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ON THE SYMMETRIC BLOCK DESIGN WITH PARAMETERS (153, 57, 21)

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In this paper it is proved that:

A) Up to isomorphism and duality there are exactly two possible orbital structures for a putative symmetric block design with parameters $(153, 57, 21)$ constructed using the Frobenius group $F_{17\cdot 16} = \langle \rho, \mu / \rho^{17} = \mu^{16} = 1 \rangle$, where the collineation ρ operates fixed-point-free and the collineation μ operates as $\mu = (1)(2,3,4,5,6,7,8,9)$.

B) Up to isomorphism and duality there are exactly 16 possible orbital structures for a putative symmetric block design with parameters $(153, 57, 21)$ constructed using the collineation group $G = \langle \rho, \mu / \rho^{19} = \mu^3 = 1 \rangle$, where the collineation ρ operates with one fixed point and the collineation μ operates as follows $\mu = (\infty)(1,2,3)(4)(5)(6,7,8)$.

1. Introduction

Investigations of symmetric block designs have found increasing interest in the field of combinatorics during the last two decades. A few methods for the construction of symmetric block designs are known and all of them have shown to be effective in certain situations. Here, we shall use the method of tactical decompositions, assuming that a certain automorphism group acts on the design

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we want to construct. This method has been suggested and used by Zvonimir Janko [5] (see also [6]).

We assume that the reader is familiar with the basic facts of design theory. For introductory material see for instance [1], [2] and [7]. Briefly, a symmetric block design with parameters (v,k,λ) is finite incidence structure consisting of two disjoint sets $\mathscr{P}$ and $\mathscr{B}$, where the elements of $\mathscr{P}$ are called points and the elements of $\mathscr{B}$ are called blocks or lines; further $|\mathscr{P}| = |\mathscr{B}|$. In addition, every block is incident with precisely k points and every two points are incident with precisely λ blocks. In this paper, for the sake of simplicity and without loss of generality, we shall say that point lies on a block or that block passes through a point if the point and the block in question are incident.

If g is an automorphism of a symmetric design $\mathscr{D}$ with parameters (v,k,λ) then g fixes an equal number of points and blocks; see [7, Theorem 3.1, p.78].

It is also known that an automorphism group G of symmetric design has the same number of orbits on the set points $\mathscr{P}$ as on the set of lines $\mathscr{B}$; see [7, Theorem 3.3, p.79].

Let $\mathscr{D}$ be a symmetric design with parameters (v,k,λ) and let G be a subgroup of the automorphism group $Aut(\mathscr{D})$ of $\mathscr{D}$. Denote the point orbits of G on $\mathscr{P}$ by $\mathscr{P}_1, \mathscr{P}_2, ..., \mathscr{P}_t$ and the line orbits of G on $\mathscr{B}$ by $\mathscr{B}_1, \mathscr{B}_2, ..., \mathscr{B}_t$. Put $|\mathscr{P}| = \omega_r$ and $|\mathscr{B}| = \Omega_i$. Obviously,

$$\sum_{r=1}^{t} \omega_r = v \quad \text{and} \quad \sum_{i=1}^{t} \Omega_i = v.$$

Let γ_{ir} be the number of points from $\mathscr{P}_r$ which lie on line $\mathscr{B}_i$. Clearly, this number does not depend on the particular line chosen. Similarly, let Γ_{js} be the number of lines from $\mathscr{B}_j$ which pass through a point from $\mathscr{P}_s$. Then, obviously,

$$\sum_{r=1}^{t} \gamma_{ir} = k \quad \text{and} \quad \sum_{j=1}^{t} \Gamma_{js} = k.$$

By [2, Lemma 5.3.1, p. 221], our partition of the point set $\mathscr{P}$ and of the block set $\mathscr{B}$ forms a tactical decomposition of the design $\mathscr{D}$ in the sense of [3, p. 210]. The integer $n = k - \lambda$ is called the order of the symmetric block design $\mathscr{D}$.

