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EXISTENCE OF CONTINUOUS SOLUTIONS TO EVOLUTIONARY QUASI-VARIATIONAL INEQUALITIES WITH APPLICATIONS

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The author presents dynamic elastic traffic equilibrium problems with data depending explicitly on time and studies under which assumptions the continuity of solutions with respect to the time can be ensured. In particular, regularity results for solutions to time-dependent quasi-variational inequalities associated to a general class of closed lower semicontinuous multifunctions will be showed. These results will be obtained making use of the property of the Mosco's convergence. At last, it will be applied an example of the dynamic elastic traffic equilibrium problem.

1. Introduction

The aim of this paper is to consider continuity results for solutions to evolutionary quasi-variational inequalities associated to linear and nonlinear strongly monotone operators. Our result is related to a general class of closed lower semicontinuous multifunctions with set-values fulfil the Mosco's convergence property. In particular, the continuity results obtained in [1] in the core of linear strongly monotone operators for the set of constraints related to the dynamic elastic traffic equilibrium problems will be generalized for this general class

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of multifunctions. Then, the continuity result for nonlinear strictly monotone evolutionary quasi-variational inequalities will be showed.

The paper is organized as follows. In Sec. 2, we introduce the time-dependent quasi-variational inequality which models the time-dependent elastic traffic equilibrium problem. In Sec. 3, we generalize Theorem 3.2 and Theorem 5.2 in [1], and we show that solutions to time-dependent quasi-variational inequalities associated to linear and nonlinear strongly monotone operators are continuous mappings from the time interval $[0,T]$ to the Euclidian space $\mathbb{R}^m_+$ (see Theorems 3.3 and 3.4). We use the last result to prove the continuity of solutions to nonlinear strictly monotone evolutionary variational inequalities, in Sec. 4. At last, in Sec. 5, we apply the shown results to an example of the dynamic elastic traffic network and the associated quasi-variational inequality.

2. The Traffic Equilibrium Problem: Time-dependent and Elastic Cases

Our purpose is to present a class of equilibrium problems which can be examined by putting to use quasi-variational inequalities. In fact, we consider time-dependent and elastic models of transportation networks. It is worth noting that the time-dependent formulation is required when data evolve in time, whereas when travel demands depend on the equilibrium distribution the elastic framework is necessary. Let us consider a traffic network where: W is the set of Origin Destination (O/D) pairs w_j, $j = 1,2,\dots,l$; $\mathscr{R}_j$ is the set of routes R_r, $r = 1,2,\dots,m$, which connect the pair w_j; Φ is the incidence matrix, whose elements are

$$\varphi_{jr} = \begin{cases} 1 \text{ if } R_r \in \mathscr{R}_j \\ 0 \text{ otherwise.} \end{cases}$$

For technical reason, the functional setting is the reflexive Banach space $L^2([0,T],\mathbb{R}^m_+)$.

Let us assume that:

- $C : [0,T] \times \mathbb{R}^m_+ \to \mathbb{R}^m_+$ is the route cost function;
- $\rho : [0,T] \times \mathbb{R}^m_+ \to \mathbb{R}^l_+$ is the elastic demand, depending on the equilibrium pattern;
- $\lambda(t), \mu(t) \in L^2([0,T],\mathbb{R}^m_+)$, $\lambda(t) < \mu(t)$ a.e. in $[0,T]$ are the capacity restrictions.

Then if D is a nonempty, compact, convex subset of $L^2([0,T],\mathbb{R}^m_+)$, the set of feasible flows is the set-valued function defined as follows:

$$\mathbf{K} : D \to L^2([0,T],\mathbb{R}^m_+),$$

$$\mathbf{K}(H) = \Bigg\{ F \in L^2([0,T],\mathbb{R}^m_+) : \ \lambda(t) \le F(t) \le \mu(t) \quad \text{a.e. in } [0,T], \tag{1}$$

$$\Phi F(t) = \frac{1}{T}\int_0^T \rho(t,H(\tau))d\tau \quad \text{a.e. in } [0,T] \Bigg\}.$$

In order to ensure the non voidness of $\mathbf{K}(H)$, we suppose that $\Phi\lambda(t) \le \Phi F(t) \le \Phi\mu(t)$ a.e. in $[0,T]$. We consider a formulation of equilibrium problems where the dependence of the flows F_r on the unknown solution H is assumed on average with respect to time, i.e.,

$$\Sigma_{r=1}^m \varphi_{jr} F_r(t) = \frac{1}{T}\int_0^T \rho_j(t,H(\tau))d\tau,$$

see [3–5] for more details. In conclusion, the elastic and time-dependent equilibrium problem is expressed by the following evolutionary quasi-variational inequality:

Find $H \in \mathbf{K}(H)$ such that

$$\int_0^T \langle C(t,H(t)), F(t) - H(t)\rangle dt \ge 0, \quad \forall F \in \mathbf{K}(H). \tag{2}$$

We set

$$\mathbf{K}(t,H) = \left\{ F(t) \in \mathbb{R}^m : \ \lambda(t) \le F(t) \le \mu(t), \ \Phi F(t) = \frac{1}{T}\int_0^T \rho_j(t,H(\tau))d\tau \right\},$$

for a.e. $t \in [0,T]$, and we observe that problem (2) (see [7]) is equivalent to the following one:

