LE MATEMATICHE
Vol. LXXVIII (2023) – Issue II, pp. 317–335
doi: 10.4418/2023.78.2.3

A STUDY ON k-COALESCENCE OF TWO GRAPHS

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The k-coalescence of two graphs is obtained by merging a k-clique of each graph. The A_α-matrix of a graph is the convex combination of its degree matrix and adjacency matrix. In this paper, we present some structural properties of a non-regular graph which is obtained from the k-coalescence of two graphs. Also, we derive the A_α-characteristic polynomial of k-coalescence of two graphs and then compute the A_α-spectra of k-coalescence of two complete graphs. In addition, we estimate the A_α-energy of k-coalescence of two complete graphs. Furthermore, we obtain some topological indices of vertex coalescence of two graphs, and as an application, we determine the Wiener, hyper-Wiener and Zagreb indices of Lollipop and Dumbbell graphs.

1. Introduction

Let G be a simple graph on n vertices with vertex set $v_1, v_2, \ldots, v_n$ and m edges. The adjacency matrix[1] $A(G) = [a_{ij}]$ of G is defined as an $n \times n$ matrix with $a_{ij} = 1$ if v_i and v_j are adjacent, 0 otherwise. The signless Laplacian matrix $Q(G)$ of G has the form $D(G) + A(G)$, where $D(G)$ is a diagonal matrix with $a_{ii} = deg(v_i)$. In [10], Nikiforov introduced a new matrix, which is a convex combination of $D(G)$ and $A(G)$, defined as $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$, where $\alpha \in [0,1]$. The A_α matrix, $A_\alpha(G)$ coincides with $A(G)$, $D(G)$ and $\frac{1}{2}Q(G)$ when $\alpha = 0, 1, \frac{1}{2}$ respectively.

Received on April 20, 2023

AMS 2010 Subject Classification: 05C50

Keywords: A_α-spectrum, Coalescence, Wiener index, hyper-Wiener index, Zagreb index

For a matrix M, $\Phi(M,\lambda)$ denotes the characteristic polynomial of M. The solution for this polynomial constitutes the spectrum of M. The adjacency energy $\varepsilon(G)$ of a graph G is defined as the sum of absolute values of its adjacency eigenvalues. If $\lambda_i(A_\alpha(G))$ denotes the A_α-eigenvalues of G, then the A_α-energy[7] is defined as $\varepsilon_\alpha = \sum_{i=1}^{n} \left| \lambda_i(A_\alpha(G)) - \frac{2\alpha m}{n} \right|$. If G is a regular graph then A_α-energy is $(1-\alpha)\varepsilon(G)$.

Let G_1 and G_2 be two graphs on n_1, n_2 vertices and m_1, m_2 edges. The k-coalescence[9] $G_1 \circ_k G_2$ of G_1 and G_2 is the graph obtained by merging a clique of order k of both G_1 and G_2. The graph $G_1 \circ_k G_2$ is non-regular with n_1+n_2-k vertices and $m_1+m_2-\frac{k(k-1)}{2}$ edges. If $k=1$, it is called the vertex coalescence and if $k=2$, it is called the edge coalescence[8]. The merged clique of order k is represented by $\mathcal{Q}$. It is difficult to calculate a general formula for A_α-energy of non-regular graphs. In this paper, we obtain a formula for the A_α-energy of vertex coalescence and edge coalescence of two complete graphs.

A topological index is a real number that is invariant under graph isomorphism and is derived from the structure of a graph. They have become prevalent due to their applications in several areas, including chemistry and networks. The most famous indices are Zagreb, Randić, Wiener, harmonic indices and their variants. Many chemists and mathematicians have extensively studied the Wiener index. In this paper, we compute certain topological indices, such as the Wiener index, hyper Wiener index, etc., of k-coalescence of two graphs.

Throughout this paper, K_n denotes the complete graph of order n. The matrix I_n denotes the identity matrix of order n, $O_{m\times n}$ denotes the 0 matrix of order $m\times n$ and $J_{m\times n}$ is the matrix of order $m\times n$ with all entries equal to one.

This paper is organised as follows. Section 2 presents some definitions and results used for our work. In Section 3, we determine some structural properties of k-coalescence of two graphs. In Section 4, we estimate the A_α-characteristic polynomial of k-coalescence of two graphs. In Section 5, A_α-spectrum and A_α-energy of k-coalescence of two complete graphs are determined. In Section 6, some topological indices of vertex coalescence of two graphs are computed.

2. Preliminaries

This section presents some definitions and theorems used to prove the main results. For basic graph theoretical definitions, the reader can refer to [1].

Definition 2.1. [1] The distance $d(u,v)$ between two vertices u and v in G is the length of the shortest path joining them, if any; otherwise, $d(u,v)=\infty$.

Definition 2.2. [1] A complete subgraph of G is called a clique of G, and a clique of G is a maximal clique of G if it is not properly contained in another clique of G. The clique number of a graph G is the number of vertices in a maximal clique of G, denoted by $\omega(G)$.

Theorem 2.3. [1] *A nontrivial connected graph G is Eulerian if and only if every vertex of G has an even degree.*

Definition 2.4. [2] Let G be a finite, undirected, connected simple graph. Wiener index $W(G)$ of a graph G is a distance based topological index, defined as the sum of the distance between all pairs of vertices in a graph G. Let $d_G(v)$ be the sum of distance between v and all other vertices of G, then

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d(u,v) = \frac{1}{2} \sum_{v\in V(G)} d_G(v).$$

Definition 2.5. [4] Let G be a finite, undirected, connected simple graph. The hyper-Wiener index $WW(G)$ of a graph G is defined as

$$WW(G) = \frac{1}{2}W(G) + \frac{1}{2} \sum_{\{u,v\}\subseteq V(G)} d^2(u,v),$$

where $d^2(u,v) = d(u,v)^2$ and $d(u,v)$ is distance from u to v. Let $d_G^2(v)$ be the sum of square of distances between v and all other vertices of G, then

$$WW(G) = \frac{1}{2}W(G) + \frac{1}{4} \sum_{v\in V(G)} d_G^2(v).$$

Definition 2.6. [3] The forgotten topological index $F(G)$ of a graph G is

$$F(G) = \sum_{v\in V(G)} deg(v)^3 = \sum_{uv\in E(G)} \left(deg(u)^2 + deg(v)^2\right).$$

Definition 2.7. [6] The first Zagreb index $M_1(G)$ of a graph G is $M_1(G) = \sum\limits_{v\in V(G)} deg(v)^2$.

Definition 2.8. [5] The Narumi - Katayama index $NK(G)$ of a graph G is $NK(G) = \prod\limits_{v\in V(G)} deg(v)$.

3. Structural properties of k-coalescence of graphs

This section estimates the structural properties of k-coalescence of graphs, namely, chromatic number, vertex connectivity, edge connectivity, etc. Throughout the section, G_i represents graphs on n_i vertices.

We represent a graph's maximum degree and minimum degree by $\Delta(G)$ and $\delta(G)$, respectively.

