LE MATEMATICHE
Vol. LXIV (2009) – Fasc. I, pp. 113–125

ON SEQUENCES OF INTEGERS FOR HANKEL PLANES IN A LINEAR SPACE Σ OF $\mathbb{P}^m$

GIOIA FAILLA

For a vector space $R \subseteq k^{m+1}$ of dimension $r+1$ on the algebraically closed field k we determine, for any $i \leq r$, the possible numbers of Hankel $i-$planes contained in the $r-$plane $\mathbb{P}(R)$, linear space in $\mathbb{P}^m$.

1. Introduction

Let $\mathbb{P}^m$ be the projective space of dimension m defined on k, algebraically closed field. Let $R \subseteq k^{m+1}$ be a $k-$vector space of dimension $r+1$ and let $\mathbb{P}(R) \subseteq \mathbb{P}^m$ be the corresponding $r-$plane.
In [2] the theory of the Hankel planes was developed. The authors give the definition of Hankel $r-$plane, starting from a matrix (called Hankel matrix) of elements of k. They give necessary and sufficient conditions for an $r-$plane to be Hankel.
An interesting problem is to find invariants for an $r-$plane with respect to a change of coordinates leaving fixed the standard rational normal curve $X_m \subseteq \mathbb{P}^m$. Our investigation in the topic of Hankel planes brings us to deduce that invariants of a linear space are given by the positive integer numbers h_i defined in the following way: h_i denotes the number of the independent Hankel $i-$planes skew to the rational normal curve X_m of $\mathbb{P}^m$ and contained in the linear space

Entrato in redazione: 20 gennaio 2009
AMS 2000 Subject Classification: 14L24,15A03.
Keywords: Hankel Planes, Geometric Invariants.

$\mathbb{P}(R)$. We shall take the term h_i as the general term of a decreasing sequence, called $h-$sequence.
An open problem, given in [2], was to describe all the possible $h-$sequences in a given linear space. In this paper, the problem has been completely solved, so we obtain new invariants associated to a linear space Σ, in terms of a $h-$sequence.
To be precise, in section 2, we give some definitions and recall some notions and results useful in the sequel. In Sections 3 and 4, considering a linear space $\Sigma \subseteq \mathbb{P}^m$ of dimension r, skew to X_m and joining t maximal Hankel planes (that can also have the same dimension), we determine the relative $h-$sequence.
We obtain simplified $h-$sequences when all the maximal Hankel planes have the same dimension, or all of them have different dimensions.
In Section 5, we introduce the *difference sequence* Δh, with general term $\Delta h_i = h_i - h_{i+1}$. We consider a decreasing sequence of positive integer numbers h_i and we show that it is the $h-$sequence of some space Σ if, and only if, Δh is not increasing. Finally, we find the maximum and minimum number of Hankel $i-$planes contained in a linear space Σ i. e. the maximal and the minimal $h-$sequences.

2. Preliminaries and Notations

Let k be an algebraically closed field, char$(k) = 0$. Starting from a matrix $A \in k^{(m+1)\times(n+1)}$, for any $p \geq 0$, we can construct a block Toeplitz matrix $T_A(p) \in k^{(m+p+1),(n+1)(p+1)}$. Let $R \subseteq k^{m+1}$ be the $k-$space of the relations among the rows of A and, $R(p) \subseteq k^{m+p+1}$ be the $k-$space of the relations among the rows of $T_A(p)$. In particular, any element of $R(p)$ gives a Hankel matrix whose rows belong to R (For more details see[2]). We recall the following:

Definition 2.1. Define *Hankel Matrix* a matrix of the following type:

$$H = \begin{pmatrix} \lambda_0 & \lambda_1 & \dots & \dots & \lambda_m \\ \lambda_1 & \lambda_2 & \dots & \lambda_m & \lambda_{m+1} \\ \dots & \dots & \dots & \dots & \dots \\ \lambda_{p-1} & \lambda_p & \dots & \dots & \lambda_{m+p-1} \\ \lambda_p & \lambda_{p+1} & \dots & \lambda_{m+p-1} & \lambda_{m+p} \end{pmatrix} \in k^{p+1,m+1}$$

Denote by $X_m \subseteq \mathbb{P}^m$ the rational normal curve, locus of points $(a,b)^m = (a^m, a^{m-1}b,...,b^m)$, $a,b \in k$.

Definition 2.2. Define *secant $s-$plane* any $s-$dimensional linear subspace $\Sigma_s \subseteq \mathbb{P}^m$ that cuts X_m in $s+1$ points (counting their multiplicity). If Σ_s cuts X_m in the point P_i with multiplicity t_i, we write $\Sigma_s =< t_0P_0, t_1P_1,...,t_hP_h >$, $(i = 0,...,h; \sum t_i = s+1)$.