Find $H \in \mathbf{K}(H)$ such that

$$\langle C(t,H(t)), F(t) - H(t)\rangle \ge 0, \quad \forall F(t) \in \mathbf{K}(t,H), \text{ a.e. in } [0,T]. \tag{3}$$

Regarding the existence of solutions, let us recall the following general result (see [11]):

Theorem 2.1. Let X be a locally convex, Hausdorff topological vector space, D a nonempty compact, convex subset of X, $C : D \to X^*$ a continuous function, $\mathbf{K} : D \to 2^D$ a closed lower semicontinuous multifunction with $\mathbf{K}(H) \subseteq D$ nonempty, compact, convex $\forall H \in D$. Then, there exists a solution to quasi-variational inequality

$$\langle C(t,H(t)), F(t) - H(t)\rangle \ge 0, \quad \forall F \in \mathbf{K}(H).$$

By Theorem 2, it follows prove the next existence result related to our equilibrium problem (see [9], Theorem 3):

Theorem 2.2. Let $C : [0,T] \times \mathbb{R}^m_+ \to \mathbb{R}^m_+$ be an operator verifying the following conditions:

$$C(t,F) \text{ is measurable in } t, \forall F \in \mathbb{R}^m_+, \text{ continuous in } F, \text{ a.e. in } [0,T],$$
$$\exists \gamma \in L^2([0,T]) : \; \|C(t,F)\|_m \leq \gamma(t) + \|F\|_m,$$

and

$$\exists \nu > 0 : \; \langle C(t,F_1) - C(t,F_2), F_1 - F_2 \rangle \geq \nu \|F_1 - F_2\|^2.$$

Let $\lambda, \mu \in L^2([0,T], \mathbb{R}^m_+)$ be vector-functions and let $\rho \in L^2([0,T] \times \mathbb{R}^m_+, \mathbb{R}^l_+)$ be an operator verifying the following conditions

$$\exists \psi \in L^1([0,T]) : \; \|\rho(t,F)\|_l \leq \psi(t) + \|F\|^2_m,$$
$$\exists \nu \in L^2([0,T]) : \; \|\rho(t,F_1) - \rho(t,F_2)\|_l \leq \nu(t) \|F_1 - F_2\|_m.$$

Then, evolutionary quasi-variational inequality (3) admits a solution.

3. Continuity results for time-dependent quasi-variational inequalities

In this section, we will generalize the theorem of continuity for solutions to evolutionary quasi-variational inequalities associated to linear strongly monotone operator and to a particular nonlinear operator proved in [1] for a general class of set-value mappings $\mathbf{K}$, and we will present analogous results for nonlinear strongly monotone operators.

In the following, we will make use the important concept of the sets convergence in Mosco's sense (see [8]).

Definiton 3.1. Let $(V, \|\cdot\|)$ be an Hilbert space and $\mathbf{K} \subset V$ a closed, nonempty, convex set. A sequence of nonempty, closed, convex sets $\mathbf{K}_n$ converges to $\mathbf{K}$, as $n \to +\infty$, in Mosco's sense, if

(M1) for any $H \in \mathbf{K}$, there exists a sequence $\{H_n\}_{n\in\mathbb{N}}$ strongly converging to H in V such that H_n lies in $\mathbf{K}_n$ for all $n \in \mathbb{N}$,

(M2) for any subsequence $\{H_{k_n}\}_{n\in\mathbb{N}}$ weakly converging to H in V, such that H_{k_n} lies in $\mathbf{K}_{k_n}$ for all $n \in \mathbb{N}$, then the weak limit H belongs to $\mathbf{K}$.

Definiton 3.2. A sequence of operators $A_n : \mathbf{K}_n \to V'$ converges to an operator $A : \mathbf{K} \to V'$ if

$$\|A_n H_n - A_n F_n\|_* \leq M \|H_n - F_n\|, \quad \forall H_n, F_n \in \mathbf{K}_n, \tag{4}$$

$$\langle A_n H_n - A_n F_n, H_n - F_n \rangle \geq \nu \|H_n - F_n\|^2, \quad \forall H_n, F_n \in \mathbf{K}_n, \tag{5}$$

hold with fixed constants $M, \nu > 0$ and

(M3) the sequence $\{A_n H_n\}_{n\in\mathbb{N}}$ strongly converges to AH in V', for any sequence $\{H_n\}_{n\in\mathbb{N}}$, such that H_n lies in $\mathbf{K}_n$ for all $n \in \mathbb{N}$, strongly converging to $H \in \mathbf{K}$.

In (4) $\|\cdot\|_*$ is the norm in the dual space of V.

It results that the set as in (1) fulfils the conditions of Definition 3.1.

Lemma 3.1. *Let $\lambda, \mu \in C([0,T], \mathbb{R}^m_+)$, let $\rho \in C([0,T], \mathbb{R}^l_+)$ and let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence such that $t_n \to t \in [0,T]$, as $n \to +\infty$. Then, the sequence of sets*

$$\mathbf{K}(t_n, H) = \Big\{F(t_n) \in \mathbb{R}^m : \lambda(t_n) \le F(t_n) \le \mu(t_n),$$

$$\Phi F(t_n) = \frac{1}{T}\int_0^T \rho(t_n, H(\tau))d\tau\Big\},$$

$\forall n \in \mathbb{N}$, *converge to*

$$\mathbf{K}(t, H) = \Big\{F(t) \in \mathbb{R}^m : \lambda(t) \le F(t) \le \mu(t),$$

$$\Phi F(t) = \frac{1}{T}\int_0^T \rho(t, H(\tau))d\tau\Big\},$$

as $n \to +\infty$, in Mosco's sense, for every $H \in L^2([0,T], \mathbb{R}^m_+)$.