In our case, the symmetric block design with parameters $(153,57,21)$ is finite incidence structure consisting of two disjoint sets $\mathscr{P}$ and $\mathscr{B}$, where points of $\mathscr{P}$ and blocks of $\mathscr{B}$ satisfy the following equality $|\mathscr{P}| = |\mathscr{B}| = 153$. Furthermore, every block of $\mathscr{B}$ is incident with precisely 57 points of $\mathscr{P}$ and every two points of $\mathscr{P}$ are incident with precisely 21 blocks of $\mathscr{B}$.

There are 18 possible parameter sets (v,k,λ) for symmetric designs of order $n = 36$, but up to now a few results are known. Among these are the parameters of the projective plane of order 36, which is the first projective plane of square order, the existence of which is still undecided.

Due to the fact that symmetric designs of order 36 have a large number of points (blocks), the study of sporadic cases is very difficult, except, possibly, when the existence of a suitable collineation group is assumed.

2. main results

A). Denote by $\mathscr{D}$ a symmetric block design with parameters

$$(v,k,\lambda) = (153,57,21)$$

admitting a collineation group which is a Frobenius group of order 272 with Frobenius kernel of order 17. Let ρ be a chosen collineation generating the Frobenius kernel. Since ρ acts fixed-point-free, we may assume for ρ the following form $\rho = (1_0 1_1 \cdots 1_{16})(2_0 2_2 \cdots 2_{16}) \cdots (9_0 9_1 \cdots 9_{16})$ where the capital numbers $1,2,...,9$ are the so called orbital numbers while the small subscripts $_{0,1,2,\ldots,16}$ are the so called indices of the points of $\mathscr{D}$.

Let $l_1 = 1_a 2_b 3_c 4_d 5_e 6_f 7_g 8_h 9_i$ be the first $\rho-$ invariant block, where numbers $a,b,...,i$ denote the multiplicity of appearance of orbital numbers in the orbital block l_1. Since $k = 57$, we have $a+b+c+d+e+f+g+h+i = 57$.

The block l_1 must satisfy a condition on the Hamming length, namely

$$\begin{aligned}&[H(l_1) = (|\rho|-1)\cdot\lambda = 336, \text{ i.e.}\\ &a(a-1)+b(b-1)+c(c-1)+d(d-1)+e(e-1)+\\ &+f(f-1)+g(g-1)+h(h-1)+i(i-1) = 336.\end{aligned}$$

From the last relation, for the multiplicity of appearance in the block l_1, we have these reductions $0 \le a,b,c,...,i \le 12$. In order to reduce isomorphic cases that may appear in the orbital structures at a later stage, we may assume that the inequalities $a \le b \le c \le d \le e \le f \le g \le h \le i$ hold for the block l_1.

Using the computer we have found that there are exactly 29 orbital types for the block l_1 that satisfy the above conditions (a program in the C^{++} programming language performing this task is attached in the Annex 1 at the end of the paper).

Now, consider the collineation μ, which operates on orbital numbers as

$$\mu = (1)(2,3,4,5,6,7,8,9).$$

Only one of the 29 $\rho-$invariant orbital types is $\mu-$invariant. That case is

$$l_1 = 1_1 2_7 3_7 4_7 5_7 6_7 7_7 8_7 9_7.$$

We continue the construction of the orbital structures with the Frobenius group

$$F_{17\cdot 16} = \langle \rho, \mu / \rho^{17} = \mu^{16} = 1 \rangle.$$

For the reduction of isomorphic cases we use the involutions

$$\begin{aligned}\alpha &= (1)(2)(3,9)(4,8)(5,7)(6)\\ \beta &= (1)(2)(3,5)(4,8)(7,9)(6)\end{aligned}$$

and their product

$$\alpha \cdot \beta = (1)(2)(3,7)(4)(5,9)(6)(8).$$

Considering the action of the collineations μ, α, β and $\alpha \cdot \beta$ the form of possible orbital structures is:

$$\begin{aligned}
l_2 \quad &= 1_{a_1}\ 2_{a_2}\ 3_{a_3}\ 4_{a_4}\ 5_{a_3}\ 6_{a_5}\ 7_{a_3}\ 8_{a_4}\ 9_{a_3}\\
l_2^{\mu} = l_3 \quad &= 1_{a_1}\ 2_{a_3}\ 3_{a_2}\ 4_{a_3}\ 5_{a_4}\ 6_{a_3}\ 7_{a_5}\ 8_{a_3}\ 9_{a_4}\\
l_2^{\mu^2} = l_4 \quad &= 1_{a_1}\ 2_{a_4}\ 3_{a_3}\ 4_{a_2}\ 5_{a_3}\ 6_{a_4}\ 7_{a_3}\ 8_{a_5}\ 9_{a_3}\\
l_2^{\mu^3} = l_5 \quad &= 1_{a_1}\ 2_{a_3}\ 3_{a_4}\ 4_{a_3}\ 5_{a_2}\ 6_{a_3}\ 7_{a_4}\ 8_{a_3}\ 9_{a_5} \qquad (1)\\
l_2^{\mu^4} = l_6 \quad &= 1_{a_1}\ 2_{a_5}\ 3_{a_3}\ 4_{a_4}\ 5_{a_3}\ 6_{a_2}\ 7_{a_3}\ 8_{a_4}\ 9_{a_3}\\
l_2^{\mu^5} = l_7 \quad &= 1_{a_1}\ 2_{a_3}\ 3_{a_5}\ 4_{a_3}\ 5_{a_4}\ 6_{a_3}\ 7_{a_2}\ 8_{a_3}\ 9_{a_4}\\
l_2^{\mu^6} = l_8 \quad &= 1_{a_1}\ 2_{a_4}\ 3_{a_3}\ 4_{a_5}\ 5_{a_3}\ 6_{a_4}\ 7_{a_3}\ 8_{a_2}\ 9_{a_3}\\
l_2^{\mu^7} = l_9 \quad &= 1_{a_1}\ 2_{a_3}\ 3_{a_4}\ 4_{a_3}\ 5_{a_5}\ 6_{a_3}\ 7_{a_4}\ 8_{a_3}\ 9_{a_2}
\end{aligned}$$

The above blocks must satisfy Game products $\left| l_1 \cap l_2^{\mu^x} \right| = 21$, $x = 0, 1, 2, ..., 7$. I.e. $Sp(l_1, l_2) = 372$ and

$$Sp\left(l_2, l_2^{\mu}\right) = Sp\left(l_2, l_2^{\mu^2}\right) = Sp\left(l_2, l_2^{\mu^3}\right) = Sp\left(l_2, l_2^{\mu^4}\right) = |\rho| \cdot \lambda = 372.$$

The other Game products between blocks of (1) are ensured by the action of the collineation μ on the block l_2.

Conditions involving the multiplicity of appearance of orbital numbers in the block l_2 and its $\mu-$images yield:

$$\begin{aligned}
&a_1 + a_2 + 4a_3 + 2a_4 + a_5 = 57\\
&a_1 + 7(a_2 + 4a_3 + 2a_4 + a_5) = 357\\
&a_1^2 + a_2^2 + 4a_3^2 + 2a_4^2 + a_5^2 = 393
\end{aligned}$$

$$\begin{aligned} a_1^2+2a_2a_3+4a_3a_4+2a_3a_5&=357 \\ a_1^2+2a_2a_4+4a_3^2+2a_4a_5&=357 \\ a_1^2+2a_2a_3+4a_3a_4+2a_3a_5&=357 \\ a_1^2+2a_2a_5+4a_3^2+2a_4^2&=357 \end{aligned} \qquad (2)$$

From the condition $H(l_2)=336$ we obtain the reductions $0 \le a_1, a_2, a_5 \le 18$, $0 \le a_4 \le 13$ and $0 \le a_3 \le 9$. Solving the system of equations (2), according to the a_1, a_2, a_3, a_4, a_5 and having in consideration $0 \le a_1, a_2, a_5 \le 18$, $0 \le a_4 \le 13$ and $0 \le a_3 \le 9$, we have found exactly two distinct orbital structures, up to isomorphism and duality:

S_1

7	1	7	7	7	7	7	7	7
7	7	7	1	7	7	7	7	7
7	7	7	1	7	7	7	7	7
7	7	7	7	1	7	7	7	7
7	7	7	7	7	1	7	7	7
7	7	7	7	7	7	1	7	7
7	7	7	7	7	7	7	1	7
7	7	7	7	7	7	7	7	1

