right\}$$

is nonempty, where $h \in (0, \frac{\lambda_1}{p})$. Indeed, we see from the continuity of the function $t \mapsto e^{-h(t-t_2)}$ that there exists $T > t_2$ such that $\eta e^{-h(t-t_2)} > \frac{1}{2}\eta$ for all $t \in [t_2,T]$, and by (4.14) we have $\|U(\cdot,t)\|_{L^\infty(\Omega)} \le \eta e^{-h(t-t_2)}$ for all $t \in [t_2,T]$, which means that $T \in \mathcal{S}$.

Now we define

$$T^* := \sup \mathcal{S} \in (t_2,\infty]$$

and taking account of the definition of $\mathcal{S}$, we observe that

$$\|U(\cdot,t)\|_{L^\infty(\Omega)} \le \eta e^{-h(t-t_2)} \quad \text{for all } t \in [t_2,T^*). \tag{4.15}$$

Then, in order to establish asymptotic stability of $\overline{u_0}$, it is sufficient to show that $T^* = \infty$.

We first derive exponential decay of $\nabla V(\cdot,t)$ in the following lemma.

Lemma 4.5. *Suppose that $n = 1$ and $p \in [2,\infty)$. Let $q \in [2,\infty)$. Assume that $\eta_0 > 0$ satisfies the condition* (4.13). *Let $h \in (0, \frac{\lambda_1}{p})$. Then for all $\eta \in (0,\eta_0)$, whenever u_0 fulfills that $m = \|u_0\|_{L^1(\Omega)} \le \eta$, V satisfies that*

$$\|\nabla V(\cdot,t)\|_{L^q(\Omega)} \le C\eta e^{-h(t-t_2)} \quad \textit{for all } t \in (t_2,T^*)$$

with some $C > 0$, where $t_2 > 0$ is the time appearing in (4.11) *and* (4.12).

Proof. On account of the representation

$$V(\cdot,t) = e^{(t-t_2)(\Delta-1)}V(\cdot,t_2) + \int_{t_2}^t e^{(t-\sigma)(\Delta-1)}U(\cdot,\sigma)\,d\sigma, \quad t \in (t_2,T^*),$$

we infer that

$$\|\nabla V(\cdot,t)\|_{L^q(\Omega)} \le e^{-(t-t_2)}\|\nabla e^{(t-t_2)\Delta}V(\cdot,t_2)\|_{L^q(\Omega)} + \int_{t_2}^t e^{-(t-\sigma)}\|\nabla e^{(t-\sigma)\Delta}U(\cdot,\sigma)\|_{L^q(\Omega)} \tag{4.16}$$

for all $t \in (t_2, T^*)$. We derive from Lemma 2.2 (iii), (4.11) and the relation $m \le \eta$ that

$$e^{-(t-t_2)}\|\nabla e^{(t-t_2)\Delta}V(\cdot,t_2)\|_{L^q(\Omega)} \le c_1 e^{-(1+\lambda_1)(t-t_2)}\|\nabla V(\cdot,t_2)\|_{L^q(\Omega)} \le c_2\eta e^{-(1+\lambda_1)(t-t_2)} \tag{4.17}$$

for all $t \in (t_2, T^*)$. Moreover, from the fact $h < \frac{\lambda_1}{p} < 1+\lambda_1$, we can estimate the second term on the right-hand side of (4.16) by using Lemma 2.2 (ii), (4.15) and Lemma 2.1 as

$$\begin{aligned}
&\int_{t_2}^{t} e^{-(t-\sigma)}\|\nabla e^{(t-\sigma)\Delta}U(\cdot,\sigma)\|_{L^q(\Omega)} \\
&\le c_3\int_{t_2}^{t}\left(1+(t-\sigma)^{-\frac{1}{2}}\right)e^{-(1+\lambda_1)(t-\sigma)}\|U(\cdot,\sigma)\|_{L^q(\Omega)}\,d\sigma \\
&\le c_3|\Omega|^{\frac{1}{q}}\int_{t_2}^{t}\left(1+(t-\sigma)^{-\frac{1}{2}}\right)e^{-(1+\lambda_1)(t-\sigma)}\|U(\cdot,\sigma)\|_{L^\infty(\Omega)}\,d\sigma \\
&\le c_3|\Omega|^{\frac{1}{q}}\eta\int_{t_2}^{t}\left(1+(t-\sigma)^{-\frac{1}{2}}\right)e^{-(1+\lambda_1)(t-\sigma)}e^{-h(\sigma-t_2)}\,d\sigma \\
&= c_3|\Omega|^{\frac{1}{q}}\eta\int_{0}^{t-t_2}\left(1+(t-t_2-\tau)^{-\frac{1}{2}}\right)e^{-(1+\lambda_1)(t-t_2-\tau)}e^{-h\tau}\,d\tau \\
&\le c_4\eta\left(1+(t-t_2)^{\min\{0,1-\frac{1}{2}\}}\right)e^{-\min\{1+\lambda_1,h\}(t-t_2)} \\
&= 2c_4\eta e^{-h(t-t_2)}
\end{aligned} \tag{4.18}$$

for all $t \in (t_2, T^*)$. The claim follows from (4.16), (4.17) and (4.18). □

Finally we derive $T^* = \infty$, which yields that $u(\cdot,t)$ converges to $\overline{u_0}$ as $t \to \infty$.