Proof. See proof of Theorem 3.2 in [1]. □

We recall an abstract stability result due to Mosco (see [10], Theorem 4.1):

Theorem 3.1. Let $\mathbf{K}_n \to \mathbf{K}$ in Mosco's sense (M1)–(M2) , $A_n \to A$ in the sense of (M3) and $B_n \to B$ in V'. Then the unique solutions H_n of

$$H_n \in \mathbf{K}_n : \quad \langle A_n H_n - B_n, F_n - H_n\rangle \ge 0, \quad \forall F_n \in \mathbf{K}_n$$

converge strongly to the solution H of the limit problem

$$H \in \mathbf{K} : \quad \langle AH - B, F - H\rangle \ge 0, \quad \forall F \in \mathbf{K},$$

i.e.,

$$H_n \to H \quad in\ V.$$

Let $\mathbf{K}$ be a set-valued mapping satisfying the following condition

(MM) $\mathbf{K} : D \to 2^{L^2([0,T],\mathbb{R}^m_+)}$ is closed lower semicontinuous with $\mathbf{K}(t,H)$, $t \in [0,T]$, $H \in L^2([0,T],\mathbb{R}^m_+)$, nonempty, convex, closed of $\mathbb{R}^m$ such that $\mathbf{K}(t_n,H)$ converges to $\mathbf{K}(t,H)$ in Mosco's sense, for any sequence $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,T]$, with $t_n \to t \in [0,T]$, as $n \to +\infty$.

Now, we can generalize the continuity result proved in [1], Theorem 4.1, for a general set satisfying condition (MM).

Theorem 3.2. Let $A \in C([0,T],\mathbb{R}^{m\times m}_+)$ be a positive definite matrix-function and let $B \in C([0,T],\mathbb{R}^m_+)$ be a vector-function. Let $\mathbf{K}$ be a set-valued mapping satisfying condition (MM). Then, the time-dependent quasi-variational inequality

$$\langle A(t)H(t)+B(t),F(t)-H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t,H),\ \textit{in}\ [0,T], \tag{6}$$

has a solution $H \in \mathbf{K}(H)$ such that $H \in C([0,T],\mathbb{R}^m_+)$.

Proof. By virtue of Theorem 2.1, we have that (6) admits a solution $H(t) \in \mathbf{K}(t,H)$, for $t \in [0,T]$ and the solution is unique in $\mathbf{K}(t,H)$ for t in $[0,T]$.

Now, we prove the continuity of solution applying Theorem 3.1. Let $t \in [0,T]$ be fixed and let $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,T]$ be a sequence, with $t_n \to t$. From the assumption of continuity of the function A, one has

$$A(t_n) \to A(t) \quad \text{in } \mathbb{R}^{m\times m},$$

moreover, if $\{F(t_n)\}_{n\in\mathbb{N}}$ is a sequence, with $F(t_n) \in \mathbf{K}(t_n)$, such that $F(t_n) \to F(t)$ in $\mathbb{R}^m$, it results

$$A(t_n)F(t_n) \to A(t)F(t) \quad \text{in } \mathbb{R}^m.$$

Finally, for the continuity of the function B we have

$$B(t_n) \to B(t) \quad \text{in } \mathbb{R}^m.$$

Taking into account that the set-valued mapping $\mathbf{K}$ satisfies condition (MM) and using the stability Theorem 3.1, we can conclude that the solutions $H(t_n)$ to quasi-variational inequalities

$$\langle A(t_n)H(t_n)+B(t_n),F(t_n)-H(t_n)\rangle \geq 0, \quad \forall F(t_n) \in \mathbf{K}(t_n,H),$$

converge strongly to the solution $H(t)$ of the limit problem (6), i.e.,

$$H(t_n) \to H(t) \quad \text{in } \mathbb{R}^m,$$

namely $H \in C([0,T],\mathbb{R}^m_+)$. □

Now, we still assume that the operator is linear with respect to the flows, but the matrix-function A depends on time and on integral average of the flow vectors, namely

$$C(t,F(t)) = A(t,F_{\mathscr{T}})F(t) + B(t),$$

for a.e. $t \in [0,T]$ and for every $F \in L^2([0,T],\mathbb{R}^m_+)$, where $A : [0,T] \times \mathbb{R}^m_+ \to \mathbb{R}^{m\times m}_+$ and $B : [0,T] \to \mathbb{R}^m_+$ are two functions, $\mathscr{T} = [0,T]$ and $F_{\mathscr{T}}$ is the integral average, i.e.

$$F_{\mathscr{T}} = \frac{\int_0^T F(\tau)d\tau}{T}.$$

We suppose that $A(t,u)$ is a bounded matrix, namely

$$\exists M > 0 : \ \|A(t,u)\|_{m\times m} \le M, \quad \text{for a.e. } t \in [0,T], \ \forall u \in \mathbb{R}^m_+. \tag{7}$$

Then we study the continuity of solutions to the following evolutionary quasi-variational inequality:

Find $H \in \mathbf{K}(H)$ such that

$$\langle A(t,F_{\mathscr{T}})H(t) + B(t), F(t) - H(t)\rangle \ge 0, \quad \forall F(t) \in \mathbf{K}(t,H), \text{ a.e. in } [0,T], \tag{8}$$

where $\mathbf{K}$ is a set-valued mapping satisfying condition (MM).

