A STUDY ON \( k \)-COALESCENCE OF TWO GRAPHS

V. K. NAJIYA - A. V. CHITHRA

The \( k \)-coalescence of two graphs is obtained by merging a \( k \)-clique of each graph. The \( A_\alpha \)-matrix of a graph is the convex combination of its degree matrix and adjacency matrix. In this paper, we present some structural properties of a non-regular graph which is obtained from the \( k \)-coalescence of two graphs. Also, we derive the \( A_\alpha \)-characteristic polynomial of \( k \)-coalescence of two graphs and then compute the \( A_\alpha \)-spectra of \( k \)-coalescence of two complete graphs. In addition, we estimate the \( A_\alpha \)-energy of \( k \)-coalescence of two complete graphs. Furthermore, we obtain some topological indices of vertex coalescence of two graphs, and as an application, we determine the Wiener, hyper-Wiener and Zagreb indices of Lollipop and Dumbbell graphs.

1. Introduction

Let \( G \) be a simple graph on \( n \) vertices with vertex set \( \{v_1, v_2, \ldots, v_n\} \) and \( m \) edges. The adjacency matrix[1] \( A(G) = [a_{ij}] \) of \( G \) is defined as an \( n \times n \) matrix with \( a_{ij} = 1 \) if \( v_i \) and \( v_j \) are adjacent, 0 otherwise. The signless Laplacian matrix \( Q(G) \) of \( G \) has the form \( D(G) + A(G) \), where \( D(G) \) is a diagonal matrix with \( a_{ii} = \text{deg}(v_i) \). In [10], Nikiforov introduced a new matrix, which is a convex combination of \( D(G) \) and \( A(G) \), defined as \( A_\alpha(G) = \alpha D(G) + (1 - \alpha)A(G) \), where \( \alpha \in [0, 1] \). The \( A_\alpha \) matrix, \( A_\alpha(G) \) coincides with \( A(G) \), \( D(G) \) and \( \frac{1}{2}Q(G) \) when \( \alpha = 0, 1, \frac{1}{2} \) respectively.

AMS 2010 Subject Classification: 05C50

Keywords: \( A_\alpha \)-spectrum, Coalescence, Wiener index, hyper-Wiener index, Zagreb index
For a matrix $M$, $\Phi(M, \lambda)$ denotes the characteristic polynomial of $M$. The solution for this polynomial constitutes the spectrum of $M$. The adjacency energy $\varepsilon(G)$ of a graph $G$ is defined as the sum of absolute values of its adjacency eigenvalues. If $\lambda_i(A_{\alpha}(G))$ denotes the $A_{\alpha}$-eigenvalues of $G$, then the $A_{\alpha}$-energy is defined as $\varepsilon_{\alpha} = \sum_{i=1}^{n} \lambda_i(A_{\alpha}(G)) - \frac{2\alpha m}{n}$. If $G$ is a regular graph then $A_{\alpha}$-energy is $(1 - \alpha)\varepsilon(G)$.

Let $G_1$ and $G_2$ be two graphs on $n_1, n_2$ vertices and $m_1, m_2$ edges. The $k$-coalescence of $G_1$ and $G_2$ is the graph obtained by merging a clique of order $k$ of both $G_1$ and $G_2$. The graph $G_1 \circ_k G_2$ is non-regular with $n_1 + n_2 - k$ vertices and $m_1 + m_2 - \frac{k(k-1)}{2}$ edges. If $k = 1$, it is called the vertex coalescence and if $k = 2$, it is called the edge coalescence. The merged clique of order $k$ is represented by $Q$. It is difficult to calculate a general formula for $A_{\alpha}$-energy of non-regular graphs. In this paper, we obtain a formula for the $A_{\alpha}$-energy of vertex coalescence and edge coalescence of two complete graphs.

A topological index is a real number that is invariant under graph isomorphism and is derived from the structure of a graph. They have become prevalent due to their applications in several areas, including chemistry and networks. The most famous indices are Zagreb, Randić, Wiener, harmonic indices and their variants. Many chemists and mathematicians have extensively studied the Wiener index. In this paper, we compute certain topological indices, such as the Wiener index, hyper Wiener index, etc., of $k$-coalescence of two graphs.

Throughout this paper, $K_n$ denotes the complete graph of order $n$. The matrix $I_n$ denotes the identity matrix of order $n$, $O_{m \times n}$ denotes the 0 matrix of order $m \times n$ and $J_{m \times n}$ is the matrix of order $m \times n$ with all entries equal to one.

This paper is organised as follows. Section 2 presents some definitions and results used for our work. In Section 3, we determine some structural properties of $k$-coalescence of two graphs. In Section 4, we estimate the $A_{\alpha}$-characteristic polynomial of $k$-coalescence of two graphs. In Section 5, $A_{\alpha}$-spectrum and $A_{\alpha}$-energy of $k$-coalescence of two complete graphs are determined. In Section 6, some topological indices of vertex coalescence of two graphs are computed.

2. Preliminaries

This section presents some definitions and theorems used to prove the main results. For basic graph theoretical definitions, the reader can refer to [1].

**Definition 2.1.** [1] The distance $d(u, v)$ between two vertices $u$ and $v$ in $G$ is the length of the shortest path joining them, if any; otherwise, $d(u, v) = \infty$.

**Definition 2.2.** [1] A complete subgraph of $G$ is called a clique of $G$, and a clique of $G$ is a maximal clique of $G$ if it is not properly contained in another
A STUDY ON $k$-COALESCENCE OF TWO GRAPHS

319
clique of $G$. The clique number of a graph $G$ is the number of vertices in a maximal clique of $G$, denoted by $\omega(G)$.

**Theorem 2.3.** [1] A nontrivial connected graph $G$ is Eulerian if and only if every vertex of $G$ has an even degree.

**Definition 2.4.** [2] Let $G$ be a finite, undirected, connected simple graph. Wiener index $W(G)$ of a graph $G$ is a distance based topological index, defined as the sum of the distance between all pairs of vertices in a graph $G$. Let $d_G(v)$ be the sum of distance between $v$ and all other vertices of $G$, then

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u,v) = \frac{1}{2} \sum_{v \in V(G)} d_G(v).$$

**Definition 2.5.** [4] Let $G$ be a finite, undirected, connected simple graph. The hyper-Wiener index $WW(G)$ of a graph $G$ is defined as

$$WW(G) = \frac{1}{2} W(G) + \frac{1}{2} \sum_{\{u,v\} \subseteq V(G)} d^2(u,v),$$

where $d^2(u,v) = d(u,v)^2$ and $d(u,v)$ is distance from $u$ to $v$. Let $d_G^2(v)$ be the sum of square of distances between $v$ and all other vertices of $G$, then

$$WW(G) = \frac{1}{2} W(G) + \frac{1}{4} \sum_{v \in V(G)} d_G^2(v).$$

**Definition 2.6.** [3] The forgotten topological index $F(G)$ of a graph $G$ is

$$F(G) = \sum_{v \in V(G)} \deg(v)^3 = \sum_{uv \in E(G)} (\deg(u)^2 + \deg(v)^2).$$

**Definition 2.7.** [6] The first Zagreb index $M_1(G)$ of a graph $G$ is $M_1(G) = \sum_{v \in V(G)} \deg(v)^2$.

**Definition 2.8.** [5] The Narumi - Katayama index $NK(G)$ of a graph $G$ is

$$NK(G) = \prod_{v \in V(G)} \deg(v).$$

3. **Structural properties of $k$-coalescence of graphs**

This section estimates the structural properties of $k$-coalescence of graphs, namely, chromatic number, vertex connectivity, edge connectivity, etc. Throughout the section, $G_i$ represents graphs on $n_i$ vertices.