Lemma 4.6. *Suppose that $n = 1$ and $p \in [2,\infty)$. Let $h \in (0, \frac{\lambda_1}{p})$. Then there exists $\eta_0 > 0$ such that for all $\eta \in (0,\eta_0)$, whenever u_0 satisfies the relation $m = \|u_0\|_{L^1(\Omega)} \le \eta$, we have*

$$\|U(\cdot,t)\|_{L^\infty(\Omega)} \le \eta e^{-h(t-t_2)}$$

for all $t \ge t_2$, where $t_2 > 0$ is the time appearing in (4.11) *and* (4.12)*.*

Proof. We choose η_0 as in (4.13). According to the variation-of-constants formula for U in (4.4), we have

$$\|U(\cdot,t)\|_{L^\infty(\Omega)} \le \|e^{(t-t_2)\Delta}U(\cdot,t_2)\|_{L^\infty(\Omega)} + \chi\int_{t_2}^{t}\|e^{(t-\sigma)\Delta}\nabla\cdot\left(u(\cdot,\sigma)|\nabla V(\cdot,\sigma)|^{p-2}\nabla V(\cdot,\sigma)\right)\|_{L^\infty(\Omega)}\,d\sigma \tag{4.19}$$

for all $t \in (t_2, T^*)$. In view of the fact $\int_\Omega U = 0$ and the assumption $m \le \eta$, a combination of Lemma 2.2 (i) and (4.12) ensures that

$$\begin{aligned} \|e^{(t-t_2)\Delta}U(\cdot,t_2)\|_{L^\infty(\Omega)} &\le c_1 e^{-\lambda_1(t-t_2)}\|U(\cdot,t_2)\|_{L^\infty(\Omega)} \\ &\le c_2\eta^p(1+\eta^\alpha+\eta^\beta)e^{-\lambda_1(t-t_2)} \end{aligned} \tag{4.20}$$

for all $t \in (t_2, T^*)$, with some $\alpha > 0$ and $\beta > 0$. We now estimate the second term on the right-hand side of (4.19). An application of Lemma 2.2 (iv) entails that

$$\begin{aligned} &\chi\int_{t_2}^t \|e^{(t-\sigma)\Delta}\nabla\cdot\big(u(\cdot,\sigma)|\nabla V(\cdot,\sigma)|^{p-2}\nabla V(\cdot,\sigma)\big)\|_{L^\infty(\Omega)}\,d\sigma \\ &\quad\le c_3\int_{t_2}^t \big(1+(t-\sigma)^{-\frac{1}{2}-\frac{1}{4}}\big)e^{-\lambda_1(t-\sigma)}\big\|u(\cdot,\sigma)|\nabla V(\cdot,\sigma)|^{p-1}\big\|_{L^2(\Omega)}\,d\sigma \\ &\quad\le c_3\int_{t_2}^t \big(1+(t-\sigma)^{-\frac{3}{4}}\big)e^{-\lambda_1(t-\sigma)}\|u(\cdot,\sigma)\|_{L^\infty(\Omega)}\|\nabla V(\cdot,\sigma)\|_{L^{2(p-1)}(\Omega)}^{p-1}\,d\sigma \end{aligned} \tag{4.21}$$

for all $t \in (t_2, T^*)$. Here, from the identity $u(\cdot,t) = U(\cdot,t) + \overline{u_0}$, (4.15) and the relation $m \le \eta$, we see that

$$\begin{aligned} \|u(\cdot,t)\|_{L^\infty(\Omega)} &\le \|U(\cdot,t)\|_{L^\infty(\Omega)} + \overline{u_0} \\ &\le \eta e^{-h(t-t_2)} + \frac{\eta}{|\Omega|} \end{aligned} \tag{4.22}$$

for all $t \in (t_2, T^*)$. Also, Lemma 4.5 implies

$$\|\nabla V(\cdot,t)\|_{L^{2(p-1)}(\Omega)} \le c_4\eta e^{-h(t-t_2)} \tag{4.23}$$

for all $t \in (t_2, T^*)$. Inserting (4.22) and (4.23) into (4.21), we can derive that