Also in this case, we can obtain a regularity result for the solutions to (8).

Theorem 3.3. Let $A \in C([0,T] \times \mathbb{R}^m_+, \mathbb{R}^{m\times m}_+)$ be a matrix-function satisfying the condition (7), and let $B \in C([0,T],\mathbb{R}^m_+)$ be a vector function. Let $\mathbf{K}$ be a set-valued mapping satisfying condition (MM). Then, evolutionary quasi-variational inequality (8) has a solution $H \in \mathbf{K}(H)$ such that $H \in C([0,T],\mathbb{R}^m_+)$.

Proof. The existence of solutions to the evolutionary quasi-variational inequality follows by Theorem 2.1. We remark that it needs to prove only that $A(t,F_{\mathscr{T}})$ is continuous in F, for $t \in [0,T]$. Hence, let $F \in \mathbf{K}(H)$ be fixed and let $\{F_n\}_{n\in\mathbb{N}} \subseteq \mathbf{K}(H)$ be a sequence, such that $F_n \to F$, in $L^2([0,T],\mathbb{R}^m_+)$. It results

$$\lim_{n\to\infty} \int_0^T F_n(\tau)d\tau = \int_0^T F(\tau)d\tau,$$

and, taking into account that $A(t,v)$ is continuous in v and bounded for $t \in [0,T]$, we get the continuity of $A(t,F_{\mathscr{T}})$ in F, for $t \in [0,T]$. Then we have the existence of a solution and the continuity of the solution to the evolutionary quasi-variational inequality from Theorem 3.2. $\square$

We present the analogous result for nonlinear strongly monotone time-dependent quasi-variational inequalities (see [2], Theorem 4.1).

Theorem 3.4. Let $C \in C([0,T] \times \mathbb{R}^m_+, \mathbb{R}^m_+)$ be an operator such that

$$\exists \gamma \in C([0,T]): \ \|C(t,F)\|_m \leq \gamma(t) + \|F\|_m, \quad \forall F \in \mathbb{R}^m_+, \text{ in } [0,T],$$

and

$$\exists \nu > 0: \ \langle C(t,F_1) - C(t,F_2), F_1 - F_2 \rangle \geq \nu \|F_1 - F_2\|^2, \quad \forall F_1, F_2 \in \mathbb{R}^m_+, \text{ in } [0,T].$$

Let $\mathbf{K}$ be a set-valued mapping satisfying condition (MM). Then, the evolutionary variational inequality

$$\langle C(t,H(t)), F(t) - H(t) \rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t,H), \ in \ [0,T],$$

admits a solution $H \in \mathbf{K}(H)$ such that $H \in C([0,T], \mathbb{R}^m_+)$.

Remark 3.1. Theorem 3.4 still holds true for the set-valued mapping defined by

$$\mathbf{K} : D \to 2^{L^2([0,T],\mathbb{R}^m_+)}$$

$$\mathbf{K}(H) = \Big\{ F \in L^2([0,T], \mathbb{R}^m) : \ \lambda(t) \leq F(t) \leq \mu(t), \ in \ [0,T],$$

$$\Phi F(t) = \frac{1}{T} \int_0^T \rho(t, H(\tau)) d\tau \ in \ [0,T] \Big\}.$$

supposing that $\lambda, \mu \in C([0,T], \mathbb{R}^m_+)$ and $\rho \in C([0,T] \times \mathbb{R}^m_+, \mathbb{R}^l_+)$, under these assumptions the family of sets satisfies condition (MM) (see Lemma 3.1).

4. Regularity results for strictly monotone evolutionary quasi-variational inequalities

Now, we consider nonlinear strictly monotone evolutionary quasi-variational inequalities and we prove that they have some continuous solutions. More precisely, let $C : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$ be an operator satisfying the following assumptions:

$$\exists \gamma \in L^2([0,T], \mathbb{R}_+) : \ \|C(t,F)\|_m \leq \gamma(t) + \|F\|_m, \quad \forall F \in \mathbb{R}^m, \text{ a.e. in } [0,T]. \quad (9)$$

and

$$\langle C((t,H) - C(t,F), H - F \rangle > 0, \quad \forall H, F \in \mathbb{R}^m, \ H \neq F, \text{ a.e. in } [0,T]. \quad (10)$$

Let us consider the following evolutionary quasi-variational inequality

Find $H \in \mathbf{K}$ such that

$$\langle C(t,H(t)), F(t) - H(t) \rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t,H), \text{ a.e. in } [0,T], \quad (11)$$

where the multifunction $\mathbf{K} : D \to 2^{L^2([0,T],\mathbb{R}^m)}$ satisfies condition (MM).

We observe that there exists a solution $\overline{H}$ to (11), and it is unique in $\mathbf{K}(\overline{H})$, for Theorem 2.2. Hence, to show that there exists a continuous solution to (11) we can prove that the unique solution $\overline{H}$ in the set $\mathbf{K}(\overline{H})$ is continuous. Then, we fix the solution $\overline{H} \in \mathbf{K}(\overline{H})$ and we work in $\mathbf{K}(\overline{H})$.