Theorem 2.3. ([2], Theorem 3.7) *Let $\pi_r \subseteq \mathbb{P}^m$ be an $r-$plane spanned by points of coordinates $(d_{i0}, \ldots, d_{im})$, $i = 0, \ldots, r$. Then the number of independent Hankel $p-$planes contained in π_r is equal to the number of independent solutions of the linear system associated to the matrix $S_p(\pi_r)$, that is*

$$\dim R(p) = (p+1)(r+1) - rank\ S_p(\pi_r)$$

where

$$S_p(\pi_r) = \begin{pmatrix} B_h & B_f & \Omega & \ldots & \Omega & \Omega \\ \Omega & B_h & B_f & \ldots & \Omega & \Omega \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \Omega & \Omega & \Omega & \ldots & B_h & B_f \end{pmatrix} \in k^{pm,(p+1)(r+1)}$$

$$B_h B_f = \left(\begin{array}{cccc|cccc} d_{01} & d_{11} & \ldots & d_{r1} & d_{00} & d_{10} & \ldots & d_{r0} \\ d_{02} & d_{12} & \ldots & d_{r2} & d_{01} & d_{11} & \ldots & d_{r1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ d_{0m} & d_{1m} & \ldots & d_{rm} & d_{0m-1} & d_{1m-1} & \ldots & d_{rm-1} \end{array} \right)$$

and $\Omega \in k^{m,r+1}$ is the null matrix.

Remark 2.4. If rank $S_p(\pi_r) = p(r+1)$, then $\dim R(p) = r+1$. So $\dim R(p) = \dim R$ for any $p \geq 0$ and π_r is a secant $r-$plane.

Definition 2.5. Let $V \subseteq k^{m+1}$ be a $k-$vector space of dimension $r+1$. V is called *Hankel space* if there exists a non zero Hankel matrix $H \in k^{r+1,m+1}$, whose $(r+1)$ rows belong to V.

Definition 2.6. An $r-$plane $\pi_r \subseteq \mathbb{P}^m$ is called *Hankel plane* if $\pi_r = \mathbb{P}(V)$, where V is a Hankel space. The $r-$plane π_r is called *non trivial* if $\pi_r \cap X_m = \emptyset$. In particular the Hankel $1-$planes are called Hankel lines.

Remark 2.7. Any point $P \in \mathbb{P}^m$ is a Hankel $0-$plane. In fact we can always construct a $1 \times (m+1)-$matrix that can be considered as a Hankel matrix.

Definition 2.8. Let $\Sigma \subseteq \mathbb{P}^m$ be a linear space. A Hankel $r-$plane $\pi_r \subseteq \Sigma$ is called *maximal* in Σ if it is not contained in any Hankel $(r+1)-$plane of Σ.

Remark 2.9. If $P \in \pi_r \cap X_m$, $P \equiv (a,b)^m$, then π_r is trivially Hankel since it is always possible to write a Hankel matrix of rank 1 just using coordinates of P:

$$H = \begin{pmatrix} a^{m+r} & a^{m+r-1}b & \ldots & a^r b^m \\ a^{m+r-1}b & a^{m+r-2}b^2 & \ldots & a^{r-1}b^{m+1} \\ \ldots & \ldots & \ldots & \\ a^m b^r & a^{m-1}b^{r+1} & \ldots & b^{m+r} \end{pmatrix}.$$

Therefore it is more interesting to study Hankel $r-$planes which are skew to X_m.

Remark 2.10. ([2], Remark 4.12) Let $\pi_r = \mathbb{P}(R)$ be a Hankel $r-$plane such that $\pi_r \cap X_m = \emptyset$, where $\dim R = r+1$. Let $H = {}^t(\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_r)$ be the Hankel matrix correspondent to π_r. H has maximal rank by Theorem 2.11 and π_r contains:two Hankel independent $(r-1)-$planes ${}^t(\mathbf{v}_0, \dots, \mathbf{v}_{r-1}), {}^t(\mathbf{v}_1, \dots, \mathbf{v}_r)$, three Hankel independent $(r-2)-$planes ${}^t(\mathbf{v}_0, \dots, \mathbf{v}_{r-2}), {}^t(\mathbf{v}_1, \dots, \mathbf{v}_{r-1}),\ {}^t(\mathbf{v}_2, \dots, \mathbf{v}_r)$,..., r Hankel independent lines ${}^t(\mathbf{v}_0, \mathbf{v}_1), {}^t(\mathbf{v}_1, \mathbf{v}_2),\ \dots, {}^t(\mathbf{v}_{r-1}, \mathbf{v}_r)$, $(r+1)$ Hankel $0-$planes $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_r$.

The following theorem characterizes the non trivial Hankel planes.