$$\begin{aligned} &\chi\int_{t_2}^t \|e^{(t-\sigma)\Delta}\nabla\cdot\big(u(\cdot,\sigma)|\nabla V(\cdot,\sigma)|^{p-2}\nabla V(\cdot,\sigma)\big)\|_{L^\infty(\Omega)}\,d\sigma \\ &\quad\le c_5\int_{t_2}^t \big(1+(t-\sigma)^{-\frac{3}{4}}\big)e^{-\lambda_1(t-\sigma)} \\ &\qquad\qquad\times\left(\eta e^{-h(\sigma-t_2)}+\frac{\eta}{|\Omega|}\right)\eta^{p-1}e^{-h(p-1)(\sigma-t_2)}\,d\sigma \\ &\quad= c_5\eta^p\int_{t_2}^t \big(1+(t-\sigma)^{-\frac{3}{4}}\big)e^{-\lambda_1(t-\sigma)}e^{-hp(\sigma-t_2)}\,d\sigma \\ &\qquad+\frac{c_5}{|\Omega|}\eta^p\int_{t_2}^t \big(1+(t-\sigma)^{-\frac{3}{4}}\big)e^{-\lambda_1(t-\sigma)}e^{-h(p-1)(\sigma-t_2)}\,d\sigma \\ &\quad=: c_5\eta^pJ_1(\cdot,t)+\frac{c_5}{|\Omega|}\eta^pJ_2(\cdot,t) \end{aligned} \tag{4.24}$$

for all $t \in (t_2, T^*)$. We estimate the terms $J_1(\cdot,t)$ and $J_2(\cdot,t)$. Lemma 2.1 yields

$$\begin{aligned}
J_1(\cdot,t) &= \int_0^{t-t_2} \left(1+(t-t_2-\tau)^{-\frac{3}{4}}\right) e^{-\lambda_1(t-t_2-\tau)} e^{-hp\tau}\, d\tau \\
&\le c_6\left(1+(t-t_2)^{\min\{0,1-\frac{3}{4}\}}\right) e^{-\min\{\lambda_1,hp\}(t-t_2)} \\
&= 2c_6 e^{-hp(t-t_2)} \qquad (4.25)
\end{aligned}$$

for all $t \in (t_2, T^*)$, where we have used the fact $hp < \lambda_1$ since $h \in (0, \frac{\lambda_1}{p})$. Similarly, we can apply Lemma 2.1 again and get

$$\begin{aligned}
J_2(\cdot,t) &= \int_0^{t-t_2} \left(1+(t-t_2-\tau)^{-\frac{3}{4}}\right) e^{-\lambda_1(t-t_2-\tau)} e^{-h(p-1)\tau}\, d\tau \\
&\le c_7\left(1+(t-t_2)^{\min\{0,1-\frac{3}{4}\}}\right) e^{-\min\{\lambda_1,h(p-1)\}(t-t_2)} \\
&= 2c_7 e^{-h(p-1)(t-t_2)} \qquad (4.26)
\end{aligned}$$

for all $t \in (t_2, T^*)$. Plugging (4.25) and (4.26) into (4.24), we obtain

$$\chi \int_{t_2}^{t} \|e^{(t-\sigma)\Delta}\nabla\cdot\left(u(\cdot,\sigma)|\nabla V(\cdot,\sigma)|^{p-2}\nabla V(\cdot,\sigma)\right)\|_{L^\infty(\Omega)}\, d\sigma \le c_8\eta^p e^{-h(p-1)(t-t_2)} \qquad (4.27)$$

for all $t \in (t_2, T^*)$. From the fact $h \le h(p-1)$ since $p \in [2,\infty)$, and the relation $h < \frac{\lambda_1}{p} < \lambda_1$, we combine (4.20) and (4.27) with (4.19) to confirm that

$$\|U(\cdot,t)\|_{L^\infty(\Omega)} \le c_9\eta^p(1+\eta^\alpha+\eta^\beta)e^{-h(t-t_2)}$$

for all $t \in (t_2, T^*)$. Now, taking η_0 in (4.13) such that

$$2c_9\eta_0^{p-1}(1+\eta_0^\alpha+\eta_0^\beta) \le 1,$$

we can see that for all $\eta \in (0,\eta_0)$, whenever $m \le \eta$ we have

$$\|U(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{1}{2}\eta e^{-h(t-t_2)}$$

for all $t \in (t_2, T^*)$. Therefore, in light of the definition of T^*, we conclude from the continuity of U that $T^* = \infty$. This proves the lemma. □

Proof of Theorem 1.3. In light of (4.12) we see that the estimate (1.6) holds for all $t \ge t_2$. The stabilization (1.7) is a result of Lemma 4.6. □

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S. KOHATSU
Department of Mathematics
Tokyo University of Science
e-mail: `sho.kht.jp@gmail.com`

T. YOKOTA
Department of Mathematics
Tokyo University of Science
e-mail: `yokota@rs.tus.ac.jp`