We represent a graph’s maximum degree and minimum degree by $\Delta(G)$ and $\delta(G)$, respectively.
Proposition 3.1. Let $G_i$ be regular graphs of order $n_i$ and regularity $r_i$ for $i = 1, 2$ and let $G = G_1 \circ_k G_2$. Then $\Delta(G) = r_1 + r_2 - k + 1$.

If $k = n_1$ or $n_2$, then $\delta(G) = \max\{r_1, r_2\}$ and if $k < n_1, n_2$, then $\delta(G) = \min\{r_1, r_2\}$.

Proof. Let $v$ be any vertex of $G_1 \circ_k G_2$. Then

$$\deg(v) = \begin{cases} 
\deg_{G_1}(v) & \text{if } v \in V(G_1 \setminus Q), \\
\deg_{G_2}(v) & \text{if } v \in V(G_2 \setminus Q), \\
\deg_{G_1}(v) + \deg_{G_2}(v) - k + 1 & \text{if } v \in Q.
\end{cases}$$

If $G_1$ and $G_2$ are regular, then the vertices in $Q$ have degree $r_1 + r_2 - k + 1$, which is greater than $r_1$ and $r_2$. Thus the maximum degree, $\Delta(G) = r_1 + r_2 - k + 1$.

Without loss of generality, assume that $k = n_1$ and $n_1 < n_2$. Then all the vertices in $G_1$ will be merged to a $k$ clique in $G_2$ resulting in $G_2$ itself. Then $\delta(G) = r_2 = \max\{r_1, r_2\}$.

Next assume $k < n_1, n_2$. Then there are vertices of degrees $r_1$ and $r_2$ in $G_1 \circ_k G_2$. Thus $\delta(G) = \min\{r_1, r_2\}$. \qed

Proposition 3.2. Let $g(G_i)$ be the girth of $G_i, i = 1, 2$. Then the girth of $G_1 \circ_k G_2$

$$g(G_1 \circ_k G_2) = \begin{cases} 
3 & \text{if } k \geq 3, \\
\min\{g(G_1), g(G_2)\} & \text{if } k \leq 2.
\end{cases}$$

Proof. If $k$ is greater than 2, then the graph $G_1 \circ_k G_2$ will have a cycle of length 3 in $Q$.

If $k \leq 2$, then the shortest cycle in $G_1 \circ_k G_2$ will be the shortest cycle in either $G_1$ or $G_2$. \qed

Proposition 3.3. Let $\omega_i$ be the clique number of $G_i, i = 1, 2$. Then the clique number of $G_1 \circ_k G_2$,

$$\omega(G_1 \circ_k G_2) = \max\{\omega_1, \omega_2\}.$$ 

Proof. The graph $G = G_1 \circ_k G_2$ has $G_1$ and $G_2$ as induced subgraphs. Thus, any clique of $G_1$ and $G_2$ is a clique of $G$ as well. Also, the merging of vertices does not produce a new clique. Hence, $\omega(G_1 \circ_k G_2) = \max\{\omega_1, \omega_2\}$. \qed

Proposition 3.4. Let $\kappa_i$ be the vertex connectivity of $G_i, i = 1, 2$. Then the vertex connectivity of $G_1 \circ_k G_2$,

$$\kappa(G_1 \circ_k G_2) = \min\{\kappa_1, \kappa_2, k\}.$$
Proof. Suppose $K_1$ and $K_2$ are greater than or equal to $k$, then $G_1 \circ_k G_2$ can be disconnected by removing $k$ vertices in $Q$. Otherwise, the minimum vertex-cut of $G_i$ belongs to $V(G_i \setminus Q)$. Therefore, the vertex connectivity of $G_1 \circ_k G_2 = \min\{K_1, K_2, k\}$. 

Proposition 3.5. Let $\lambda_i$ be the edge connectivity of $G_i, i = 1, 2$. Then the edge connectivity of $G_1 \circ_k G_2$, 

$$\lambda_i(G_1 \circ_k G_2) = \min\{\lambda_1, \lambda_2\}.$$ 

Proof. If the minimum edge-cut of $G_1$ and $G_2$ does not belong to $Q$, then edge connectivity of $G_1 \circ_k G_2 = \min\{\lambda_1, \lambda_2\}$. If the minimum edge-cut of $G_1$ or $G_2$ is in $Q$, then it is same as the minimum edge-cut of $G_1 \circ_k G_2$, therefore $G_1 \circ_k G_2 = \min\{\lambda_1, \lambda_2\}$. 

Proposition 3.6. Let $G_1$ and $G_2$ be Eulerian graphs. Then the graph $G_1 \circ_k G_2$ is Eulerian if and only if $k$ is odd.

Proof. If $G_1$ and $G_2$ are Eulerian, then by Theorem 2.3, every vertex of $G_1$ and $G_2$ are of even degree. For a vertex $v$ in $G_1 \circ_k G_2$

$$\deg(v) = \begin{cases} \deg_{G_1}(v) & \text{if } v \in V(G_1 \setminus Q), \\ \deg_{G_2}(v) & \text{if } v \in V(G_2 \setminus Q), \\ \deg_{G_1}(v) + \deg_{G_2}(v) - k + 1 & \text{if } v \in Q. \end{cases}$$

Then $G_1 \circ_k G_2$ is Eulerian if and only if $\deg_{G_1}(v) + \deg_{G_2}(v) - k + 1$ is even, that is $k$ is odd. 

![Figure 1: $C_4 \circ_2 C_4$ is not Eulerian whereas $C_4 \circ_1 C_4$ is Eulerian.](image)

Proposition 3.7. For $k > 1$, the graph $G_1 \circ_k G_2$ is Hamiltonian if and only if both $G_1$ and $G_2$ are Hamiltonian. If $k = 1$, then $G_1 \circ_k G_2$ is not Hamiltonian.
Proof. If $k = 1$, the vertex in $Q$ is a vertex cut. Then $G_1 \circ_k G_2$ is not Hamiltonian.