Theorem 2.11. (See [2], Theorem 4.11) *Let $\pi_r \subseteq \mathbb{P}^m$ be a Hankel $r-$plane. Then $\pi_r \cap X_m = \emptyset$ if, and only if, you can construct a unique maximal rank Hankel matrix with $r+1$ rows, coordinates of points of π_r.*

3. Hankel maximal $r-$planes

We want to focus our study on Hankel maximal $r-$planes of a linear space $\Sigma \subseteq \mathbb{P}^m$. We need these preliminary results:

Proposition 3.1. *Let $R \subseteq k^{m+1}$ be a vector space,* $\dim R = r+1$, *such that* $\mathbb{P}(R)$ *is skew to $X_m \subseteq \mathbb{P}^m$. If* $\dim R(1) = r$, *then* $\dim R(2) = r-1$, …, $\dim R(r) = 1$, $\dim R(r+1) = 0$ *and $\pi_r = \mathbb{P}(R) \subseteq \mathbb{P}^m$ is a Hankel $r-$plane.*

Proof. Consider a basis of $R(1)$:

$$L_1 = \begin{pmatrix} P_1 \\ Q_1 \end{pmatrix}, L_2 = \begin{pmatrix} P_2 \\ Q_2 \end{pmatrix}, \dots, L_r = \begin{pmatrix} P_r \\ Q_r \end{pmatrix}$$

with $P_i = (a_{i0}, \dots, a_{im}), Q_i = (a_{i1}, \dots, a_{i(m+1)})$ elements of R. Since the line L_i corresponds to the point $(a_{i0}, \dots, a_{i(m+1)}) \in \mathbb{P}^{m+1}$ ([2], Remark 4.15), we look for the Hankel $2-$planes as solutions of the homogeneous linear system associated to the matrix

$$(B_h B_f) = (Q_1\, Q_2 \dots, Q_r \mid P_1\, P_2 \dots, P_r).$$

We have that $r \leq \text{rank } (B_h B_f) \leq r+1$. rank $(B_h B_f)$ cannot be r otherwise $\mathbb{P}(R)$ should be a secant $r-$plane (Remark 2.4). Then it is $r+1$ and the linear system has $2r-(r+1) = r-1$ independent solutions. To determine the Hankel $3-$planes, we know that $R(1)$ has dimension equal to r and $R(2)$ has dimension $r-1$. By the same argument, $R(3) \cong R(2)(1)$ has dimension $r-2$, and so on. In particular there exists a non-zero element of $R(r)$. So π_r contains $(r-1)$ independent Hankel $2-$planes, $(r-2)$ independent Hankel $3-$planes, …, 2 independent Hankel $(r-1)-$planes, 1 Hankel $r-$ Hankel plane. Then π_r is a Hankel $r-$plane. □

Example 3.2. Consider in $\mathbb{P}^6$ a Hankel $2-$plane π_2 with Hankel matrix:

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & -1 & 1 & 2 & 0 \\ 0 & 0 & -1 & 1 & 2 & 0 & -1 \end{pmatrix},$$

and a Hankel line π_1 with Hankel matrix:

$$\begin{pmatrix} v_4 \\ v_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 & 0 & 2 & 3 & 1 \\ 0 & -1 & 0 & 2 & 3 & 1 & 0 \end{pmatrix}.$$

We observe that π_1 and π_2 are not skew since $v_4 = v_1 + v_2 + v_3$. Let $\pi = \pi_1 + \pi_2$ be their joining space, $\dim \pi = 3$. π contains 3 independent Hankel lines $(1,0,0,0,-1,1,2,0)$, $(0,0,0,-1,1,2,0,1)$, $(1,0,-1,0,2,3,1,0)$. Consider the matrix $(B_h B_f)$ to compute the Hankel
$2-$planes:

$$(B_h B_f) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 \\ -1 & 1 & 2 & 0 & -1 & 0 \\ 1 & 2 & 3 & -1 & 1 & 2 \\ 2 & 0 & 1 & 1 & 2 & 3 \\ 0 & -1 & 0 & 2 & 0 & 1 \end{pmatrix}.$$

By simple computation, we have that rank $(B_h B_f) = 4$ and there are two Hankel independent $2-$planes in π. Hence π_1 is not maximal in π.