Proposition 3.1. *Let G_i be regular graphs of order n_i and regularity r_i for $i = 1,2$ and let $G = G_1 \circ_k G_2$. Then $\Delta(G) = r_1 + r_2 - k + 1$.*

If $k = n_1$ or n_2, then $\delta(G) = max\{r_1, r_2\}$ and if $k < n_1, n_2$, then $\delta(G) = min\{r_1, r_2\}$.

Proof. Let v be any vertex of $G_1 \circ_k G_2$. Then

$$deg(v) = \begin{cases} deg_{G_1}(v) & \text{if } v \in V(G_1 \setminus \mathcal{Q}), \\ deg_{G_2}(v) & \text{if } v \in V(G_2 \setminus \mathcal{Q}), \\ deg_{G_1}(v) + deg_{G_2}(v) - k + 1 & \text{if } v \in \mathcal{Q}. \end{cases}$$

If G_1 and G_2 are regular, then the vertices in $\mathcal{Q}$ have degree $r_1 + r_2 - k + 1$, which is greater than r_1 and r_2. Thus the maximum degree, $\Delta(G) = r_1 + r_2 - k + 1$.

Without loss of generality, assume that $k = n_1$ and $n_1 < n_2$. Then all the vertices in G_1 will be merged to a k clique in G_2 resulting in G_2 itself. Then $\delta(G) = r_2 = max\{r_1, r_2\}$.

Next assume $k < n_1, n_2$. Then there are vertices of degrees r_1 and r_2 in $G_1 \circ_k G_2$. Thus $\delta(G) = min\{r_1, r_2\}$. □

Proposition 3.2. *Let $g(G_i)$ be the girth of $G_i, i = 1,2$. Then the girth of $G_1 \circ_k G_2$*

$$g(G_1 \circ_k G_2) = \begin{cases} 3 & \text{if } k \geq 3, \\ min\{g(G_1), g(G_2)\} & \text{if } k \leq 2. \end{cases}$$

Proof. If k is greater than 2, then the graph $G_1 \circ_k G_2$ will have a cycle of length 3 in $\mathcal{Q}$.

If $k \leq 2$, then the shortest cycle in $G_1 \circ_k G_2$ will be the shortest cycle in either G_1 or G_2. □

Proposition 3.3. *Let ω_i be the clique number of $G_i, i = 1,2$. Then the clique number of $G_1 \circ_k G_2$,*

$$\omega(G_1 \circ_k G_2) = max\{\omega_1, \omega_2\}.$$

Proof. The graph $G = G_1 \circ_k G_2$ has G_1 and G_2 as induced subgraphs. Thus, any clique of G_1 and G_2 is a clique of G as well. Also, the merging of vertices does not produce a new clique. Hence, $\omega(G_1 \circ_k G_2) = max\{\omega_1, \omega_2\}$. □

Proposition 3.4. *Let $\mathcal{K}_i$ be the vertex connectivity of $G_i, i = 1,2$. Then the vertex connectivity of $G_1 \circ_k G_2$,*

$$\mathcal{K}(G_1 \circ_k G_2) = min\{\mathcal{K}_1, \mathcal{K}_2, k\}.$$

Proof. Suppose $\mathcal{K}_1$ and $\mathcal{K}_2$ are greater than or equal to k, then $G_1 \circ_k G_2$ can be disconnected by removing k vertices in $\mathcal{Q}$. Otherwise, the minimum vertex-cut of G_i belongs to $V(G_i \setminus \mathcal{Q})$. Therefore, the vertex connectivity of $G_1 \circ_k G_2 = min\{\mathcal{K}_1, \mathcal{K}_2, k\}$. □

Proposition 3.5. *Let λ_i be the edge connectivity of $G_i, i = 1,2$. Then the edge connectivity of $G_1 \circ_k G_2$,*

$$\lambda_i(G_1 \circ_k G_2) = min\{\lambda_1, \lambda_2\}.$$

Proof. If the minimum edge-cut of G_1 and G_2 does not belong to $\mathcal{Q}$, then edge connectivity of $G_1 \circ_k G_2 = min\{\lambda_1, \lambda_2\}$. If the minimum edge-cut of G_1 or G_2 is in $\mathcal{Q}$, then it is same as the minimum edge-cut of $G_1 \circ_k G_2$, therefore $G_1 \circ_k G_2 = min\{\lambda_1, \lambda_2\}$. □

Proposition 3.6. *Let G_1 and G_2 be Eulerian graphs. Then the graph $G_1 \circ_k G_2$ is Eulerian if and only if k is odd.*

Proof. If G_1 and G_2 are Eulerian, then by Theorem 2.3, every vertex of G_1 and G_2 are of even degree. For a vertex v in $G_1 \circ_k G_2$

$$deg(v) = \begin{cases} deg_{G_1}(v) & \text{if } v \in V(G_1 \setminus \mathcal{Q}), \\ deg_{G_2}(v) & \text{if } v \in V(G_2 \setminus \mathcal{Q}), \\ deg_{G_1}(v) + deg_{G_2}(v) - k + 1 & \text{if } v \in \mathcal{Q}. \end{cases}$$

Then $G_1 \circ_k G_2$ is Eulerian if and only if $deg_{G_1}(v) + deg_{G_2}(v) - k + 1$ is even, that is k is odd. □

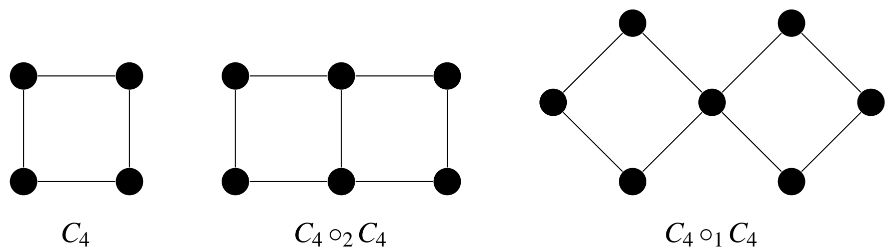

Figure 1: $C_4 \circ_2 C_4$ is not Eulerian whereas $C_4 \circ_1 C_4$ is Eulerian.

Proposition 3.7. *For $k > 1$, the graph $G_1 \circ_k G_2$ is Hamiltonian if and only if both G_1 and G_2 are Hamiltonian. If $k = 1$, then $G_1 \circ_k G_2$ is not Hamiltonian.*

Proof. If $k=1$, the vertex in Q is a vertex cut. Then $G_1 \circ_k G_2$ is not Hamiltonian.

Consider $k \geq 2$. Let n_i be the order of G_i, $i=1,2$. Assume G_1 and G_2 are Hamiltonian, then they have a Hamiltonian cycle $u_1u_2\cdots u_{n_1}u_1$ and $v_1v_2\cdots v_{n_2}v_1$ respectively, where u_i's are the vertices of G_1 and v_i's are the vertices of G_2.