Consider $k \geq 2$. Let $n_i$ be the order of $G_i$, $i = 1, 2$. Assume $G_1$ and $G_2$ are Hamiltonian, then they have a Hamiltonian cycle $u_1u_2 \cdots u_{n_1}u_1$ and $v_1v_2 \cdots v_{n_2}v_1$ respectively, where $u_i$'s are the vertices of $G_1$ and $v_i$'s are the vertices of $G_2$.

Let $u_r, u_{r+1}, \cdots, u_{r+k}$ and $v_1, v_2, \cdots, v_k$ be the vertices merging in $G_1 \circ_k G_2$. We denote the resulting vertices as $w_1, w_2, \cdots, w_k$. The merging is in such a way that $v_1$ merge with $u_{r+m+1}$ for some $m \in \{r, r+1, \cdots, r+k\}$ and is denoted as $w_{m+1}$, $v_2$ merge with $u_{r+m+2}$ and is denoted as $w_{m+2}$ and so on (see Figure 2).

Then we can construct a new Hamiltonian cycle

$$u_1u_2 \cdots w_1w_2 \cdots w_mv_{k+1}v_{k+2} \cdots v_{n_2}w_{m+1} \cdots w_k \cdots u_{n_1}u_1.$$

Hence $G_1 \circ_k G_2$ is Hamiltonian.

Conversely, if $G_1 \circ_k G_2$ is Hamiltonian, then there exists a Hamiltonian cycle $u_1u_2 \cdots w_1w_2 \cdots w_mv_{k+1}v_{k+2} \cdots v_{n_2}w_{m+1} \cdots w_k \cdots u_{n_1}u_1$. In this cycle, consider the path $w_{m+1} \cdots w_k \cdots u_{n_1}u_1u_2 \cdots w_1w_2 \cdots w_m$. Since there is an edge between $w_m$ and $w_{m+1}$, adding this edge to the path will produce a cycle containing all the vertices of $G_1$. Therefore $G_1$ is Hamiltonian. Similarly, we can show that $G_2$ is also Hamiltonian.

Figure 2: Hamiltonian cycle in $G_1 \circ_k G_2$.

The following proposition gives us a lower and upper bound for the independence number of $k$-coalescence of two graphs.
Proposition 3.8. Let $\beta_0(G_i)$ be the independence number of $G_i, i = 1, 2$. Then the independence number of $G = G_1 \circ_k G_2$ satisfies

$$\beta_0(G_1) + \beta_0(G_2) - 2 \leq \beta_0(G) \leq \beta_0(G_1) + \beta_0(G_2).$$

Proof. Let $G = G_1 \circ_k G_2$

Case 1: Both $G_1$ and $G_2$ are complete graphs.

Then the vertices in $V(G_1 \setminus Q)$ are not adjacent to vertices in $V(G_2 \setminus Q)$. Thus $\beta_0(G) = 2 = \beta_0(G_1) + \beta_0(G_2)$.

Case 2: Either $G_1$ or $G_2$ is complete.

Without loss of generality, assume that $G_1$ is complete and $G_2$ is not. If the independent set of $G_2$ contains a vertex in $Q$, then $\beta_0(G) = \beta_0(G_2) + 1 = \beta_0(G_2) + \beta_0(G_1)$. If the independent set of $G_2$ does not contain a vertex in $Q$, then the independent set of $G$ contains independent vertices of $G_2$ along with a vertex from $G_1 \setminus Q$. Thus $\beta_0(G_1 \circ_k G_2) = \beta_0(G_2) + 1 = \beta_0(G_2) + \beta_0(G_1)$.

Case 3: Neither $G_1$ nor $G_2$ is complete.

If both $G_i$’s have an independent set disjoint from $Q$ then their union gives the independent set for $G$, that is, $\beta_0(G) = \beta_0(G_1) + \beta_0(G_2)$. If one of the $G_i$’s has a vertex common in its independent set and $Q$, then $\beta_0(G) = \beta_0(G_1) + \beta_0(G_2) - 1$. If both $G_i$’s have vertices common in their independent set and $Q$, then $\beta_0(G) = \beta_0(G_1) + \beta_0(G_2) - 2$. \hfill $\Box$

Proposition 3.9. Let $\chi_i$ be the chromatic number of $G_i, i = 1, 2$. Then the chromatic number of $G_1 \circ_k G_2$,

$$\chi(G_1 \circ_k G_2) = k + \max\{\chi_1 - k, \chi_2 - k\}.$$

Proof. We need $k$ different colours to colour the vertices in $Q$. Since the vertices in $V(G_1 \setminus Q)$ and $V(G_2 \setminus Q)$ are not adjacent, they can be coloured using $\max\{\chi_1 - k, \chi_2 - k\}$ colours. Thus chromatic number of $G_1 \circ_k G_2 = k + \max\{\chi_1 - k, \chi_2 - k\}$. \hfill $\Box$

4. $A_\alpha$-characteristic polynomial of $k$-coalescence of graphs

This section computes the $A_\alpha$-characteristic polynomial of $k$-coalescence of two graphs. Using that, the $A_\alpha$-characteristic polynomial of Lollipop graphs is estimated.

Let $G$ be a graph containing a $k$-clique. Then we partition the adjacency matrix of $G$ into the form $A(G) = \begin{bmatrix} B & C^T \\ C & A(K_k) \end{bmatrix}$. 

**Proposition 4.1.** Let $G_1$ and $G_2$ be two graphs of order $n_1$ and $n_2$ respectively such that $n_1 + n_2 > 3k$. Then the $A_\alpha$-characteristic polynomial of $G_1 \circ_k G_2$ is

$$\Phi(A_\alpha(G_1 \circ_k G_2), \lambda) = \Phi(A_\alpha(G_1), \lambda) \Phi(A_\alpha(G_2 \setminus Q), \lambda) + \Phi(A_\alpha(G_2), \lambda) \Phi(A_\alpha(G_1 \setminus Q), \lambda)$$

$$- \Phi(A_\alpha(G_1 \setminus Q), \lambda) \Phi(A_\alpha(G_2 \setminus Q), \lambda) \left( \alpha |D_1(Q) - (k - 1)I| + \alpha |D_2(Q) - (k - 1)I| + |\lambda - \alpha(D_1(Q) + D_2(Q) - (k - 1)I) - (1 - \alpha)A(K_k)| \right),$$

where $D_i(Q)$ represents the degree matrix of the $k$ vertices in $Q$ of $G_i$, $i = 1, 2$.