Corollary 3.3. *Let $\pi_r, \pi_s \subseteq \Sigma$ be maximal Hankel planes in the linear space Σ. Then $\pi_r \cap \pi_s = \emptyset$.*

Proof. If π_s is a Hankel $s-$plane, then for any point $P \in \pi_s$ there exists a Hankel line L, passing through P. In fact, π_s contains two Hankel $(s-1)-$planes; in the sheaf generated by these two Hankel $(s-1)-$ planes consider the Hankel π_{s-1} plane passing through P. π_{s-1} contains two Hankel $(s-2)-$planes. In the sheaf generated by these two Hankel $(s-2)-$ planes consider the Hankel π_{s-2} plane passing through P and so on. Finally we find a Hankel line L passing through P. Suppose $\pi_r \cap \pi_s \neq \emptyset$ and $P \in \pi_r \cap \pi_s$. Since for any point of π_s there is a Hankel line L, then $\pi_r \cap L \neq \emptyset$. Consider the Hankel matrix that represents the Hankel π_r-plane:

$$H_1 = \begin{pmatrix} A_0 \\ A_1 \\ \dots \\ A_r \end{pmatrix} = \begin{pmatrix} a_0 & a_1 & \dots & a_m \\ a_1 & a_2 & \dots & a_{m+1} \\ \dots & \dots & \dots & \dots \\ a_r & a_{r+1} & \dots & a_{m+r} \end{pmatrix}$$

and the Hankel line

$$L = \begin{pmatrix} B_0 \\ B_1 \end{pmatrix} = \begin{pmatrix} b_0 & b_1 & \dots & b_m \\ b_1 & b_2 & \dots & b_{m+1} \end{pmatrix}.$$

Suppose that $B_0 \in < A_0, \dots, A_r, B_1 >$. Consider the joining space $\pi_r + L = < A_0, A_1, \dots, A_r, B_1 >$. By Proposition 2.3, consider the matrix $(B_h B_f)$ to find the Hankel lines:

$$\begin{pmatrix} a_1 & a_2 & \dots & a_{r+1} & b_2 & a_0 & a_1 & \dots & a_r & b_1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & & \\ a_m & a_{m+1} & \dots & a_{m+r} & b_{m+1} & a_{m-1} & a_m & \dots & a_{m+r-1} & b_m \end{pmatrix} \in k^{m,2r+4}.$$

In the matrix $(B_h B_f)$ the columns arising from the Hankel matrix H_1 are $2(r+1)$ whose $r+2$ are independent. Since the column ${}^t(b_1, \dots, b_m) \in \langle {}^t(a_1, \dots, a_m), \dots, {}^t(a_{r+1}, \dots, a_{m+r}), {}^t(b_2, \dots, b_{m+1}) \rangle$, then rank $(B_h B_f) = r+3$. Thus, there are $r+1$ Hankel independent lines in $\pi_r + L$ and so, by Proposition 3.1, $\pi_r + L$ is a Hankel $(r+1)-$plane that contains a Hankel maximal $r-$plane. Contradiction. □

Remark 3.4. Let π_r and π_s be two Hankel planes contained in a linear space Σ, then the Hankel $h-$planes that they contain can be dependent $(h < \min(r,s))$. In this case π_r and π_s are not maximal in their joining space.

Theorem 3.5. *Let $\Sigma \subseteq \mathbb{P}^m$ be a linear space of dimension r. Consider t Hankel maximal r_i-planes $\pi_i = \mathbb{P}(V_i) \subseteq \Sigma$, $\dim \pi_i = r_i$, such that $\pi_i \cap X_m = \emptyset$ $(i = 1, ..., t)$. Then $V_1 + ... + V_t = V_1 \oplus ... \oplus V_t$, i.e., $\dim < \pi_1, ..., \pi_t > = \sum_{i=1}^t \dim V_i - 1$.*

Proof. Consider the sum space $V = \sum_{i=1}^{t} V_i$. Let $\mathbb{P}(V)$ be the joining linear space of all the Hankel π_i-planes. In π_i there are r_i- Hankel lines and $(r_i - 1)$ Hankel $2-$planes (Remark 2.10). Then in $\mathbb{P}(V)$ there are at most $\sum_{i=1}^{t} r_i - t$ Hankel $2-$planes. On the other hand, the Hankel r_i-lines in π_i are:

$$\begin{pmatrix} P_{i1} \\ Q_{i1} \end{pmatrix}, \begin{pmatrix} P_{i2} \\ Q_{i2} \end{pmatrix}, \dots, \begin{pmatrix} P_{ir_i} \\ Q_{ir_i} \end{pmatrix} \qquad P_{ij}, Q_{ij} \in V_i \quad j = 1, ..., r_i, i = 1, \dots, t.$$

Compute the Hankel $2-$planes in $\mathbb{P}(V)$ writing the matrix $(B_h B_f)$. Since $R(2) \cong R(1)(1)$, then we have $S_1(\mathbb{P}(R(2))) =$

$$({}^tQ_{11} \dots {}^tQ_{1r_1}\, {}^tQ_{21} \dots {}^tQ_{2r_2} \dots {}^tQ_{t1} \dots {}^tQ_{tr_t} \mid {}^tP_{11} \dots {}^tP_{1r_1}\, {}^tP_{21} \dots {}^tP_{2r_2} \dots {}^tP_{t1} \dots {}^tP_{tr_t}).$$