Let $u_r, u_{r+1}, \cdots, u_{r+k}$ and $v_1, v_2, \cdots, v_k$ be the vertices merging in $G_1 \circ_k G_2$. We denote the resulting vertices as $w_1, w_2, \cdots, w_k$. The merging is in such a way that v_1 merge with u_{r+m+1} for some $m \in \{r, r+1, \cdots, r+k\}$ and is denoted as w_{m+1}, v_2 merge with u_{r+m+2} and is denoted as w_{m+2} and so on(see Figure 2). Then we can construct a new Hamiltonian cycle

$$u_1u_2\cdots w_1w_2\cdots w_m v_{k+1}v_{k+2}\cdots v_{n_2}w_{m+1}\cdots w_k\cdots u_{n_1}u_1.$$

Hence $G_1 \circ_k G_2$ is Hamiltonian.

Conversely, if $G_1 \circ_k G_2$ is Hamiltonian, then there exists a Hamiltonian cycle $u_1u_2\cdots w_1w_2\cdots w_m v_{k+1}v_{k+2}\cdots v_{n_2}w_{m+1}\cdots w_k\cdots u_{n_1}u_1$. In this cycle, consider the path $w_{m+1}\cdots w_k\cdots u_{n_1}u_1u_2\cdots w_1w_2\cdots w_m$. Since there is an edge between w_m and w_{m+1}, adding this edge to the path will produce a cycle containing all the vertices of G_1. Therefore G_1 is Hamiltonian. Similarly, we can show that G_2 is also Hamiltonian.

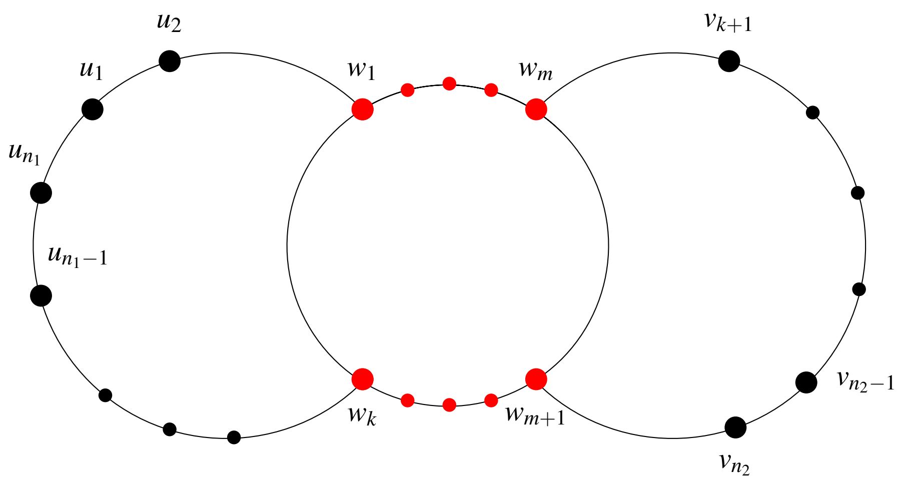

Figure 2: Hamiltonian cycle in $G_1 \circ_k G_2$.

□

The following proposition gives us a lower and upper bound for the independence number of k-coalescence of two graphs.

Proposition 3.8. *Let $\beta_0(G_i)$ be the independence number of $G_i, i = 1,2$. Then the independence number of $G = G_1 \circ_k G_2$ satisfies*

$$\beta_0(G_1) + \beta_0(G_2) - 2 \leq \beta_0(G) \leq \beta_0(G_1) + \beta_0(G_2).$$

Proof. Let $G = G_1 \circ_k G_2$
Case 1: Both G_1 and G_2 are complete graphs.
Then the vertices in $V(G_1 \setminus \mathcal{Q})$ are not adjacent to vertices in $V(G_2 \setminus \mathcal{Q})$. Thus $\beta_0(G) = 2 = \beta_0(G_1) + \beta_0(G_2)$.

Case 2: Either G_1 or G_2 is complete.
Without loss of generality, assume that G_1 is complete and G_2 is not. If the independent set of G_2 contains a vertex in $\mathcal{Q}$, then $\beta_0(G) = \beta_0(G_2) = \beta_0(G_2) + \beta_0(G_1) - 1$. If the independent set of G_2 does not contain a vertex in $\mathcal{Q}$, then the independent set of G contains independent vertices of G_2 along with a vertex from $G_1 \setminus \mathcal{Q}$. Thus $\beta_0(G_1 \circ_k G_2) = \beta_0(G_2) + 1 = \beta_0(G_2) + \beta_0(G_1)$.

Case 3: Neither G_1 nor G_2 is complete.
If both G_i's have an independent set disjoint from $\mathcal{Q}$ then their union gives the independent set for G, that is, $\beta_0(G) = \beta_0(G_1) + \beta_0(G_2)$. If one of the G_i's has a vertex common in its independent set and $\mathcal{Q}$, then $\beta_0(G) = \beta_0(G_1) + \beta_0(G_2) - 1$. If both G_i's has vertices common in their independent set and $\mathcal{Q}$, then $\beta_0(G) = \beta_0(G_1) + \beta_0(G_2) - 2$. □

Proposition 3.9. *Let χ_i be the chromatic number of $G_i, i = 1,2$. Then the chromatic number of $G_1 \circ_k G_2$,*

$$\chi(G_1 \circ_k G_2) = k + max\{\chi_1 - k, \chi_2 - k\}.$$

Proof. We need k different colours to colour the vertices in $\mathcal{Q}$. Since the vertices in $V(G_1 \setminus \mathcal{Q})$ and $V(G_2 \setminus \mathcal{Q})$ are not adjacent, they can be coloured using $max\{\chi_1 - k, \chi_2 - k\}$ colours. Thus chromatic number of $G_1 \circ_k G_2 = k + max\{\chi_1 - k, \chi_2 - k\}$. □

4. A_α-characteristic polynomial of k-coalescence of graphs

This section computes the A_α-characteristic polynomial of k-coalescence of two graphs. Using that, the A_α-characteristic polynomial of Lollipop graphs is estimated.

Let G be a graph containing a k-clique. Then we partition the adjacency matrix of G into the form $A(G) = \begin{bmatrix} B & C^T \\ C & A(K_k) \end{bmatrix}$.

Proposition 4.1. *Let G_1 and G_2 be two graphs of order n_1 and n_2 respectively such that $n_1+n_2>3k$. Then the A_α-characteristic polynomial of $G_1\circ_k G_2$ is*

$$\begin{aligned}\Phi(A_\alpha(G_1\circ_k G_2),\lambda) =&\Phi(A_\alpha(G_1),\lambda)\Phi(A_\alpha(G_2\backslash\mathcal{Q}),\lambda)+\Phi(A_\alpha(G_2),\lambda)\Phi(A_\alpha(G_1\backslash\mathcal{Q}),\lambda)\\ &-\Phi(A_\alpha(G_1\backslash\mathcal{Q}),\lambda)\Phi(A_\alpha(G_2\backslash\mathcal{Q}),\lambda)\Big(\alpha|D_1(\mathcal{Q})-(k-1)I|+\alpha|D_2(\mathcal{Q})-(k-1)I|\\ &+|\lambda-\alpha(D_1(\mathcal{Q})+D_2(\mathcal{Q})-(k-1)I)-(1-\alpha)A(K_k)|\Big),\end{aligned}$$

where $D_i(\mathcal{Q})$ represents the degree matrix of the k vertices in $\mathcal{Q}$ of G_i, $i=1,2$.