**Proof.** The $A_\alpha$-matrix of $G_1 \circ_k G_2$ with proper labelling has the form

$$A_\alpha(G_1 \circ_k G_2) = \begin{bmatrix} D & R_1^T & R_2^T \\ R_1 & A_\alpha(G_1 \setminus Q) & O \\ R_2 & O & A_\alpha(G_2 \setminus Q) \end{bmatrix},$$

where $D = \alpha |D_1(Q) + D_2(Q) - (k - 1)I| + (1 - \alpha)A(K_k)$ and $R_i = (1 - \alpha)C_i$, where $C_i$ is the block matrix in the adjacency matrix of $G_i$. Then,

$$\Phi(A_\alpha(G_1 \circ_k G_2), \lambda) = \begin{vmatrix} \lambda - D & -R_1^T & -R_2^T \\ -R_1 & \lambda - A_\alpha(G_1 \setminus Q) & O \\ -R_2 & O & \lambda - A_\alpha(G_2 \setminus Q) \end{vmatrix} + \begin{vmatrix} \lambda - D & O & -R_2^T \\ -R_1 & O & O \\ -R_2 & O & \lambda - A_\alpha(G_2 \setminus Q) \end{vmatrix} + \begin{vmatrix} \lambda - D & O & O \\ -R_1 & \lambda - A_\alpha(G_1 \setminus Q) & O \\ -R_2 & O & \lambda - A_\alpha(G_2 \setminus Q) \end{vmatrix}.$$

Adding and subtracting

$$\begin{vmatrix} \lambda - D & O & O \\ -R_1 & \lambda - A_\alpha(G_1 \setminus Q) & O \\ -R_2 & O & \lambda - A_\alpha(G_2 \setminus Q) \end{vmatrix}$$

to $\Phi(A_\alpha(G_1 \circ_k G_2), \lambda)$, we get

$$\Phi(A_\alpha(G_1 \circ_k G_2), \lambda) = \begin{vmatrix} \lambda - D & -R_1^T & O \\ -R_1 & \lambda - A_\alpha(G_1 \setminus Q) & O \\ -R_2 & O & \lambda - A_\alpha(G_2 \setminus Q) \end{vmatrix} + \begin{vmatrix} \lambda - D & O & -R_2^T \\ -R_1 & \lambda - A_\alpha(G_1 \setminus Q) & O \\ -R_2 & O & \lambda - A_\alpha(G_2 \setminus Q) \end{vmatrix} + \begin{vmatrix} \lambda - D & O & O \\ -R_1 & \lambda - A_\alpha(G_1 \setminus Q) & O \\ -R_2 & O & \lambda - A_\alpha(G_2 \setminus Q) \end{vmatrix}$$

$$- \begin{vmatrix} \lambda - D & O & -R_1^T \\ -R_1 & \lambda - A_\alpha(G_1 \setminus Q) & O \\ -R_2 & O & \lambda - A_\alpha(G_2 \setminus Q) \end{vmatrix} + \begin{vmatrix} \lambda - D & O & O \\ -R_1 & \lambda - A_\alpha(G_1 \setminus Q) & O \\ -R_2 & O & \lambda - A_\alpha(G_2 \setminus Q) \end{vmatrix} - \begin{vmatrix} \lambda - A_\alpha(G_1 \setminus Q) & \lambda - D & -R_2^T \\ -R_2 & \lambda - A_\alpha(G_2 \setminus Q) & \lambda - A_\alpha(G_1 \setminus Q) \end{vmatrix} + \begin{vmatrix} \lambda - A_\alpha(G_1 \setminus Q) & \lambda - D & -R_1^T \\ -R_2 & \lambda - A_\alpha(G_2 \setminus Q) & \lambda - A_\alpha(G_1 \setminus Q) \end{vmatrix}.$$
Here,
\[
\begin{vmatrix}
\lambda - D & -R_1^T \\
-R_1 & \lambda - A_\alpha(G_1 \setminus Q)
\end{vmatrix} = \begin{vmatrix}
\lambda - \alpha(A_1(Q) + D_2(Q) - (k-1)I) - (1-\alpha)A(K_k) & -R_1^T \\
-R_1 & \lambda - A_\alpha(G_1 \setminus Q)
\end{vmatrix}
\]
\[
= \begin{vmatrix}
\lambda - \alpha A_1(Q) - (1-\alpha)A(K_k) & -R_1^T \\
-R_1 & \lambda - A_\alpha(G_1 \setminus Q)
\end{vmatrix}
\]
\[
+ \begin{vmatrix}
-\alpha(A_2(Q) - (k-1)I) & -R_1^T \\
0 & \lambda - A_\alpha(G_1 \setminus Q)
\end{vmatrix}
\]
\[
= [\lambda - A_\alpha(G_1) - \alpha D_2(Q) - (k-1)I] |\lambda - A_\alpha(G_1 \setminus Q)|.
\]

Similarly
\[
\begin{vmatrix}
\lambda - D & -R_2^T \\
-R_2 & \lambda - A_\alpha(G_2 \setminus Q)
\end{vmatrix} = [\lambda - A_\alpha(G_2) - \alpha D_1(Q) - (k-1)I] |\lambda - A_\alpha(G_2 \setminus Q)|.
\]

Therefore,
\[
\Phi(A_\alpha(G_1 \circ_k G_2), \lambda) = [\lambda - A_\alpha(G_1)] |\lambda - A_\alpha(G_2 \setminus Q)| + [\lambda - A_\alpha(G_2)] |\lambda - A_\alpha(G_1 \setminus Q)|
\]
\[
- [\lambda - A_\alpha(G_1 \setminus Q)] [\lambda - A_\alpha(G_2 \setminus Q)] \left( \alpha \left( |D_1(Q) - (k-1)I| + |D_2(Q) - (k-1)I| \right) + |\lambda - D| \right)
\]
\[
= \Phi(A_\alpha(G_1), \lambda) \Phi(A_\alpha(G_2 \setminus Q), \lambda) + \Phi(A_\alpha(G_2), \lambda) \Phi(A_\alpha(G_1 \setminus Q), \lambda)
\]
\[
- \Phi(A_\alpha(G_1 \setminus Q), \lambda) \Phi(A_\alpha(G_2 \setminus Q), \lambda) \left( \alpha \left( |D_1(Q) - (k-1)I| + |D_2(Q) - (k-1)I| \right) + |\lambda - D| \right).
\]

\[
\square
\]

Corollary 4.2. Let \( G_1 \) and \( G_2 \) be two graphs of order \( n_1 \) and \( n_2 \) respectively such that \( n_1 + n_2 > 3k \). Then the adjacency characteristic polynomial of \( G_1 \circ_k G_2 \) is
\[
\Phi(A_\alpha(G_1 \circ_k G_2), \lambda) = \Phi(A(G_1), \lambda) \Phi(A(G_2 \setminus Q), \lambda) + \Phi(A(G_2), \lambda) \Phi(A(G_1 \setminus Q), \lambda)
\]
\[
- (\lambda - k + 1)(n+1)^{k-1} \Phi(A(G_1 \setminus Q), \lambda) \Phi(A(G_2 \setminus Q), \lambda).
\]