The first step of the proof of the continuity result is to show a regularization lemma. We recall that if the operator C is monotone it results that the set $\mathbf{X}(\overline{H})$ of solutions to evolutionary quasi-variational inequality (11) is closed, convex and nonempty.

Let $I : L^2([0,T],\mathbb{R}^m) \to L^2([0,T],\mathbb{R}^m)$ be the identity operator and let us consider the following evolutionary quasi-variational inequality

$$\langle IH(t), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{X}(t,\overline{H}), \text{ a.e. in } [0,T]. \tag{12}$$

Then, evolutionary quasi-variational inequality (12) admits a unique solution in the set $\mathbf{X}(\overline{H})$. Further, for every $\varepsilon > 0$, let us consider the perturbed evolutionary quasi-variational inequality

$$\langle C(t,H(t)) + \varepsilon IH(t), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t,\overline{H}), \text{ a.e. in } [0,T], \tag{13}$$

and we prove the following preliminary result.

Lemma 4.1. *Let $C \in C([0,T]\times\mathbb{R}^m,\mathbb{R}^m)$ be a monotone matrix-function satisfying condition* (9)*. Let $D \subseteq L^2([0,T],\mathbb{R}^m)$ be a nonempty, compact, convex subset. Let $\mathbf{K} : D \to 2^{L^2([0,T],\mathbb{R}^m)}$ be a multifunction such that have uniformly bounded set-values and satisfying condition* (MM)*. If $H_\varepsilon(t)$, $\forall \varepsilon > 0$, is a solution to* (13)*, it results*

$$\lim_{\varepsilon\to 0} H_\varepsilon(t) = H(t), \quad \text{in } [0,T],$$

and

$$\lim_{\varepsilon\to 0} \|H_\varepsilon(t) - H(t)\|^2_{L^2([0,T],\mathbb{R}^m)} = 0,$$

where H is a solution to the evolutionary quasi-variational inequality (11).

Proof. Let H be the solution to (12), therefore $H \in \mathbf{X}(\overline{H})$ and

$$\langle IH(t), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{X}(t,\overline{H}), \text{ in } [0,T]. \tag{14}$$

Let H_ε be the solution to (13), namely $H_\varepsilon \in \mathbf{K}(\overline{H})$ and

$$\langle C(t,H_\varepsilon(t)) + \varepsilon IH_\varepsilon(t), F(t) - H_\varepsilon(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t,\overline{H}), \text{ in } [0,T]. \tag{15}$$

Setting $F(t) = H_\varepsilon(t)$, for $t \in [0,T]$, in (13) and $F(t) = H(t)$, for $t \in [0,T]$, in (12) and adding we get

$$\langle C(t,H(t)) - C(t,H_\varepsilon(t)), H_\varepsilon(t) - H(t)\rangle + \varepsilon\langle H_\varepsilon(t), H(t) - H_\varepsilon(t)\rangle \geq 0, \quad (16)$$

in $[0,T]$. Being C monotone, we have

$$\langle C(t,H(t)) - C(t,H_\varepsilon(t)), H_\varepsilon(t) - H(t)\rangle \leq 0, \quad \text{in } [0,T],$$

then, by (16), we obtain

$$\varepsilon\langle H_\varepsilon(t), H(t) - H_\varepsilon(t)\rangle \geq 0, \quad \text{in } [0,T],$$

and dividing by $\varepsilon > 0$, it results

$$\langle H_\varepsilon(t), H(t) - H_\varepsilon(t)\rangle \geq 0, \quad \text{in } [0,T]. \quad (17)$$

Taking into account (17), one has

$$\|H_\varepsilon(t)\|_m^2 \leq \langle H_\varepsilon(t), H(t)\rangle \leq \|H(t)\|_m \|H_\varepsilon(t)\|_m, \quad \text{in } [0,T],$$

then

$$\|H_\varepsilon(t)\|_m \leq \|H(t)\|_m, \quad \text{in } [0,T].$$

We remark that $H(t) \in \mathbf{X}(t,\overline{H}) \subseteq \mathbf{K}(t,\overline{H})$, in $[0,T]$, and $\mathbf{K}(t,\overline{H})$, $t \in [0,T]$, is a family of uniformly bounded sets of $\mathbb{R}^m$, then

$$\|H(t)\|_m \leq C, \quad \text{in } [0,T],$$

with C a constant independent on ε and on $t \in [0,T]$, namely

$$\|H_\varepsilon(t)\|_m \leq C, \quad \forall \varepsilon > 0, \text{ in } [0,T].$$

Then, there exists a subsequence $\{H_\eta(t)\}_\eta$ converging in $\mathbb{R}^m$ to an element $\widehat{H}(t)$ of $\mathbb{R}^m$, in [0,T], and thus

$$\lim_{\eta \to 0} H_\eta(t) = \widehat{H}(t), \quad \text{in } [0,T].$$

Under the assumption that $\mathbf{K}(t,\overline{H})$ is a closed set of $\mathbb{R}^m$, it results

$$\widehat{H}(t) \in \mathbf{K}(t,\overline{H}), \text{ in } [0,T].$$

It remains to prove that

$$\widehat{H}(t) = H(t), \quad \text{in } [0,T].$$

Then, setting $\varepsilon = \eta$ in (15), we obtain

$$\langle C(t,H_\eta(t)),F(t)-H_\eta(t)\rangle + \eta\langle H_\eta(t),F(t)-H_\eta(t)\rangle \geq 0, \tag{18}$$