Proof. The A_α-matrix of $G_1\circ_k G_2$ with proper labelling has the form

$$A_\alpha(G_1\circ_k G_2)=\begin{bmatrix}D & R_1^T & R_2^T\\ R_1 & A_\alpha(G_1\backslash\mathcal{Q}) & O\\ R_2 & O & A_\alpha(G_2\backslash\mathcal{Q})\end{bmatrix},$$

where $D=\alpha(D_1(\mathcal{Q})+D_2(\mathcal{Q})-(k-1)I)+(1-\alpha)A(K_k)$ and $R_i=(1-\alpha)C_i$, where C_i is the block matrix in the adjacency matrix of G_i. Then,

$$\begin{aligned}\Phi(A_\alpha(G_1\circ_k G_2),\lambda) =&|\lambda-A_\alpha(G_1\circ_k G_2)|\\ =&\begin{vmatrix}\lambda-D & -R_1^T & -R_2^T\\ -R_1 & \lambda-A_\alpha(G_1\backslash\mathcal{Q}) & O\\ -R_2 & O & \lambda-A_\alpha(G_2\backslash\mathcal{Q})\end{vmatrix}\end{aligned}$$

$$=\begin{vmatrix}\lambda-D & -R_1^T & -R_2^T\\ -R_1 & O & O\\ -R_2 & O & O\end{vmatrix}+\begin{vmatrix}\lambda-D & -R_1^T & O\\ -R_1 & O & O\\ -R_2 & O & \lambda-A_\alpha(G_2\backslash\mathcal{Q})\end{vmatrix}+\begin{vmatrix}\lambda-D & O & -R_2^T\\ -R_1 & \lambda-A_\alpha(G_1\backslash\mathcal{Q}) & O\\ -R_2 & O & \lambda-A_\alpha(G_2\backslash\mathcal{Q})\end{vmatrix}.$$

Adding and subtracting $\begin{vmatrix}\lambda-D & O & O\\ -R_1 & \lambda-A_\alpha(G_1\backslash\mathcal{Q}) & O\\ -R_2 & O & \lambda-A_\alpha(G_2\backslash\mathcal{Q})\end{vmatrix}$

to $\Phi(A_\alpha(G_1\circ_k G_2),\lambda)$, we get

$$\begin{aligned}\Phi(A_\alpha(G_1\circ_k G_2),\lambda) =&\begin{vmatrix}\lambda-D & -R_1^T & O\\ -R_1 & \lambda-A_\alpha(G_1\backslash\mathcal{Q}) & O\\ -R_2 & O & \lambda-A_\alpha(G_2\backslash\mathcal{Q})\end{vmatrix}\\ &-\begin{vmatrix}\lambda-D & O & -R_2^T\\ -R_1 & \lambda-A_\alpha(G_1\backslash\mathcal{Q}) & O\\ -R_2 & O & \lambda-A_\alpha(G_2\backslash\mathcal{Q})\end{vmatrix}+\begin{vmatrix}\lambda-D & O & O\\ -R_1 & \lambda-A_\alpha(G_1\backslash\mathcal{Q}) & O\\ -R_2 & O & \lambda-A_\alpha(G_2\backslash\mathcal{Q})\end{vmatrix}\\ =&\left|\lambda-A_\alpha(G_2\backslash\mathcal{Q})\right|\begin{vmatrix}\lambda-D & -R_1^T\\ -R_1 & \lambda-A_\alpha(G_1\backslash\mathcal{Q})\end{vmatrix}\\ &+\left|\lambda-A_\alpha(G_1\backslash\mathcal{Q})\right|\begin{vmatrix}\lambda-D & -R_2^T\\ -R_2 & \lambda-A_\alpha(G_2\backslash\mathcal{Q})\end{vmatrix}-\left|\lambda-D\right|\left|\lambda-A_\alpha(G_1\backslash\mathcal{Q})\right|\left|\lambda-A_\alpha(G_2\backslash\mathcal{Q})\right|.\end{aligned}$$

Here,

$$\begin{aligned}
\begin{vmatrix} \lambda - D & -R_1^T \\ -R_1 & \lambda - A_\alpha(G_1\backslash\mathcal{Q}) \end{vmatrix} &= \begin{vmatrix} \lambda - \alpha(D_1(\mathcal{Q})+D_2(\mathcal{Q})-(k-1)I)-(1-\alpha)A(K_k) & -R_1^T \\ -R_1 & \lambda - A_\alpha(G_1\backslash\mathcal{Q}) \end{vmatrix} \\
&= \begin{vmatrix} \lambda - \alpha D_1(\mathcal{Q})-(1-\alpha)A(K_k) & -R_1^T \\ -R_1 & \lambda - A_\alpha(G_1\backslash\mathcal{Q}) \end{vmatrix} \\
&\quad + \begin{vmatrix} -\alpha(D_2(\mathcal{Q})-(k-1)I) & -R_1^T \\ O & \lambda - A_\alpha(G_1\backslash\mathcal{Q}) \end{vmatrix} \\
&= \left|\lambda - A_\alpha(G_1)\right| - \alpha\left|D_2(\mathcal{Q})-(k-1)I\right|\left|\lambda - A_\alpha(G_1\backslash\mathcal{Q})\right|.
\end{aligned}$$

Similarly

$$\begin{vmatrix} \lambda - D & -R_2^T \\ -R_2 & \lambda - A_\alpha(G_2\backslash\mathcal{Q}) \end{vmatrix} = \left|\lambda - A_\alpha(G_2)\right| - \alpha\left|D_1(\mathcal{Q})-(k-1)I\right|\left|\lambda - A_\alpha(G_2\backslash\mathcal{Q})\right|.$$

Therefore,

$$\begin{aligned}
\Phi(A_\alpha(G_1\circ_k G_2),\lambda) &= \left|\lambda - A_\alpha(G_1)\right|\left|\lambda - A_\alpha(G_2\backslash\mathcal{Q})\right| + \left|\lambda - A_\alpha(G_2)\right|\left|\lambda - A_\alpha(G_1\backslash\mathcal{Q})\right| \\
&- \left|\lambda - A_\alpha(G_1\backslash\mathcal{Q})\right|\left|\lambda - A_\alpha(G_2\backslash\mathcal{Q})\right|\Big(\alpha\left(\left|D_1(\mathcal{Q})-(k-1)I\right| + \left|D_2(\mathcal{Q})-(k-1)I\right|\right) + \left|\lambda - D\right|\Big) \\
&= \Phi(A_\alpha(G_1),\lambda)\Phi(A_\alpha(G_2\backslash\mathcal{Q}),\lambda) + \Phi(A_\alpha(G_2),\lambda)\Phi(A_\alpha(G_1\backslash\mathcal{Q}),\lambda) \\
&- \Phi(A_\alpha(G_1\backslash\mathcal{Q}),\lambda)\Phi(A_\alpha(G_2\backslash\mathcal{Q}),\lambda)\Big(\alpha\left(\left|D_1(\mathcal{Q})-(k-1)I\right| + \left|D_2(\mathcal{Q})-(k-1)I\right|\right) + \left|\lambda - D\right|\Big).
\end{aligned}$$