In general, we have $\dim \sum_{i=1}^{t} V_i \leq \sum_{i=1}^{t}(r_i+1)$. Suppose, by absurd, that the sum of subspaces V_i is not direct. It follows that rank $S_1(\mathbb{P}(R(2))) \leq \sum_{i=1}^{t} r_i + t - 1$. On the other hand, the number of independent solutions of the homogeneous linear system, having as associated matrix $S_1(\mathbb{P}(R(2)))$, is

$$\dim R(2) \geq 2\sum_{i=1}^{t} r_i - \sum_{i=1}^{t} r_i - t + 1 = \sum_{i=1}^{t} r_i - t + 1.$$

Since $\dim R(2)$ represents the dimension of $2-$planes, we find a contradiction. In fact, we know that the number of $2-$planes is at most $\sum_{i=1}^{t} r_i - t$. □

4. h−sequences

For a generic linear space $\Sigma \subseteq \mathbb{P}^m$ we want to find all the possible decreasing sequences $h_0, h_1, \ldots, h_i, \ldots$, where h_i is the maximal number of Hankel $i-$planes contained in Σ. In this direction we give the following definitions:

Definition 4.1. Define *h-sequence*, relative to a linear space $\Sigma \subseteq \mathbb{P}^m$, a sequence of integers, whose term h_i denotes the maximal number of independent non trivial Hankel $i-$planes contained in Σ.

Theorem 4.2. *Let $\Sigma \subseteq \mathbb{P}^m$ be a linear space of dimension r, $\Sigma \cap X_m = \emptyset$. Suppose that Σ is the joining space of Hankel maximal planes $\pi_\beta \subseteq \Sigma$ ($\beta = 1,...,t$), with $\dim \pi_1 \leq \dim \pi_2 \leq \ldots \leq \dim \pi_t$. Then, the $h-$sequence relative to Σ is*

$$h_i = \begin{cases} \sum_{j=1}^{s} a_j\alpha_j - t(i-1) & 0 \leq i \leq 1+\alpha_1 \\ \sum_{j=2}^{s} a_j\alpha_j - (t-a_1)(i-1) & 2+\alpha_1 \leq i \leq 1+\alpha_2 \\ \sum_{j=3}^{s} a_j\alpha_j - (t-a_1-a_2)(i-1) & 2+\alpha_2 \leq i \leq 1+\alpha_3 \\ \cdots\cdots & \cdots\cdots \\ \sum_{j=l}^{s} a_j\alpha_j - (t-a_1-\cdots-a_{l-1})(i-1) & 2+\alpha_{l-1} \leq i \leq 1+\alpha_l < \alpha_s \\ a_s & 2+\alpha_{s-1} \leq i \leq \alpha_s \\ 0 & i > \alpha_s \end{cases}$$

where α_l is the generic term in the sequence

$$1 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_l < \ldots < \alpha_s = \dim \pi_t$$

of the dimensions of the Hankel $\pi_\beta-$planes that have different dimensions, and a_l is the number of the independent Hankel α_l- planes.

Proof. By Theorem 3.5, the t maximal planes π_β contained in Σ are mutually skew. We begin to compute the number h_0 of $0-$Hankel planes contained in Σ. It is given by the sum of the generators of Hankel maximal $\pi_\beta-$planes. To be precise, by hypothesis we have a_1 maximal Hankel α_1-planes, a_2 maximal Hankel α_2-planes, ..., a_s Hankel maximal α_s-planes, where $a_1+a_2+\ldots+a_s=t$. A Hankel maximal α_1-plane contains (α_1+1) Hankel $0-$planes (Remark 2.10). So we have $a_1(\alpha_1+1)$ Hankel $0-$planes in π_1. Then, for all Hankel maximal α_j-planes, we have $h_0=\sum_{j=1}^{s} a_j\alpha_j+t$. By the same procedure, we obtain $h_i=\sum_{j=1}^{s} a_j\alpha_j-t(i-1)$ up to $i=\alpha_1+1$. In fact, when we compute the Hankel $i-$planes contained in Σ, any Hankel maximal α_j-plane gives a contribution that decreases by one at any step. The Hankel maximal α_1-planes are the first to finish, that is, they cannot produce Hankel $r-$planes for $r>\alpha_1$. Each α_1-plane gives contribution 1 at level h_{α_1} and contribution 0 at level h_{α_1+1}, just one less than the previous step. The α_1-planes produce no Hankel $i-$planes for $i=\alpha_1+2$.
If $s=1$, then $a_1=t$ and all Hankel planes have the same dimension α_1.