Remark 4.3. The Lollipop graph, \( L(m, n-1) \) is obtained from the coalescence of a vertex from a cycle \( C_m \) and a pendant vertex from a path \( P_n \). The \( A_\alpha \)-characteristic polynomial of the Lollipop graph is, 
\[
\Phi(A_\alpha(L(m, n-1)), \lambda) = \Phi(A_\alpha(P_n), \lambda) \Phi(A_\alpha(P_{m-1}), \lambda) + \Phi(A_\alpha(C_m), \lambda) \Phi(A_\alpha(P_{n-1}), \lambda) - \lambda \Phi(A_\alpha(P_{n-1}), \lambda) \Phi(A_\alpha(P_{m-1}), \lambda).
\]

Using the Remark 4.3, we can calculate the \( A_\alpha \)-characteristic polynomial of Lollipop graphs and hence find their spectrum.
Example 4.4. The $A_\alpha$-characteristic polynomial of the Lollipop graph $L(4,3)$ is, \( \Phi(A_\alpha(L(4,3)), \lambda) = \Phi(A_\alpha(P_3), \lambda) \Phi(A_\alpha(P_3), \lambda) + \Phi(A_\alpha(C_4), \lambda) \Phi(A_\alpha(P_2), \lambda) - \lambda \Phi(A_\alpha(P_2), \lambda) \Phi(A_\alpha(P_3), \lambda). \)

5. $A_\alpha$-spectrum of $k$-coalescence of complete graphs

In this section, we compute the $A_\alpha$-spectrum and $A_\alpha$-energy of $K_m \odot_k K_n$.

Proposition 5.1. For $m, n > 1$, the $A_\alpha$-characteristic polynomial of $K_m \odot_k K_n$ is 
\[
\Phi(A_\alpha(K_m \odot_k K_n), \lambda) = (\lambda - \alpha(m+n-k+1))^{k-1} \left( \lambda - \alpha m + 1 \right)^{m-k-1} \left( \lambda - \alpha n + 1 \right)^{n-k-1} \\
\left( \lambda - m + 1 + (1 - \alpha)k \right) \left( \lambda - n + 1 + (1 - \alpha)k \right) \left( \lambda - \alpha (m+n-2k) + 1 - k \right) - (1 - \alpha)^2 k \left( (m+n-2k)\lambda - (m+n-2k)\alpha k - (m-k)(n-k-1) - (n-k)(m-k-1) \right).
\]

Proof. The degree matrix of $K_m \odot_k K_n$ with proper labelling has the form
\[
D(K_m \odot_k K_n) = \begin{bmatrix}
(m+n-1-k)I_k & O_{k \times m-k} & O_{k \times n-k} \\
O_{m-k \times k} & (m-1)I_{m-k} & O_{m-k \times n-k} \\
O_{n-k \times k} & O_{n-k \times m-k} & (n-1)I_{n-k}
\end{bmatrix}.
\]

The adjacency matrix of $K_m \odot_k K_n$ has the form
\[
A(K_m \odot_k K_n) = \begin{bmatrix}
A(K_k) & J_{k \times m-k} & J_{k \times n-k} \\
J_{m-k \times k} & A(K_{m-k}) & O_{m-k \times n-k} \\
J_{n-k \times k} & O_{n-k \times m-k} & A(K_{n-k})
\end{bmatrix}.
\]

Thus the $A_\alpha$-matrix of $K_m \odot_k K_n$ is 
\[
A_\alpha(K_m \odot_k K_n) = \begin{bmatrix}
\beta_1 & (1 - \alpha)J_{k \times m-k} & (1 - \alpha)J_{k \times n-k} \\
(1 - \alpha)J_{m-k \times k} & \beta_2 & O_{m-k \times n-k} \\
(1 - \alpha)J_{n-k \times k} & O_{n-k \times m-k} & \beta_3
\end{bmatrix},
\]
where \( \beta_1 = \alpha(m + n - 1 - k)I_k + (1 - \alpha)A(K_k), \)
\( \beta_2 = \alpha(m - 1)I + (1 - \alpha)A(K_{m-k}) \) and
\( \beta_3 = \alpha(n-1)I + (1 - \alpha)A(K_{n-k}). \)

Then the characteristic polynomial of \( K_m \circ_k K_n \) is
\[
| \lambda I - A_{\alpha}(K_m \circ_k K_n) | = \begin{vmatrix} \lambda I_k - \beta_1 & -(1 - \alpha)J_{k \times m-k} & -(1 - \alpha)J_{k \times n-k} \\ -(1 - \alpha)J_{m-k \times k} & \lambda I_k - \beta_2 & O \\ -(1 - \alpha)J_{n-k \times k} & O & \lambda I_k - \beta_3 \end{vmatrix}.
\]

In the above determinant, performing
\[
C_i \to C_i + \frac{1 - \alpha}{\lambda - m + 1 + (1 - \alpha)k} \sum_{i=k+1}^{m} C_i + \frac{1 - \alpha}{\lambda - n + 1 + (1 - \alpha)k} \sum_{j=m+1}^{m+n-k} C_j
\]
for \( l = 1, 2, \cdots, k \) columns we get,
\[
| \lambda I - A_{\alpha}(K_m \circ_k K_n) | = \begin{vmatrix} \beta_4 & -(1 - \alpha)J_{k \times m-k} & -(1 - \alpha)J_{k \times n-k} \\ O & \lambda I_k - \beta_2 & O \\ O & O & \lambda I_k - \beta_3 \end{vmatrix},
\]
where \( \beta_4 = (\lambda - \alpha(m+n-k) + 1)I_k - (1 - \alpha) \left[ \frac{(1 - \alpha)(m-k)}{\lambda - m + 1 + (1 - \alpha)k} + \frac{(1 - \alpha)(n-k)}{\lambda - n + 1 + (1 - \alpha)k} + 1 \right] J_k. \)

\[
| \lambda I - A_{\alpha}(K_m \circ_k K_n) | = |(\lambda - \alpha(m+n-k) + 1)I_k - (1 - \alpha)XJ_k|
\]
\[
|(\lambda - \alpha(m-1))I - (1 - \alpha)A(K_{m-k})|(\lambda - \alpha(n-1))I - (1 - \alpha)A(K_{n-k})|,
\]
where \( X = \left[ \frac{(1 - \alpha)(m-k)}{\lambda - m + 1 + (1 - \alpha)k} + \frac{(1 - \alpha)(n-k)}{\lambda - n + 1 + (1 - \alpha)k} + 1 \right] \)

Thus
\[
\Phi(A_{\alpha}(K_m \circ_k K_n), \lambda) = (\lambda - \alpha(m+n-k) + 1)^{k-1}(\lambda - \alpha m + 1)^{m-k-1}(\lambda - \alpha n + 1)^{n-k-1} \left( (\lambda - m + 1 + (1 - \alpha)k)(\lambda - n + 1 + (1 - \alpha)k)(\lambda - \alpha(m+n-2k) + 1 - k) - (1 - \alpha)^2 k (m+n-2k) \lambda - (m+n-2k) \alpha k - (m-k)(n-k-1) - (n-k)(m-k-1) \right).
\]

\[ \square \]

Now, in the following corollary, we obtain the \( A_{\alpha} \)-eigenvalues of \( K_m \circ_k K_n \).