□

Corollary 4.2. *Let G_1 and G_2 be two graphs of order n_1 and n_2 respectively such that $n_1+n_2 > 3k$. Then the adjacency characteristic polynomial of $G_1\circ_k G_2$ is*

$$\begin{aligned}
\Phi(A(G_1\circ_k G_2),\lambda) =& \Phi(A(G_1),\lambda)\Phi(A(G_2\backslash\mathcal{Q}),\lambda) + \Phi(A(G_2),\lambda)\Phi(A(G_1\backslash\mathcal{Q}),\lambda) \\
&- (\lambda - k+1)(x+1)^{k-1}\Phi(A(G_1\backslash\mathcal{Q}),\lambda)\Phi(A(G_2\backslash\mathcal{Q}),\lambda).
\end{aligned}$$

Remark 4.3. The Lollipop graph, $L(m,n-1)$ is obtained from the coalescence of a vertex from a cycle C_m and a pendant vertex from a path P_n. The A_α-characteristic polynomial of the Lollipop graph is, $\Phi(A_\alpha(L(m,n-1)),\lambda) =$ $\Phi(A_\alpha(P_n),\lambda)\Phi(A_\alpha(P_{m-1}),\lambda) + \Phi(A_\alpha(C_m),\lambda)\Phi(A_\alpha(P_{n-1}),\lambda) - \lambda\Phi(A_\alpha(P_{n-1}),\lambda)\Phi(A_\alpha(P_{m-1}),\lambda)$.

Using the Remark 4.3, we can calculate the A_α-characteristic polynomial of Lollipop graphs and hence find their spectrum.

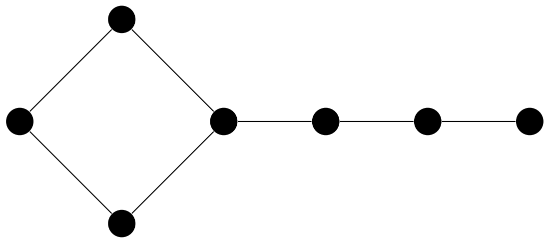

Figure 3: $L(4,3)$

Example 4.4. The A_α-characteristic polynomial of the Lollipop graph $L(4,3)$ is, $\Phi(A_\alpha(L(4,3)),\lambda) =$
$\Phi(A_\alpha(P_3),\lambda)\Phi(A_\alpha(P_3),\lambda)+\Phi(A_\alpha(C_4),\lambda)\Phi(A_\alpha(P_2),\lambda)-\lambda\Phi(A_\alpha(P_2),\lambda)\Phi(A_\alpha(P_3),\lambda)$.

5. A_α-spectrum of k-coalescence of complete graphs

In this section, we compute the A_α-spectrum and A_α-energy of $K_m \circ_k K_n$.

Proposition 5.1. *For $m,n > 1$, the A_α-characteristic polynomial of $K_m \circ_k K_n$ is*
$\Phi(A_\alpha(K_m \circ_k K_n),\lambda) = \Big(\lambda-\alpha(m+n-k)+1\Big)^{k-1}\Big(\lambda-\alpha m+1\Big)^{m-k-1}\Big(\lambda-\alpha n+1\Big)^{n-k-1}$
$\bigg(\Big(\lambda-m+1+(1-\alpha)k\Big)\Big(\lambda-n+1+(1-\alpha)k\Big)\Big(\lambda-\alpha(m+n-2k)+1-k\Big)-(1-$
$\alpha)^2k\Big((m+n-2k)\lambda-(m+n-2k)\alpha k-(m-k)(n-k-1)-(n-k)(m-k-1)\Big)\bigg)$.

Proof. The degree matrix of $K_m \circ_k K_n$ with proper labelling has the form

$$D(K_m \circ_k K_n) = \begin{bmatrix} (m+n-1-k)I_k & O_{k\times m-k} & O_{k\times n-k} \\ O_{m-k\times k} & (m-1)I_{m-k} & O_{m-k\times n-k} \\ O_{n-k\times k} & O_{n-k\times m-k} & (n-1)I_{n-k} \end{bmatrix}.$$

The adjacency matrix of $K_m \circ_k K_n$ has the form

$$A(K_m \circ_k K_n) = \begin{bmatrix} A(K_k) & J_{k\times m-k} & J_{k\times n-k} \\ J_{m-k\times k} & A(K_{m-k}) & O_{m-k\times n-k} \\ J_{n-k\times k} & O_{n-k\times m-k} & A(K_{n-k}) \end{bmatrix}.$$

Thus the A_α-matrix of $K_m \circ_k K_n$ is $A_\alpha(K_m \circ_k K_n) =$

$$\begin{bmatrix} \beta_1 & (1-\alpha)J_{k\times m-k} & (1-\alpha)J_{k\times n-k} \\ (1-\alpha)J_{m-k\times k1} & \beta_2 & O_{m-k\times n-k} \\ (1-\alpha)J_{n-k\times k} & O_{n-k\times m-k} & \beta_3 \end{bmatrix},$$

where $\beta_1 = \alpha(m+n-1-k)I_k + (1-\alpha)A(K_k)$,
$\beta_2 = \alpha(m-1)I + (1-\alpha)A(K_{m-k})$ and
$\beta_3 = \alpha(n-1)I + (1-\alpha)A(K_{n-k})$.

Then the characteristic polynomial of $K_m \circ_k K_n$ is

$$|\lambda I - A_\alpha(K_m \circ_k K_n)| = \begin{vmatrix} \lambda I_k - \beta_1 & -(1-\alpha)J_{k\times m-k} & -(1-\alpha)J_{k\times n-k} \\ -(1-\alpha)J_{m-k\times k} & \lambda I_k - \beta_2 & O \\ -(1-\alpha)J_{n-k\times k} & O & \lambda I_k - \beta_3 \end{vmatrix}.$$

In the above determinant, performing

$$C_l \to C_l + \frac{1-\alpha}{\lambda - m + 1 + (1-\alpha)k}\sum_{i=k+1}^{m} C_i + \frac{1-\alpha}{\lambda - n + 1 + (1-\alpha)k}\sum_{j=m+1}^{m+n-k} C_j$$

for $l = 1, 2, \cdots, k$ columns we get,

$$|\lambda I - A_\alpha(K_m \circ_k K_n)| = \begin{vmatrix} \beta_4 & -(1-\alpha)J_{k\times m-k} & -(1-\alpha)J_{k\times n-k} \\ O & \lambda I - \beta_2 & O \\ O & O & \lambda I - \beta_3 \end{vmatrix},$$

where $\beta_4 = (\lambda - \alpha(m+n-k)+1)I_k - (1-\alpha)\left[\frac{(1-\alpha)(m-k)}{\lambda-m+1+(1-\alpha)k} + \frac{(1-\alpha)(n-k)}{\lambda-n+1+(1-\alpha)k} + 1\right] J_k$.