Suppose that $a_1<t$ and consider the set made up by the Hankel $i-$ planes that have dimension $i>\alpha_1+1$. Compute the Hankel (α_1+2) planes. Any maximal α_j-plane contains Hankel $(\alpha_1+2)-$planes in number equal to $(\alpha_2-(\alpha_1+2-1))$ (Remark 2.10). Since the Hankel maximal α_2-planes are a_2, then there are

$$a_2[\alpha_2-(\alpha_1+1)] \qquad (\alpha_1+2)-\textit{ Hankel planes}.$$

The procedure is valid for $j=2,\ldots,s$ and so we have $h_{\alpha_1+2}=a_2[\alpha_2-(\alpha_1+1)]+\ldots+a_s[\alpha_s-(\alpha_1+1)]=\sum_{j=2}^{s} a_j\alpha_j-(t-a_1)(\alpha_1+1)$. In the same way, we obtain

$$h_i=\sum_{j=2}^{s} a_j\alpha_j-(t-a_1)(i-1) \qquad 2+\alpha_1 \leq i \leq 1+\alpha_2.$$

$$h_i=\sum_{j=l}^{s} a_j\alpha_j-(t-a_1-\cdots-a_{l-1})(i-1) \qquad 2+\alpha_{l-1} \leq i \leq \alpha_l+1 < \alpha_s.$$

The last term of the sequence is given by the number a_s of Hankel maximal α_s-planes. We obtain a decreasing sequence whose general term is h_i. $\square$

Corollary 4.3. *If the Hankel maximal planes have all the same dimension,* $\dim \pi_i = r \geq 1\,(i = 1,\ldots,t)$*, then the sequence h_i is the following:*

$$h_i = t(r-i+1) \qquad 0 \leq i \leq r$$

Supposing that $\dim \pi_1 < \dim \pi_2 < \ldots < \dim \pi_t$, we have the following corollary. The simplification is obtained putting $a_j = 1$ and $i = 1,\ldots,s = t$.

Corollary 4.4. *With the hypothesis of Theorem 4.2, suppose that $s = t$. Then the sequence of h_i is the following:*

$$h_i = \begin{cases} \sum_{j=1}^{t} \alpha_j - t(i-1) & 0 \leq i \leq 1+\alpha_1 \\ \sum_{j=2}^{t} \alpha_j - (t-1)(i-1) & 2+\alpha_1 \leq i \leq 1+\alpha_2 \\ \sum_{j=3}^{t} \alpha_j - (t-2)(i-1) & 2+\alpha_2 \leq i \leq 1+\alpha_3 \\ \cdots\cdots & \cdots\cdots \\ \sum_{j=l}^{t} \alpha_j - (t-l+1)(i-1) & 2+\alpha_{l-1} \leq i \leq 1+\alpha_l \\ \cdots\cdots & \cdots\cdots \\ 1 & i = \alpha_t \\ 0 & i > \alpha_t \end{cases}$$

5. Maximal and minimal sequence

In this section first we establish when a sequence of positive integers can be a $h-$sequence. Then, for a given linear space Σ, we characterize the $h-$sequences, in particular the maximal and the minimal ones.

Definition 5.1. Given a sequence of integers $h_0,\ldots h_i$, define Δh the sequence with the general term $\Delta h_i = h_i - h_{i+1}$.

Theorem 5.2. *Let $h = h_0, h_1, \ldots, h_\alpha, 0, \ldots$ be a sequence of integers such that $h_i > h_{i+1}$ for $i \leq \alpha, h_i = 0$ for $i > \alpha$ and $\Delta h_0 = \Delta h_1$. Then h is the $h-$sequence of a linear space $\Sigma \subseteq \mathbb{P}^m$ spanned by $t = \Delta h_0$ mutually skew Hankel planes of positive dimension if and only if Δh is not increasing.*

Proof. (⇒) Since $\Delta h_0 = \Delta h_1$, Σ is the joining space of t maximal planes of positive dimension and we can choose as generators of Σ h_0 points such that none of them is a Hankel maximal $0-$plane.
By hypothesis, we have a $h-$sequence determined by t mutually skew Hankel planes. Suppose that they are so organized: a_1 Hankel α_1-planes, a_2 Hankel α_2-planes, $\dots, a_s$ Hankel α_s-planes.

By simple computation, you see that $\Delta h_0 = \Delta h_1 = \dots = \Delta h_{\alpha_1} = t$ and in the $h-$sequence the contribution a_1 of Hankel maximal α_1-planes is missing from h_{α_1+2}. Then $h_{\alpha_1+2} = h_{\alpha_1+1} - (t - a_1)$ and $\Delta h_{\alpha_1+1} = t - a_1$, that implies $\Delta h_{\alpha_1} > \Delta h_{\alpha_1+1}$. By the same procedure, we compute the other terms of the $h-$sequence and finally we obtain $\Delta h_{\alpha_s-1} = t - \sum_{i=0}^{s-1} a_i = \Delta h_{\alpha_s} = h_{\alpha_s}$ that is the last non zero term of the sequence Δh. In fact $\Delta h_{\alpha_s+1} = 0 = t - \sum_{i=0}^{s} a_i = \Delta h_{\alpha_s} - a_s$ and $\Delta h_{\alpha_s} > \Delta h_{\alpha_s+1}$.