$\forall F(t) \in \mathbf{K}(t,\overline{H})$, in $[0,T]$, and taking account that

$$\lim_{\eta\to 0}\langle H_\eta(t),H_\eta(t)\rangle = \langle \widehat{H}(t),\widehat{H}(t)\rangle, \quad \text{in } [0,T],$$

and that

$$\lim_{\eta\to 0}\langle C(t,H_\eta(t)),H_\eta(t)\rangle = \langle C(t,\widehat{H}(t)),\widehat{H}(t)\rangle, \quad \text{in } [0,T],$$

from (18), we have

$$\langle C(t,\overline{H}(t)),F(t)-\widehat{H}(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t,\overline{H}), \text{ in } [0,T]. \tag{19}$$

Hence (19) implies that $\widehat{H}$ is a solution to (11), in $[0,T]$, namely

$$\overline{H} \in \mathbf{X}(\overline{H}).$$

If the solution to (11) is unique, then the proof is concluded. Now, we suppose that the solution to (11) is not unique. Setting $\varepsilon = \eta$ in (17) we obtain

$$\langle H_\eta(t),H(t)-H_\eta(t)\rangle \geq 0, \quad \text{in } [0,T],$$

and passing to the limit for $\eta \to 0$, we get

$$\langle \widehat{H}(t),H(t)-\widehat{H}(t)\rangle \geq 0, \quad \text{in } [0,T]. \tag{20}$$

Setting $F = \widehat{H} \in \mathbf{X}(\overline{H})$ in (14), it results

$$\langle H(t),\widehat{H}(t)-H(t)\rangle \geq 0, \quad \text{in } [0,T], \tag{21}$$

and adding (20) and (21), we have

$$\langle \widehat{H}(t)-H(t),H(t)-\widehat{H}(t)\rangle \geq 0, \quad \text{in } [0,T].$$

Then

$$\langle \widehat{H}(t)-H(t),H(t)-\widehat{H}(t)\rangle = 0, \quad \text{in } [0,T],$$

that implies

$$\widehat{H}(t) = H(t), \quad \text{in } [0,T].$$

We have proved that every convergent subsequence converges to the same limit $H(t)$ and then

$$\lim_{\varepsilon\to 0} H_\varepsilon(t) = H(t), \quad \text{in } [0,T].$$

Moreover, it results

$$\|H_\varepsilon(t)-H(t)\|_m^2 \le 2\big(\|H_\varepsilon(t)\|_m^2+\|H(t)\|_m^2\big) \le 4C^2 \in C([0,T],\mathbb{R}^m), \quad \text{in } [0,T],$$

hence, by virtue of Lebesgue's Theorem we have

$$\lim_{\varepsilon\to 0}\|H_\varepsilon(t)-H(t)\|^2_{L^2([0,T],\mathbb{R}^m)}=0.$$

□

Now, we are able to show the continuity of solutions to nonlinear strictly monotone evolutionary quasi-variational inequalities.

Theorem 4.1. Let $C\in C([0,T]\times\mathbb{R}^m,\mathbb{R}^m)$ be a vector-function satisfying conditions (9) and (10). Let $D\subseteq L^2([0,T],\mathbb{R}^m)$ be a nonempty, compact, convex subset. Let $\mathbf{K}: D\to 2^{L^2([0,T],\mathbb{R}^m)}$ be a multifunction with uniformly bounded set-values and satisfying condition (MM). Then, evolutionary quasi-variational inequality (11) has a solution $H\in\mathbf{K}(H)$ such that $H\in C([0,T],\mathbb{R}^m)$.

Proof. By Theorem 2.2, it follows that (11) has a solution $\overline{H}\in\mathbf{K}(\overline{H})$, furthermore the solution is unique in the set $\mathbf{K}(\overline{H})$. Then, we fix the set $\mathbf{K}(\overline{H})$.

Let $t\in[0,T]$ be fixed and let $\{t_n\}_{n\in\mathbb{N}}\subseteq[0,T]$ be a sequence, such that $t_n\to t$, as $n\to+\infty$.

Let $\overline{H}(t)$ be the solution to quasi-variational inequality (11) in $t\in[0,T]$ and let $H(t_n)$, $\forall n\in\mathbb{N}$, be the solutions to quasi-variational inequalities

$$\langle C(t_n,H(t_n)),F(t_n)-H(t_n)\rangle\ge 0, \quad \forall F(t_n)\in\mathbf{K}(t_n,\overline{H}),\ \forall n\in\mathbb{N}. \tag{22}$$

Let $H_\varepsilon(t)\in\mathbf{K}(t,\overline{H})$ be the solution to the following perturbed strongly monotone quasi-variational inequality

$$\langle C(t,H_\varepsilon(t))+\varepsilon I H_\varepsilon(t),F(t)-H_\varepsilon(t)\rangle\ge 0, \quad \forall F(t)\in\mathbf{K}(t,\overline{H}).$$