$$\begin{aligned} |\lambda I - A_\alpha(K_m \circ_k K_n)| &= |(\lambda - \alpha(m+n-k)+1)I_k - (1-\alpha)XJ_k| \\ &\quad |(\lambda - \alpha(m-1))I - (1-\alpha)A(K_{m-k})|\,|(\lambda - \alpha(n-1))I - (1-\alpha)A(K_{n-k})|, \end{aligned}$$

where $X = \left[\frac{(1-\alpha)(m-k)}{\lambda-m+1+(1-\alpha)k} + \frac{(1-\alpha)(n-k)}{\lambda-n+1+(1-\alpha)k} + 1\right]$

Thus

$\Phi(A_\alpha(K_m \circ_k K_n), \lambda) = \Big(\lambda - \alpha(m+n-k)+1\Big)^{k-1}\Big(\lambda - \alpha m + 1\Big)^{m-k-1}\Big(\lambda - \alpha n + 1\Big)^{n-k-1}\bigg(\Big(\lambda - m + 1 + (1-\alpha)k\Big)\Big(\lambda - n + 1 + (1-\alpha)k\Big)\Big(\lambda - \alpha(m+n-2k) + 1 - k\Big) - (1-\alpha)^2 k\Big((m+n-2k)\lambda - (m+n-2k)\alpha k - (m-k)(n-k-1) - (n-k)(m-k-1)\Big)\bigg).$

$\square$

Now, in the following corollary, we obtain the A_α-eigenvalues of $K_m \circ_k K_n$.

Corollary 5.2. *The A_α-eigenvalues of $K_m \circ_k K_n$ are*

1. $\alpha(m+n-k) - 1$ *repeated* $k-1$ *times,*
2. $\alpha m - 1$ *repeated* $m-k-1$ *times,*
3. $\alpha n - 1$ *repeated* $n-k-1$ *times,*

4. *three roots of the equation* $\Big(\Big(\lambda - m + 1 + (1-\alpha)k\Big)\Big(\lambda - n + 1 + (1-\alpha)k\Big)\Big(\lambda - \alpha(m+n-2k) + 1 - k\Big) - (1-\alpha)^2 k\Big((m+n-2k)\lambda - (m+n-2k)\alpha k - (m-k)(n-k-1) - (n-k)(m-k-1)\Big)\Big) = 0.$

The following corollary helps us to determine the A_α-energy of non-regular graph $K_m \circ_k K_n$.

Corollary 5.3. *The A_α-energy of $K_m \circ_k K_n$ is*

$$\varepsilon_\alpha(K_m \circ_k K_n) = (k-1)\left|\alpha(1-2k) + \frac{2\alpha mn}{m+n-1} - 1\right| + (m-k-1)\left|\alpha(1-k) + \frac{\alpha n(m-n+k)}{m+n-k} - 1\right| + (n-k-1)\left|\alpha(1-k) + \frac{\alpha m(n-m+k)}{m+n-k} - 1\right| + |\beta - X_1| + |\gamma - X_1| + |\delta - X_1|,$$

where $X_1 = \dfrac{\alpha\left(m^2+n^2-k^2-(m+n-k)\right)}{m+n-k}$ *and β, γ, δ are roots of the equation* $\Big(\Big(\lambda - m + 1 + (1-\alpha)k\Big)\Big(\lambda - n + 1 + (1-\alpha)k\Big)\Big(\lambda - \alpha(m+n-2k) + 1 - k\Big) - (1-\alpha)^2 k\Big((m+n-2k)\lambda - (m+n-2k)\alpha k - (m-k)(n-k-1) - (n-k)(m-k-1)\Big)\Big) = 0.$

Corollary 5.4. *The A_α-energy of $K_m \circ_k K_m$ is*

$$\varepsilon_\alpha(K_m \circ_k K_m) = (k-1)\left|\alpha(1-2k) + \frac{2\alpha m^2}{2m-1} - 1\right| + 2(m-k-1)\left|\alpha(1-k) + \frac{\alpha mk}{2m-k} - 1\right| + \Big|\beta - X_2\Big| + \Big|\gamma - X_2| + \Big|\delta - X_2|,$$

where $X_2 = \dfrac{\alpha\left(2m^2-k^2-(2m-k)\right)}{2m-k}$ *and β, γ and δ are roots of the equation* $(\lambda - m + 1 + (1-\alpha)k)^2(\lambda - 2\alpha(m-k) + 1 - k) - (1-\alpha)^2 k((2m-k)\lambda - 2\alpha k(m-k) - 2(m-k)(m-k-1)) = 0.$

Corollary 5.5. *For $m, n > 1$, the A_α-characteristic polynomial of $K_m \circ_1 K_n$ is*

$$\Phi(A_\alpha(K_m \circ_1 K_n), \lambda) = (\lambda - \alpha m + 1)^{m-2}(\lambda - \alpha n + 1)^{n-2}\Big((\lambda - m + 2 - \alpha)(\lambda - n + 2 - \alpha)(\lambda - \alpha(m+n-2)) - (1-\alpha)^2\Big((m+n-2)\lambda - (m+n-2)\alpha - (m-1)(n-2) - (m-2)(n-1)\Big)\Big).$$

Corollary 5.6. *The A_α-eigenvalues of $K_m \circ_1 K_n$ are*

1. $\alpha m-1$ *repeated* $m-2$ *times,*

2. $\alpha n-1$ *repeated* $n-2$ *times,*

3. *three roots of the equation* $(\lambda-m+2-\alpha)(\lambda-n+2-\alpha)(\lambda-\alpha(m+n-2))-(1-\alpha)^2\Big[(m+n-2)\lambda-(m+n-2)\alpha-(m-1)(n-2)-(m-2)(n-1)\Big]=0.$

The following corollary helps us to determine the A_α-energy of a non-regular graph $K_m \circ_1 K_n$.

Corollary 5.7. *The* A_α*-energy of* $K_m \circ_1 K_n$ *is*
$\varepsilon_\alpha(K_m \circ_1 K_n)=\frac{m-2}{m+n-1}\Big|\alpha n(m-n+1)-(m+n-1)\Big|+\frac{n-2}{m+n-1}\Big|\alpha m(n-m+1)-(m+n-1)\Big|+|\beta-X_3|+|\gamma-X_3|+|\delta-X_3|,$
where $X_3=\frac{\alpha(m^2+n^2-m-n)}{m+n-1}$ *and* β,γ,δ *are roots of the equation* $(\lambda-m+2-\alpha)(\lambda-n+2-\alpha)(\lambda-\alpha(m+n-2))-(1-\alpha)^2\Big[(m+n-2)\lambda-(m+n-2)\alpha-(m-1)(n-2)-(m-2)(n-1)\Big]=0.$

Corollary 5.8. *The* A_α*-energy of* $K_m \circ_1 K_m$ *is*
$\varepsilon_\alpha(K_m \circ_1 K_m)=\frac{2(m-2)}{2m-1}(m(2-\alpha)-1)+\Big|\frac{2m^2(1-\alpha)-5m+2-\alpha}{2m-1}\Big|+\Big|\beta-\frac{2m\alpha(m-1)}{2m-1}\Big|+\Big|\gamma-\frac{2m\alpha(m-1)}{2m-1}\Big|,$
where β *and* γ *are roots of the equation* $\lambda^2-(m-2+\alpha(2m-1))\lambda+2(m-1)(\alpha m-1)=0.$

Corollary 5.9. *For* $m,n>2$*, the* A_α*-characteristic polynomial of* $K_m \circ_2 K_n$ *is*
$\Phi(A_\alpha(K_m \circ_2 K_n),\lambda)=(\lambda-\alpha m+1)^{m-3}(\lambda-\alpha n+1)^{n-3}(\lambda-\alpha(m+n-2)+1)\Bigg((\lambda-m+3-2\alpha)(\lambda-n+3-2\alpha)(\lambda-\alpha(m+n-4)-1)-2(1-\alpha)^2\Big((m+n-4)\lambda-(m+n-4)2\alpha-(m-2)(n-3)-(m-3)(n-2)\Big)\Bigg).$

Now, in the following corollary, we obtain the A_α-eigenvalues of $K_m \circ_2 K_n$.