Since the last non zero term of the $h-$sequence is h_{α_s}, $\Delta h_{\alpha_s+1} = 0$ and $\Delta h_i = 0$ for $i > \alpha_s$. Therefore Δh is a non increasing finite sequence.
(⇐) Suppose that $\Delta h_0 = \Delta h_1 = \dots = \Delta h_{\alpha_1} = t$ and $\Delta h_{\alpha_1} > \Delta h_{\alpha_1+1}$. We can build a linear space Σ spanned by t mutually skew Hankel planes: $a_1 = \Delta h_{\alpha_1} - \Delta h_{\alpha_1+1}$ of them are α_1-planes. Assuming $\Delta h_{\alpha_1+1} = \dots = \Delta h_{\alpha_2} > \Delta h_{\alpha_2+1}$ we have $a_2 = \Delta h_{\alpha_2} - \Delta h_{\alpha_2+1}$ Hankel α_2-planes and so on, up to $\alpha_s = \alpha$; we have $h_{\alpha_s} = \Delta h_{\alpha_s}$ gives the number of Hankel α_s-planes. Altogether the above Hankel planes build the required linear space Σ. □

Example 5.3. Consider the mutually skew Hankel planes π_2, π_2', π_3, π_4, π_4', π_6, π_6' in the linear space they span: two $2-$planes, one $3-$plane, two $4-$planes, two $6-$planes. Therefore $s = 4$, $\alpha_s = 6$ and $t = 7$.
Then the $h-$sequence is: $h_0 = 34$, $h_1 = 27$, $h_2 = 20$, $h_3 = 13$, $h_4 = 8$, $h_5 = 4$, $h_6 = 2$.
Conversely, given the above sequence, the Δh sequence is $\Delta h_0 = 7$, $\Delta h_1 = 7$, $\Delta h_2 = 7$, $\Delta h_3 = 5$, $\Delta h_4 = 4$,$\Delta h_5 = 2$, $\Delta h_6 = 2$. Since $\Delta h_1 = \Delta h_2 > \Delta h_3 > \Delta h_4 > \Delta h_5 = \Delta h_6 > \Delta h_7$, there exist two Hankel maximal $2-$planes, one Hankel maximal $3-$plane, two Hankel maximal $4-$planes, two Hankel maximal $6-$planes. There are not maximal Hankel lines and maximal Hankel $5-$planes.

Now, we study cases relative to a chosen linear space Σ, not necessarily skew to X_m.

Theorem 5.4. *Let $\Sigma \subseteq \mathbb{P}^m$ be a linear space of dimension r and $deg(\Sigma \cap X_m) = s$. Suppose that in Σ there are t Hankel maximal planes of positive dimension, and that Σ' is their joining space. Then the $h-$sequence h_i relative to Σ and the $h-$sequence h_i' relative to Σ' are different only for the first term: $h_0 = h_0' + \beta$, where $\beta \leq \frac{m}{2}$ denotes the number of the maximal $0-$planes contained in Σ.*

Proof. Consider the linear space $\Sigma' \subseteq \Sigma$, skew to X_m and joining the maximal $t-$planes in Σ. The $h-$sequence relative to Σ' is $h'_0, h'_1, \ldots, h'_\gamma, 0, \ldots, 0, \ldots 0$, where h'_i is given by Theorem 3.5 ($i = 0, \ldots \gamma$) and h'_γ is the last term of the sequence, where there exist Hankel $i-$planes skew to X_m. The two $h-$sequences relative to Σ and to Σ' are different for the maximal $0-$planes skew to X_m contained in Σ. Then $h_0 = h'_0 + \beta \quad h_i = h'_i \quad i \geq 1$, where β is the number of the maximal $0-$planes skew to X_m. We observe that they do not produce Hankel i-planes in the other terms of the $h-$sequence. Then the linear system with associated matrix $(B_h B_f)$ has no solution. As a consequence, since $(B_h B_f) \in k^{m,2\beta}$, it must be $\beta \leq \dfrac{m}{2}$.

□

Theorem 5.5. (maximal sequence) *Let $\Sigma \subseteq \mathbb{P}^m$ be a linear space of dimension r. Suppose that deg$(\Sigma \cap X_m) = s$. Then the maximal h-sequence relative to Σ is*

$$h_0 = r + 1 - s, h_1 = r - s, \ldots, h_{r-s} = 1, h_{r-s+1} = 0, \ldots.$$

Proof. The maximal $h-$sequence is obtained when Σ is the join of a non-trivial Hankel $(r-s)-$plane π and s points of X_m. Of course π has $r-s+1$ generators, contains $r-s$ Hankel lines, and so on. □

Remark 5.6. In any term of the $h-$sequence it appears a contribution equal to s, if Σ cuts the rational normal curve X_m in s points, or equal to zero, if Σ is skew to X_m.