**Corollary 5.2.** The \( A_{\alpha} \)-eigenvalues of \( K_m \circ_k K_n \) are

1. \( \alpha(m+n-k) - 1 \) repeated \( k-1 \) times,
2. \( \alpha m - 1 \) repeated \( m-k-1 \) times,
3. \( \alpha n - 1 \) repeated \( n-k-1 \) times,
where \( X \)

\[
\text{Corollary 5.5.} \quad \text{For } m > 2, \quad \text{three roots of the equation } \left( \lambda - m + 1 + (1 - \alpha)k \right) \left( \lambda - n + 1 + (1 - \alpha)k \right) \left( \lambda - \alpha(m+n-2k) + 1 - k \right) - (1 - \alpha)^2k \left( m+n-2k \right) \lambda - \alpha(m+n-2k) - (m-k)(n-k-1) = 0.
\]

The following corollary helps us to determine the \( A_\alpha \)-energy of non-regular graph \( K_m \circ_k K_n \).

\textbf{Corollary 5.3.} \quad \text{The } A_\alpha\text{-energy of } K_m \circ_k K_n \text{ is}

\[
\varepsilon_\alpha(K_m \circ_k K_n) =
(k-1) \left| \alpha(1-2k) + \frac{2\alpha\alpha m}{m+n-1} - 1 \right| + (m-k-1) \left| \alpha(1-k) + \frac{\alpha n(m-n+k)}{m+n-k} - 1 \right| +
(n-k-1) \left| \alpha(1-k) + \frac{\alpha m(n-m+k)}{m+n-k} - 1 \right| + |\beta - X_1| + |\gamma - X_1| + |\delta - X_1|,
\]

where \( X_1 = \frac{\alpha(m^2+n^2-k^2-(m+n-k))}{m+n-k} \) and \( \beta, \gamma, \delta \) are roots of the equation

\[
\left( \frac{\lambda - m + 1 + (1 - \alpha)k}{\lambda - n + 1 + (1 - \alpha)k} \right) \left( \lambda - \alpha(m+n-2k) + 1 - k \right) - (1 - \alpha)^2k \left( m+n-2k \right) \lambda - \alpha(m+n-2k) - (m-k)(n-k-1) = 0.
\]

\textbf{Corollary 5.4.} \quad \text{The } A_\alpha\text{-energy of } K_m \circ_k K_m \text{ is}

\[
\varepsilon_\alpha(K_m \circ_k K_m) =
(k-1) \left| \alpha(1-2k) + \frac{2\alpha m^2}{2m-1} - 1 \right| + 2(m-k-1) \left| \alpha(1-k) + \frac{\alpha m k}{2m-k} - 1 \right| +
|\beta - X_2| + |\gamma - X_2| + |\delta - X_2|,
\]

where \( X_2 = \frac{\alpha(2m^2-k^2-(2m-k))}{2m-k} \) and \( \beta, \gamma, \delta \) are roots of the equation

\[
\left( \lambda - m + 1 + (1 - \alpha)k \right)^2 \left( \lambda - 2\alpha(m-k) + 1 - k \right) - (1 - \alpha)^2k \left( 2m-k \right) \lambda - 2\alpha(k(m-k) - 2(m-k)(m-k-1)) = 0.
\]

\textbf{Corollary 5.5.} \quad \text{For } m, n > 1, \text{ the } A_\alpha\text{-characteristic polynomial of } K_m \circ_1 K_n \text{ is}

\[
\Phi(A_\alpha(K_m \circ_1 K_n), \lambda) = (\lambda - \alpha m + 1)^{m-2} (\lambda - \alpha n + 1)^{n-2} \left( \lambda - m + 2 - \alpha \right) (\lambda - n + 2 - \alpha) (\lambda - \alpha(m+n-2)) - (1 - \alpha)^2 \left( m+n-2 \lambda - (m+n-2) \alpha - (m-1)(n-2) - (m-2)(n-1) \right).
\]

\textbf{Corollary 5.6.} \quad \text{The } A_\alpha\text{-eigenvalues of } K_m \circ_1 K_n \text{ are}
1. $\alpha m - 1$ repeated $m - 2$ times,

2. $\alpha n - 1$ repeated $n - 2$ times,

3. three roots of the equation $(\lambda - m + 2 - \alpha)(\lambda - n + 2 - \alpha)(\lambda - \alpha(m + n - 2)) - (1 - \alpha)^2 \left[(m + n - 2)\lambda - (m + n - 2)\alpha - (m - 1)(n - 2) - (m - 2)(n - 1)\right] = 0.$

The following corollary helps us to determine the $A_{\alpha}$-energy of a non-regular graph $K_m \circ_1 K_n$.

**Corollary 5.7.** The $A_{\alpha}$-energy of $K_m \circ_1 K_n$ is

$$e_\alpha(K_m \circ_1 K_n) = \frac{m-2}{m+n-1} |\alpha n(m-n+1) - (m+n-1)| + \frac{n-2}{m+n-1} |\alpha m(n-m+1) - (m+n-1)| + |\beta - X_3| + |\gamma - X_3| + |\delta - X_3|,$$

where $X_3 = \frac{\alpha(m^2n^2-m-n)}{m+n-1}$ and $\beta, \gamma, \delta$ are roots of the equation $(\lambda - m + 2 - \alpha)(\lambda - n + 2 - \alpha)(\lambda - \alpha(m + n - 2)) - (1 - \alpha)^2 \left[(m + n - 2)\lambda - (m + n - 2)\alpha - (m - 1)(n - 2) - (m - 2)(n - 1)\right] = 0.$

**Corollary 5.8.** The $A_{\alpha}$-energy of $K_m \circ_1 K_m$ is

$$e_\alpha(K_m \circ_1 K_m) = \frac{2(m-2)}{2m-1} (m(2 - \alpha) - 1) + \left|\frac{2m^2(1-\alpha)-5m+2-\alpha}{2m-1}\right| + \left|\beta - \frac{2m\alpha(m-1)}{2m-1}\right| + \left|\gamma - \frac{2m\alpha(m-1)}{2m-1}\right|,$$

where $\beta$ and $\gamma$ are roots of the equation $\lambda^2 - (m - 2 + \alpha(2m - 1))\lambda + 2(m - 1)(\alpha m - 1) = 0$.