Corollary 5.10. *The* A_α*-eigenvalues of* $K_m \circ_2 K_n$ *are*

1. $\alpha m-1$ *repeated* $m-3$ *times,*

2. $\alpha n-1$ *repeated* $n-3$ *times,*

3. $\alpha(m+n-2)-1$,

4. *three roots of the equation* $(\lambda-m+3-2\alpha)(\lambda-n+3-2\alpha)(\lambda-\alpha(m+n-4)-1)-2(1-\alpha)^2\Big[(m+n-4)\lambda-(m+n-4)2\alpha-(m-2)(n-3)-(m-3)(n-2)\Big]=0$.

The following corollary helps us to determine the A_α-energy of a non-regular graph $K_m \circ_2 K_n$.

Corollary 5.11. *The* A_α*-energy of* $K_m \circ_2 K_n$ *is*
$\varepsilon_\alpha(K_m \circ_2 K_n) = \frac{m-3}{m+n-2}\Big|\alpha[(m-n)(n-1)+2]-1\Big| + \frac{n-3}{m+n-2}\Big|\alpha[(n-m)(m-1)+2]-1\Big| + \frac{1}{m+n-2}\Big|\alpha(2mn-3m-3n+6)-1\Big| + |\beta - X_4| + |\gamma - X_4| + |\delta - X_4|$,
where $X_4 = \frac{\alpha[m(m-1)+n(n-1)-2]}{m+n-2}$ *and* β,γ,δ *are roots of the equation* $(\lambda-m+3-2\alpha)(\lambda-n+3-2\alpha)(\lambda-\alpha(m+n-4)-1)-2(1-\alpha)^2\Big[(m+n-4)\lambda-(m+n-4)2\alpha-(m-2)(n-3)-(m-3)(n-2)\Big]=0$.

Corollary 5.12. *The* A_α*-energy of* $K_m \circ_2 K_m$ *is*
$\varepsilon_\alpha(K_m \circ_2 K_m) = \frac{m-3}{m-1}\Big|2\alpha-1\Big| + \frac{1}{2m-2}\Big|\alpha(2m^2-6m+6)-1\Big| + \Big|\frac{m^2(1-\alpha)-m(4-3\alpha)-\alpha+3}{m-1}\Big| + \Big|\beta - \frac{\alpha(m(m-1)-1)}{m-1}\Big| + \Big|\gamma - \frac{\alpha(m(m-1)-1)}{m-1}\Big|$,
where β *and* γ *are roots of the equation* $\lambda^2-(m-2+2\alpha(m-1))\lambda+2\alpha m^2-m(2\alpha+3)-2\alpha+5=0$.

6. Topological indices of vertex coalescence of graphs

In this section, some topological indices of vertex coalescence of graphs are computed. We calculate the Wiener index, hyper-Wiener and Zagreb indices of the Lollipop and Dumbbell graphs using the results.

Proposition 6.1. *Wiener index of* $G_1 \circ_1 G_2$ *is*

$$W(G_1 \circ_1 G_2) = W(G_1) + W(G_2) + (n_2-1)d_{G_1}(v) + (n_1-1)d_{G_2}(v),$$

where v *is the vertex that is merged in* $G_1 \circ_1 G_2$.

Proof. Let $G = G_1 \circ_1 G_2$ and v be the vertex merging in G. From Definition 2.4,

$$\begin{aligned} W(G) &= \sum_{\{u,w\}\in V(G_1)} d(u,w) + \sum_{\{u,w\}\in V(G_2)} d(u,w) + \sum_{\substack{u\in V(G_1)\\ w\in V(G_2)}} d(u,w) \\ &= W(G_1) + W(G_2) + (n_2-1)d_{G_1}(v) + (n_1-1)d_{G_2}(v). \end{aligned}$$

□

Remark 6.2. The Wiener index of cycle is $W(C_m) = \begin{cases} \frac{m^3}{8} & \text{if } m \text{ is even} \\ \frac{m(m^2-1)}{8} & \text{if } m \text{ is odd,} \end{cases}$

and Wiener index of path is $W(P_n) = \frac{n(n^2-1)}{6}$. Thus the Wiener index of Lollipop graph $L(m,n-1)$ is

$$W(L(m,n-1)) = \begin{cases} \frac{m^3}{8} + \frac{n(n^2-1)}{6} + (n-1)\left(\frac{m^2+2n(m-1)}{4}\right) & \text{if } m \text{ is even} \\ \frac{m(m^2-1)}{8} + \frac{n(n^2-1)}{6} + \frac{(n-1)(m-1)(m+1+2n)}{4} & \text{if } m \text{ is odd.} \end{cases}$$

Remark 6.3. The Dumbbell graph, denoted by $D_{l,m,n-3}$, is obtained from the coalescence of a cycle C_l and the pendant vertex of a Lollipop graph $L(m,n-1)$.

The Wiener index of Dumbbell graph $D_{m,m,n-3}$ is

$$W(D_{m,m,n-3}) = \begin{cases} \frac{m^3}{4} + \frac{n(n^2-1)}{6} + \frac{m(m^2+3mn-4m+4)+n(4-6m+2mn-2n)-2}{2} & \text{if } m \text{ is even} \\ \frac{m(m^2-1)}{4} + \frac{n(n^2-1)}{6} + (m-1)\frac{m^2-3m+3mn-3n+4n^2}{2} & \text{if } m \text{ is odd.} \end{cases}$$

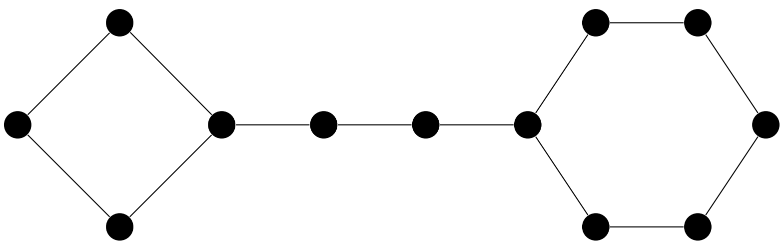

Figure 4: $D_{4,6,1}$

Proposition 6.4. *Hyper-Wiener index of $G_1 \circ_1 G_2$ is*

$$\begin{aligned} WW(G_1 \circ_1 G_2) &= WW(G_1) + WW(G_2) \\ &\quad + \frac{1}{2}\Big((n_2-1)(d_{G_1}(v) + d^2_{G_1}(v)) + (n_1-1)(d_{G_2}(v) + d^2_{G_2}(v)) + 2d_{G_1}(v)d_{G_2}(v)\Big), \end{aligned}$$

where v is the vertex that is merged in $G_1 \circ_1 G_2$.