From Theorem 3.4, it follows that H_ε is continuous in $[0,T]$, then we have that the solutions $H_\varepsilon(t_n)$, $\forall n\in\mathbb{N}$, to the evolutionary quasi-variational inequalities

$$\langle C(t_n,H_\varepsilon(t_n))+\varepsilon I(t_n)H_\varepsilon(t_n),F(t_n)-H_\varepsilon(t_n)\rangle\ge 0, \quad \forall F(t_n)\in\mathbf{K}(t_n,\overline{H}), \tag{23}$$

$\forall n\in\mathbb{N}$, converge to $H_\varepsilon(t)$, as $n\to+\infty$. Setting $F(t_n)=H(t_n)$, $\forall n\in\mathbb{N}$, in (23) and $F(t_n)=H_\varepsilon(t_n)$, $\forall n\in\mathbb{N}$, in (22) and adding it results, $\forall n\in\mathbb{N}$

$$\langle C(t_n,H_\varepsilon(t_n))-C(t_n,H(t_n)),H(t_n)-H_\varepsilon(t_n)\rangle+\varepsilon\langle H_\varepsilon(t_n),H(t_n)-H_\varepsilon(t_n)\rangle\ge 0. \tag{24}$$

Moreover, from the strict monotonicity of the function C it follows

$$\langle C(t_n, H_\varepsilon(t_n)) - C(t_n, H(t_n)), H(t_n) - H_\varepsilon(t_n)\rangle < 0, \quad \forall n \in \mathbb{N}. \tag{25}$$

Then, using (24) and (25) we obtain

$$\varepsilon\langle H_\varepsilon(t_n), H(t_n) - H_\varepsilon(t_n)\rangle \geq 0, \quad \forall n \in \mathbb{N},$$

and dividing by $\varepsilon > 0$, we get

$$\langle H_\varepsilon(t_n), H(t_n) - H_\varepsilon(t_n)\rangle \geq 0, \quad \forall n \in \mathbb{N}. \tag{26}$$

From (26), it follows

$$\|H_\varepsilon(t_n)\|_m^2 \leq \langle H_\varepsilon(t_n), H(t_n)\rangle \leq \|H(t_n)\|_m \|H_\varepsilon(t_n)\|_m, \quad \forall n \in \mathbb{N},$$

then

$$\|H_\varepsilon(t_n)\|_m \leq \|H(t_n)\|_m, \quad \forall n \in \mathbb{N}.$$

Since $H(t_n) \in \mathbf{X}(t_n, \overline{H}) \subseteq \mathbf{K}(t_n, \overline{H})$, for $n \in \mathbb{N}$, and $\mathbf{K}(t_n, \overline{H})$, for $n \in \mathbb{N}$, are uniformly bounded sets of $\mathbb{R}^m$, it results

$$\|H(t_n)\|_m \leq C, \quad \forall n \in \mathbb{N},$$

where C is a constant independent on ε and on $n \in \mathbb{N}$, then

$$\|H_\varepsilon(t_n)\|_m \leq C, \quad \forall \varepsilon > 0,\ \forall n \in \mathbb{N}. \tag{27}$$

By Lemma 4.1, we get

$$\lim_{\varepsilon \to 0} H_\varepsilon(t_n) = \widetilde{H}(t_n), \quad \forall n \in \mathbb{N},$$

where $\widetilde{H}(t_n) \in \mathbf{K}(t_n, \overline{H})$, $\forall n \in \mathbb{N}$, and such that

$$\langle C(t_n, \widetilde{H}(t_n)), F(t_n) - \widetilde{H}(t_n)\rangle \geq 0, \quad \forall F(t_n) \in \mathbf{K}(t_n, \overline{H}),\ \forall n \in \mathbb{N}.$$

For the uniqueness of the solution to (22) in the set $\mathbf{K}(t_n, \overline{H})$, it results

$$\widetilde{H}(t_n) = H(t_n), \quad \forall n \in \mathbb{N},$$

and, passing to the limit as $\varepsilon \to 0$ in (27), it follows

$$\|H(t_n)\|_m \leq C, \quad \forall n \in \mathbb{N},$$

namely the sequence $\{H(t_n)\}_{n\in\mathbb{N}}$ is bounded. Hence, there exists a subsequence $\{H(t_{k_n})\}_{n\in\mathbb{N}}$, with $H(t_{k_n}) \in \mathbf{K}(t_{k_n}, \overline{H})$, $\forall n \in \mathbb{N}$, converging in $\mathbb{R}^m$ to an element $\widehat{H}(t)$ of $\mathbb{R}^m$, namely

$$\lim_{n\to+\infty} H(t_{k_n}) = \widehat{H}(t).$$

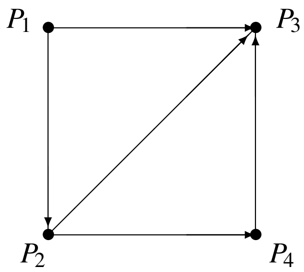

Figure 1: A network model.

Moreover, by (22) it obtains

$$\langle C(t,\widehat{H}(t)),F(t)-\widehat{H}(t)\rangle \geq 0, \quad \forall F(t)\in \mathbf{K}(t,\overline{H}),$$

and, for the uniqueness of the solution to (11) in $\mathbf{K}(t,\overline{H})$, it follows

$$\widehat{H}(t)=\overline{H}(t).$$

The same result holds for each subsequence and so

$$\lim_{n\to+\infty} H(t_n)=\overline{H}(t),$$

namely our assert. $\square$

5. An example

Let us consider the network as in Figure 1, where $N=\{P_1,P_2,P_3,P_4\}$ is the set of nodes and $L=\{(P_1,P_2),(P_1,P_3),(P_2,P_3),(P_2,P_4),(P_4,P_3)\}$ is the set of links.