**Corollary 5.9.** For $m, n > 2$, the $A_{\alpha}$-characteristic polynomial of $K_m \circ_2 K_n$ is

$$\Phi(A_{\alpha}(K_m \circ_2 K_n), \lambda) = (\lambda - \alpha m + 1)^{n-3}(\lambda - \alpha n + 1)^{n-3}(\lambda - \alpha(m + n - 2) + 1) \left[(\lambda - m + 3 - 2\alpha)(\lambda - n + 3 - 2\alpha)(\lambda - \alpha(m + n - 4) - 1) - 2(1 - \alpha)^2 \left((m + n - 4)\lambda - (m + n - 4)2\alpha - (m - 2)(n - 3) - (m - 3)(n - 2)\right)\right].$$

Now, in the following corollary, we obtain the $A_{\alpha}$-eigenvalues of $K_m \circ_2 K_n$.

**Corollary 5.10.** The $A_{\alpha}$-eigenvalues of $K_m \circ_2 K_n$ are

1. $\alpha m - 1$ repeated $m - 3$ times,
2. $\alpha n - 1$ repeated $n - 3$ times,

3. $\alpha(m + n - 2) - 1$,

4. three roots of the equation \((\lambda - m + 3 - 2\alpha)(\lambda - n + 3 - 2\alpha)(\lambda - \alpha(m + n - 4) - 1) - 2(1 - \alpha)^2 \left[ (m+n-4)\lambda - (m+n-4)2\alpha - (m-2)(n-3) - (m-3)(n-2) \right] = 0.

The following corollary helps us to determine the $A_\alpha$-energy of a non-regular graph $K_m \circ_2 K_n$.

**Corollary 5.11.** The $A_\alpha$-energy of $K_m \circ_2 K_n$ is

$$\epsilon_{\alpha}(K_m \circ_2 K_n) = \frac{m-3}{m+n-2} \left[ \alpha(2mn - 3m - 3n + 6) - 1 \right] + \alpha[(m-n)(n-1) + 2] - 1 + \alpha[(n-m)(m-1) + 2],$$

where $X_4 = \frac{\alpha[m(m-1)+n(n-1)-2]}{m+n-2}$ and $\beta, \gamma, \delta$ are roots of the equation \((\lambda - m + 3 - 2\alpha)(\lambda - n + 3 - 2\alpha)(\lambda - \alpha(m + n - 4) - 1) - 2(1 - \alpha)^2 \left[ (m+n-4)\lambda - (m+n-4)2\alpha - (m-2)(n-3) - (m-3)(n-2) \right] = 0.$

**Corollary 5.12.** The $A_\alpha$-energy of $K_m \circ_2 K_m$ is

$$\epsilon_{\alpha}(K_m \circ_2 K_m) = \frac{m-3}{m-1} \left[ \alpha(2m^2 - 6m + 6) - 1 \right] + \alpha[(m-2) + 2\alpha(m-1)] \lambda + 2\alpha m^2 - m(2\alpha + 3) - 2\alpha + 5 = 0.$$

### 6. Topological indices of vertex coalescence of graphs

In this section, some topological indices of vertex coalescence of graphs are computed. We calculate the Wiener index, hyper-Wiener and Zagreb indices of the Lollipop and Dumbbell graphs using the results.

**Proposition 6.1.** Wiener index of $G_1 \circ_1 G_2$ is

$$W(G_1 \circ_1 G_2) = W(G_1) + W(G_2) + (n_2 - 1)d_{G_1}(v) + (n_1 - 1)d_{G_2}(v),$$

where $v$ is the vertex that is merged in $G_1 \circ_1 G_2$. 

Proof. Let $G = G_1 \circ_1 G_2$ and $v$ be the vertex merging in $G$. From Definition 2.4,

$$W(G) = \sum_{\{u,w\} \in V(G_1)} d(u, w) + \sum_{\{u,w\} \in V(G_2)} d(u, w) + \sum_{u \in V(G_1), w \in V(G_2)} d(u, w)$$

$$= W(G_1) + W(G_2) + (n_2 - 1)d_{G_1}(v) + (n_1 - 1)d_{G_2}(v).$$

\[\square\]

Remark 6.2. The Wiener index of cycle is $W(C_m) = \begin{cases} \frac{m^3}{8} & \text{if } m \text{ is even} \\ \frac{m(m^2 - 1)}{8} & \text{if } m \text{ is odd} \end{cases}$, and Wiener index of path is $W(P_n) = \frac{n(n^2 - 1)}{6}$. Thus the Wiener index of Lollipop graph $L(m, n - 1)$ is

$$W(L(m, n - 1)) = \begin{cases} \frac{m^3}{8} + n(n^2 - 1) - (n - 1)\left(\frac{m^2 + 2n(m-1)}{4}\right) & \text{if } m \text{ is even} \\ \frac{m(m^2 - 1)}{8} + n(n^2 - 1) + (n-1)(m-1)(m+1+2n) & \text{if } m \text{ is odd} \end{cases}$$

Remark 6.3. The Dumbbell graph, denoted by $D_{l,m,n-3}$, is obtained from the coalescence of a cycle $C_l$ and the pendant vertex of a Lollipop graph $L(m, n - 1)$.

The Wiener index of Dumbbell graph $D_{m,m,n-3}$ is

$$W(D_{m,m,n-3}) = \begin{cases} \frac{m^3}{4} + \frac{n(n^2 - 1)}{6} + \frac{m(m^2 + 3mn - 4m + 4) + n(4 - 6m + 2m - 2n) - 2}{2} & \text{if } m \text{ is even} \\ \frac{m(m^2 - 1)}{4} + \frac{n(n^2 - 1)}{6} + (m-1)m^2 - 3m + 3mn - 3n + 4n^2 & \text{if } m \text{ is odd} \end{cases}$$

![Figure 4: D_{4,6,1}](image)

Proposition 6.4. Hyper-Wiener index of $G_1 \circ_1 G_2$ is

$$WW(G_1 \circ_1 G_2) = WW(G_1) + WW(G_2)$$

$$+ \frac{1}{2}\left((n_2 - 1)(d_{G_1}(v) + d^2_{G_1}(v)) + (n_1 - 1)(d_{G_2}(v) + d^2_{G_2}(v)) + 2d_{G_1}(v)d_{G_2}(v)\right),$$

where $v$ is the vertex that is merged in $G_1 \circ_1 G_2$. 
Proof. Let $G = G_1 \circ_1 G_2$ and $v$ be the vertex merging in $G$. From Definition 2.5,

$$WW(G) = \frac{1}{2}(W(G_1) + W(G_2) + (n_2 - 1)d_{G_1}(v) + (n_1 - 1)d_{G_2}(v))$$

$$+ \frac{1}{2} \left( \sum_{\{u,w\} \in V(G_1)} d^2(u,w) + \sum_{\{u,w\} \in V(G_2)} d^2(u,w) + \sum_{u \in V(G_1)} \sum_{w \in V(G_2)} d^2(u,w) \right)$$

$$= WW(G_1) + WW(G_2)$$

$$+ \frac{1}{2} \left( (n_2 - 1)(d_{G_1}(v) + d^2_{G_1}(v)) + (n_1 - 1)(d_{G_2}(v) + d^2_{G_2}(v)) + 2d_{G_1}(v)d_{G_2}(v) \right).$$

\[\square\]