Theorem 5.7. (minimal sequence) *Let $\Sigma = \mathbb{P}(R) \subseteq \mathbb{P}^m$ be an $r-$ plane skew with X_m. Let s be the maximum integer such that $(s+1)(r+1) - sm > 0$, then the minimal $h-$sequence relative to Σ is:*

i) $h_0 = r+1, h_1 = 0, \ldots$ *if* $\quad m \geq 2(r+1)$

ii) $h_0 = r+1, h_1 = 2(r+1) - m, \ldots, h_s = (s+1)(r+1) - sm, 0, \ldots$ *if* $\quad m < 2(r+1)$.

In particular, Σ contains $\Delta h_{s-1} - \Delta h_s$ Hankel maximal $(s-1)-$planes and h_s Hankel maximal $s-$ planes.

Proof. Of course the minimal $h-$sequence is obtained when Σ is a general $r-$plane. Consider the matrix $(B_h B_f) \in k^{m,2(r+1)}$ relative to a minimal set of generators of Σ. So we can suppose that $(B_h B_f)$ has maximal rank. If $m \geq 2(r+1)$, then there are no Hankel lines. As a consequence, there are no Hankel $2-$planes and so on, then the $h-$sequence is zero for $i \geq 1$. If $m < 2(r+1)$, there exist $2(r+1) - m$ Hankel lines. Then $R(1) = \langle w_1, \ldots, w_{2r+2-m} \rangle$ and $\dim R(1) =$

$2r+2-m$. Now we can consider a general matrix $H \in k^{(m+1),(m-r)}$ such that R is the space of the row relations; of course the rank of H is $m-r$. Consider the block Toeplitz matrix $T_H(1) \in k^{(m+2),2(m-r)}$. Since $m+2 > 2(m-r)$, we have $2(r+1)-m$ Hankel lines in $\mathbb{P}(R)$, as we know. Consider $T_H(2) \in k^{(m+3),3(m-r)}$ to establish how many $2-$planes there must be in Σ. If $m+3 > 3(m-r)$, there are at least $3(r+1)-2m$ Hankel $2-$planes. Hankel $i-$planes necessarily appear until the number of rows is greater than the number of columns, i. e. $(i+1)(r+1)-im \leq 0$. Note that in the matrix $T_H(s)$ there are more rows then columns, but this is not true for $T_H(s+1)$: $(s+2)(r+1)-(s+1)m \leq 0$.

As a consequence, we deduce that there must be in Σ at least $(s+1)(r+1)-sm = h_s$ Hankel $s-$planes and $\Delta h_{s-1} - \Delta h_s$ Hankel $(s-1)-$planes. Finally, starting from any choice of such (mutually skew) Hankel planes, we can construct the linear space Σ spanned by these Hankel planes: this gives the required $h-$sequence; moreover, this $h-$sequence is obtained for a general $r-$plane in $\mathbb{P}^m$ □

Example 5.8. In $\mathbb{P}^9$ consider a space R of dimension 7. We construct a Hankel matrix of maximal rank, $H \in k^{10,3}$. To be precise, we have $h_0 = 7$, then $T_H(1) \in k^{11,6}$ and $h_1 = 5$, $T_H(2) \in k^{12,9}$ $h_2 = 3$, $T_H(3) \in k^{13,12}$ and $h_3 = 1$, $T_H(4) \in k^{14,15}$ and $h_4 = 0$. In this case $s = 3$ since the number of the rows is less than the number of the columns in $T_H(4) \in k^{14,15}$. Then we have one Hankel maximal $3-$plane and one Hankel maximal $2-$plane. In fact $\Delta h_0 = \Delta h_1 = \Delta h_2$ but $\Delta h_3 = h_3 = 1$ and $\Delta h_4 = 0$.

Acknowledgements

I am grateful to Prof. S. Giuffrida for stimulating discussions and useful comments. I thank the referee for his careful reading of this paper, his deep understanding and the useful suggestions that helped me in improving the paper.

REFERENCES

[1] G. Failla, *Varietà di Hankel, sottovarietà di* $\mathbb{G}(r,m)$, Tesi di Dottorato di Ricerca in Matematica, Messina, 2008.

[2] S. Giuffrida, R. Maggioni, *Hankel Planes*, J. of Pure and Appl. Algebra 209 (2007), 119–138.

[3] J. Harris, *First Course of Algebraic Geometry*, Springer, 1992.

GIOIA FAILLA
Department of Mathematics
University of Messina,
C.da Papardo, salita Sperone, 31,
98166 Messina, Italy
e-mail: gioiafailla@hotmail.it