The origin-destination pair is represented by (P_1,P_3), so that the paths are the following:

$$\begin{aligned} R_1 &= (P_1,P_3),\\ R_2 &= (P_1,P_2)\cup(P_2,P_3),\\ R_3 &= (P_1,P_2)\cup(P_2,P_4)\cup(P_4,P_3). \end{aligned}$$

Let us assume that the path costs are the following:

$$\begin{aligned} C_1(H(t)) &= \alpha H_1(t)+\beta,\\ C_2(H(t)) &= \alpha H_2(t)+\gamma,\\ C_3(H(t)) &= \alpha H_2(t)+\alpha H_3(t)+\delta, \end{aligned}$$

where $\alpha, \beta, \gamma, \delta \geq 0$.

The set of feasible flows is given by:

$$\mathbf{K}(H) = \Big\{ F \in L^2([0,T],\mathbb{R}^3) : \ F_1(t), F_2(t), F_3(t) \geq 0,$$

$$F_1(t) + F_2(t) + F_3(t) = \frac{1}{T}\int_0^T (\varepsilon t + \zeta H_1(\tau))d\tau \ \text{ a.e. in } [0,T] \Big\},$$

where $\varepsilon \geq 0$, $\zeta \in [0,2[$.

The equilibrium flow is the solution of the evolutionary quasi-variational inequality:

$$\sum_{p=1}^{3} C_p(H(t))(F_p(t) - H_p(t)) \geq 0, \ \ \forall F(t) \in \mathbf{K}(t,H), \ \text{a.e. in } [0,T]. \qquad (28)$$

Following the procedure shown in [3–6], we have:

$$F_3(t) = \frac{1}{T}\int_0^T (\varepsilon t + \zeta H_1(\tau))d\tau - F_1(t) - F_2(t);$$

$$\widetilde{\mathbf{K}}(H) = \Big\{ \widetilde{F} \in L^2([0,T],\mathbb{R}^2) : \ F_1(t), F_2(t) \geq 0,$$

$$F_1(t) + F_2(t) \leq \frac{1}{T}\int_0^T (\varepsilon t + \zeta H_1(\tau))d\tau \ \text{ a.e. in } [0,T] \Big\}.$$

Let us consider:

$$\begin{aligned}
\Gamma_1(\widetilde{F}(t), \widetilde{H}(t)) &= C_1(\widetilde{F}(t), \widetilde{H}(t)) - C_3(\widetilde{F}(t), \widetilde{H}(t)) = \\
&= 2\alpha F_1(t) - \frac{\alpha}{T}\int_0^T (\varepsilon t + \zeta H_1(\tau))d\tau + \beta - \delta, \\
\Gamma_2(\widetilde{F}(t), \widetilde{H}(t)) &= C_2(\widetilde{F}(t), \widetilde{H}(t)) - C_3(\widetilde{F}(t), \widetilde{H}(t)) = \\
&= \alpha F_1(t) + \alpha F_2(t) - \frac{\alpha}{T}\int_0^T (\varepsilon t + \zeta H_1(\tau))d\tau + \gamma - \delta.
\end{aligned}$$

Thus, the quasi-variational inequality problem (28) may be written as:

$$\sum_{p=1}^{3} \Gamma_p(\widetilde{H}(t))(\widetilde{F}_p(t) - \widetilde{H}_p(t)) \geq 0, \ \ \forall \widetilde{F}(t) \in \widetilde{\mathbf{K}}(t,\widetilde{H}), \ \text{a.e. in } [0,T].$$

It is immediate to show that if $\widetilde{H}$ satisfies the following system:

$$\begin{cases} \Gamma_1(\widetilde{H}(t), \widetilde{H}(t)) = 0 \\ \Gamma_2(\widetilde{H}(t), \widetilde{H}(t)) = 0 \\ \widetilde{H} \in \widetilde{\mathbf{K}}(\widetilde{H}) \end{cases}$$

then it solves the quasi-variational inequality (28). We find that:

$$\int_0^T H_1(\tau)d\tau = \frac{T}{2\alpha}\frac{\alpha\varepsilon T - 2\beta + 2\gamma}{2-\zeta},$$

and

$$\begin{aligned} H_1(t) &= \frac{\varepsilon}{2}t + \frac{\zeta}{4\alpha}\frac{\alpha\varepsilon T - 2\beta + 2\gamma}{2-\zeta} - \frac{\beta-\gamma}{2\alpha}, \\ H_2(t) &= \frac{\varepsilon}{2}t + \frac{\zeta}{4\alpha}\frac{\alpha\varepsilon T - 2\beta + 2\gamma}{2-\zeta} + \frac{\beta+\delta-\gamma}{2\alpha}, \end{aligned}$$

under condition that:

$$H_1(t) + H_2(t) \leq \frac{1}{T}\int_0^T (\varepsilon t + \zeta H_1(\tau))d\tau.$$

From which, we obtain

$$H_3(t) = -\frac{\delta-\gamma}{\alpha},$$

Then, we have obtained that the flow-vector is continuous.

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