Proof. Let $G = G_1 \circ_1 G_2$ and v be the vertex merging in G. From Definition 2.5,

$$\begin{aligned}
WW(G) =& \frac{1}{2}(W(G_1) + W(G_2) + (n_2-1)d_{G_1}(v) + (n_1-1)d_{G_2}(v)) \\
&+ \frac{1}{2}\left(\sum_{\{u,w\}\in V(G_1)} d^2(u,w) + \sum_{\{u,w\}\in V(G_2)} d^2(u,w) + \sum_{\substack{u\in V(G_1)\\ w\in V(G_2)}} d^2(u,w) \right) \\
=& WW(G_1) + WW(G_2) \\
&+ \frac{1}{2}\Big((n_2-1)(d_{G_1}(v) + d^2_{G_1}(v)) + (n_1-1)(d_{G_2}(v) + d^2_{G_2}(v)) + 2d_{G_1}(v)d_{G_2}(v)\Big).
\end{aligned}$$

□

Remark 6.5. The hyper-Wiener index of of Lollipop graph $L(m,n-1)$ is

$$WW(L(m,n-1)) =$$

$$\begin{cases} \frac{m^2(m+1)(m+2)}{48} + \frac{n^4+2n^3-n^2-2n}{24} + \frac{(n-1)(m(m^2+3m+2)+4n(m-1)(n+1)+3m^2n)}{24} & \text{if } m \text{ is even} \\ \frac{m(m^2-1)(m+3)}{48} + \frac{n^4+2n^3-n^2-2n}{24} + \frac{(m-1)(n-1)((m+1)(m+3)+4n(n+1)+3n(m+1))}{24} & \text{if } m \text{ is odd.} \end{cases}$$

Remark 6.6. The hyper-Wiener index of Dumbbell graph $D_{m,m,n-3}$ is

$$WW(D_{m,m,n-3}) =$$

$$\begin{cases} \frac{m^2(m+1)(m+2)}{24} + \frac{n^4+2n^3-n^2-2n}{24} \\ \quad + \frac{7m^4+4m^3(-5+7n)-8n\left(1-3n+2n^2\right)+4m^2\left(2-12n+9n^2\right)+8m\left(-2+5n-6n^2+2n^3\right)}{48} & \text{if } m \text{ is even} \\ \frac{m(m^2-1)(m+3)}{24} + \frac{n^4+2n^3-n^2-2n}{24} \\ \quad + \frac{(-1+m)\left(-3+7m^3-16n-12n^2+16n^3+m^2(-13+28n)+m\left(-23-20n+36n^2\right)\right)}{48} & \text{if } m \text{ is odd.} \end{cases}$$

Proposition 6.7. *The forgotten topological index of $G_1 \circ_1 G_2$ is*

$$F(G_1 \circ_1 G_2) = F(G_1) + F(G_2) + 3deg_{G_1}(v)deg_{G_2}(v)(deg_{G_1}(v) + deg_{G_2}(v)),$$

where v is the vertex that is merged in $G_1 \circ_1 G_2$.

Proof. Let $G = G_1 \circ_1 G_2$ and v be the vertex merging in G. From Definition 2.6,

$$\begin{aligned}
F(G) =& \sum_{u\in V(G_1)} deg^3_{G_1}(u) - deg^3_{G_1}(v) + \sum_{u\in V(G_2)} deg^3_{G_2}(u) - deg^3_{G_2}(v) \\
&+ (deg_{G_1}(v) + deg_{G_2}(v))^3 \\
=& F(G_1) + F(G_2) + 3deg_{G_1}(v)deg_{G_2}(v)(deg_{G_1}(v) + deg_{G_2}(v)).
\end{aligned}$$

□

Proposition 6.8. *First Zagreb index of $G_1 \circ_1 G_2$ is*

$$M_1(G_1 \circ_1 G_2) = M_1(G_1) + M_1(G_2) + 2deg_{G_1}(v)deg_{G_2}(v),$$

where v is the vertex that is merged in $G_1 \circ_1 G_2$.

Proof. Let $G = G_1 \circ_1 G_2$ and v be the vertex merging in G. From Definition 2.7,

$$\begin{aligned} M_1(G) = &\sum_{u\in V(G_1)} deg^2_{G_1}(u) - deg^2_{G_1}(v) + \sum_{u\in V(G_2)} deg^2_{G_2}(u) - deg^2_{G_2}(v) \\ &+ (deg_{G_1}(v) + deg_{G_2}(v))^2 \\ = &M_1(G_1) + M_1(G_2) + 2deg_{G_1}(v)deg_{G_2}(v). \end{aligned}$$

□

Remark 6.9. The first Zagreb index of of Lollipop graph $L(m,n-1)$ is

$$M_1(L(m,n-1)) = 4(m+n) - 2.$$

Remark 6.10. The first Zagreb index of Dumbbell graph $D_{m,m,n-3}$ is

$$M_1(D_{m,m,n-3}) = 4(2m+n) + 2.$$

Proposition 6.11. *Narumi-Katayama index of $G_1 \circ_1 G_2$ is*

$$NK(G_1 \circ_1 G_2) = NK(G_1)NK(G_2)\frac{deg_{G_1}(v) + deg_{G_2}(v)}{deg_{G_1}(v)deg_{G_2}(v)},$$

where v is the vertex that is merged in $G_1 \circ_1 G_2$.

Proof. Let $G = G_1 \circ_1 G_2$ and v be the vertex merging in G. From Definition 2.8,

$$\begin{aligned} NK(G) &= \frac{\prod_{u\in V(G_1)} deg_{G_1}(u) \prod_{u\in V(G_2)} deg_{G_2}(u)}{deg_{G_1}(v)deg_{G_2}(v)} (deg_{G_1}(v) + deg_{G_2}(v)) \\ &= NK(G_1)NK(G_2)\frac{deg_{G_1}(v) + deg_{G_2}(v)}{deg_{G_1}(v)deg_{G_2}(v)}. \end{aligned}$$

□

7. Conclusion

This paper estimates some structural properties of a non-regular graph obtained from the k-coalescence of two graphs. Also, the A_α-characteristic polynomial of k-coalescence of two graphs is determined. Moreover, the A_α-spectrum and A_α-energy of k-coalescence of two complete graphs are computed. In addition, some topological indices of vertex coalescence of two graphs are estimated. The Wiener, hyper-Wiener and Zagreb indices of Lollipop and Dumbbell graphs are derived as an application.

Acknowledgments

The first author gratefully acknowledges the financial support of the University Grants Commission(UGC), India.

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