**Remark 6.5.** The hyper-Wiener index of of Lollipop graph $L(m,n - 1)$ is

$$WW(L(m,n - 1)) =$$

$$\begin{cases}
\frac{m^2(m+1)(m+2)}{24} + \frac{n^4+2n^3-n^2-2n}{24} + \frac{(n-1)(m^2+3m+2)+4m(m-1)(n+1)+3m^2n}{24} & \text{if } m \text{ is even} \\
\frac{m(m^2-1)(m+3)}{24} + \frac{n^4+2n^3-n^2-2n}{24} + \frac{(m-1)(n+1)(m+3)+4n(n+1)+3m(m+1)}{24} & \text{if } m \text{ is odd.}
\end{cases}$$

**Remark 6.6.** The hyper-Wiener index of Dumbbell graph $D_{m,m,n-3}$ is

$$WW(D_{m,m,n-3}) =$$

$$\begin{cases}
\frac{m^2(m+1)(m+2)}{24} + \frac{n^4+2n^3-n^2-2n}{24} + \frac{7m^4+4m^3(-5+7n)-8n(3-n+2n^2)+4m^2(2-12n+9n^2)+8m(-2+5n-6n^2+2n^3)}{48} & \text{if } m \text{ is even} \\
\frac{m(m^2-1)(m+3)}{24} + \frac{n^4+2n^3-n^2-2n}{24} + \frac{(-1+m)(3+7m^3-16n-12n^2+16n^3+m^2(-13+28n)+m(-23-20n+36n^2))}{48} & \text{if } m \text{ is odd.}
\end{cases}$$

**Proposition 6.7.** The forgotten topological index of $G_1 \circ_1 G_2$ is

$$F(G_1 \circ_1 G_2) = F(G_1) + F(G_2) + 3\deg_{G_1}(v)\deg_{G_2}(v)(\deg_{G_1}(v) + \deg_{G_2}(v)),$$

where $v$ is the vertex that is merged in $G_1 \circ_1 G_2$.

**Proof.** Let $G = G_1 \circ_1 G_2$ and $v$ be the vertex merging in $G$. From Definition 2.6,

$$F(G) = \sum_{u \in V(G_1)} \deg^3_{G_1}(u) - \deg^3_{G_1}(v) + \sum_{u \in V(G_2)} \deg^3_{G_2}(u) - \deg^3_{G_2}(v)$$

$$+ (\deg_{G_1}(v) + \deg_{G_2}(v))^3$$

$$= F(G_1) + F(G_2) + 3\deg_{G_1}(v)\deg_{G_2}(v)(\deg_{G_1}(v) + \deg_{G_2}(v)).$$

\[\square\]
Proposition 6.8. First Zagreb index of \( G_1 \circ_1 G_2 \) is

\[
M_1(G_1 \circ_1 G_2) = M_1(G_1) + M_1(G_2) + 2\deg_{G_1}(v)\deg_{G_2}(v),
\]

where \( v \) is the vertex that is merged in \( G_1 \circ_1 G_2 \).

Proof. Let \( G = G_1 \circ_1 G_2 \) and \( v \) be the vertex merging in \( G \). From Definition 2.7,

\[
M_1(G) = \sum_{u \in V(G_1)} \deg_{G_1}^2(u) - \deg_{G_1}(v) + \sum_{u \in V(G_2)} \deg_{G_2}^2(u) - \deg_{G_2}(v) + (\deg_{G_1}(v) + \deg_{G_2}(v))^2
\]

\[
= M_1(G_1) + M_1(G_2) + 2\deg_{G_1}(v)\deg_{G_2}(v).
\]

\[\square\]

Remark 6.9. The first Zagreb index of of Lollipop graph \( L(m,n-1) \) is

\[
M_1(L(m,n-1)) = 4(m+n) - 2.
\]

Remark 6.10. The first Zagreb index of Dumbbell graph \( D_{m,m,n-3} \) is

\[
M_1(D_{m,m,n-3}) = 4(2m+n) + 2.
\]

Proposition 6.11. Narumi-Katayama index of \( G_1 \circ_1 G_2 \) is

\[
NK(G_1 \circ_1 G_2) = NK(G_1)NK(G_2)\frac{\deg_{G_1}(v) + \deg_{G_2}(v)}{\deg_{G_1}(v)\deg_{G_2}(v)},
\]

where \( v \) is the vertex that is merged in \( G_1 \circ_1 G_2 \).

Proof. Let \( G = G_1 \circ_1 G_2 \) and \( v \) be the vertex merging in \( G \). From Definition 2.8,

\[
NK(G) = \prod_{u \in V(G_1)} \deg_{G_1}(u) \prod_{u \in V(G_2)} \deg_{G_2}(u) \frac{(\deg_{G_1}(v) + \deg_{G_2}(v))}{\deg_{G_1}(v)\deg_{G_2}(v)}
\]

\[
= NK(G_1)NK(G_2)\frac{\deg_{G_1}(v) + \deg_{G_2}(v)}{\deg_{G_1}(v)\deg_{G_2}(v)}.
\]

\[\square\]

7. Conclusion

This paper estimates some structural properties of a non-regular graph obtained from the \( k \)-coalescence of two graphs. Also, the \( A_\alpha \)-characteristic polynomial of \( k \)-coalescence of two graphs is determined. Moreover, the \( A_\alpha \)-spectrum and \( A_\alpha \)-energy of \( k \)-coalescence of two complete graphs are computed. In addition, some topological indices of vertex coalescence of two graphs are estimated. The Wiener, hyper-Wiener and Zagreb indices of Lollipop and Dumbbell graphs are derived as an application.
Acknowledgments

The first author gratefully acknowledges the financial support of the University Grants Commission (UGC), India.

REFERENCES


**V. K. NAJIYA**

*Department of Mathematics*

*National Institute of Technology Calicut*

*India*

*e-mail: najiya.p190046ma@nitc.ac.in*
A STUDY ON $k$-COALESCENCE OF TWO GRAPHS

A. V. CHITHRA
Department of Mathematics
National Institute of Technology Calicut
India

e-mail: chithra@nitc.ac.in