S_2

1	7	7	7	7	7	7	7	7
7	10	7	4	7	4	7	4	7
7	7	10	7	4	7	4	7	4
7	4	7	10	7	4	7	4	7
7	7	4	7	10	7	4	7	4
7	4	7	4	7	10	7	4	7
7	7	4	7	4	7	10	7	4
7	4	7	4	7	4	7	10	7
7	7	4	7	4	7	4	7	10

(Respective program in the C^{++} programming language is attached in the Annex 2 at the end of the paper). In this way we have proved the following:

Theorem 2.1. *Up to isomorphism and duality there are exactly two orbital structures for a symmetric block design with parameters* $(153, 57, 21)$ *admitting the Frobenius Group* $F_{17\cdot 16} = \langle \rho, \mu / \rho^{17} = \mu^{16} = 1 \rangle$ *of order 272 as a collineation group.*

The fixed-point-free collineation ρ *of order 17 fixes all orbital numbers, whereas the collineation* μ *operates as* $\mu = (1)(2,3,4,5,6,7,8,9)$.

Note. *The actual indexing of these two orbital structures in order to produce an example is still an open problem.*

B). Denote by $\mathscr{D}$ a symmetric block design with parameters $(v, k, \lambda) = (153, 57, 21)$ admitting a collineation group $G = \langle \rho, \mu / \rho^{19} = \mu^3 = 1 \rangle$. The collineation ρ of order 19 acts on the 153 points of $\mathscr{D}$ with one fixed point and eight orbits of length 19 each. If we denote by ∞ the fixed point of ρ and by $1, 2, \ldots, 8$ the orbital points of length 19, we may assume for ρ the following form

$$\rho = (\infty)(1_0 1_1 \cdots 1_{18})(2_0 2_1 \cdots 2_{18}) \cdots (8_0 8_1 \cdots 8_{18}).$$

We may assume that μ acts on orbital numbers as follows:

$$\mu = (\infty)(1,2,3)(4)(5)(6,7,8).$$

The orbital form of the $\rho-$fixed line l_1 can be written in the form:

$$l_1 = 1_{19}2_{19}3_{19}.$$

There are exactly three $\rho-$orbits of blocks of length 19 passing through the fixed point ∞. Considering the action of the collineation μ on the orbital numbers we can write

$$\begin{aligned} l_2 \quad &= \infty\ 1_{a_1}\ 2_{a_2}\ 3_{a_3}\ 4_{a_4}\ 5_{a_5}\ 6_{a_6}\ 7_{a_7}\ 8_{a_8} \\ l_2^{\mu} = l_3 \quad &= \infty\ 1_{a_3}\ 2_{a_1}\ 3_{a_2}\ 4_{a_4}\ 5_{a_5}\ 6_{a_8}\ 7_{a_6}\ 8_{a_7} \\ l_2^{\mu^2} = l_4 \quad &= \infty\ 1_{a_2}\ 2_{a_3}\ 3_{a_1}\ 4_{a_4}\ 5_{a_5}\ 6_{a_7}\ 7_{a_8}\ 8_{a_6} \end{aligned}$$

The multiplicity of appearances of orbital numbers satisfies the following condition

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 = 56.$$

Because $|l_1 \cap l_2| = 21$ we have $a_1 + a_2 + a_3 = 21$, and consequently $a_4 + a_5 + a_6 + a_7 + a_8 = 35$. Through the fixed point ∞ and every other orbital point there pass exactly $\lambda = 21$ blocks, so we have $a_4 = 7, a_5 = 7$. Consequently $a_6 + a_7 + a_8 = 21$.

For the reduction of isomorphic cases for the line l_2 we use the collineations $\xi = (1,2,3)$, $\eta = (6,7,8)$, $\zeta = (4,5)$ and $\nu = (1)(2,3)(5)(6)(7,8)$ which centralize μ.

The line l_2 must satisfy Hamming length

$$H(l_2) = (|\rho| - 1)(\lambda - 1) = 360$$

and Game products

$$Sp(l_i, l_j) = |\rho|(\lambda - 1) = 380 \quad (i, j \in \{2,3,4\}, i \neq j).$$

Using the computer we have proved that there exist only four different types for the line l_2 satisfying the above mentioned conditions:

$$\begin{array}{ll} 1) & l_2 = \infty\ 1_3\ 2_9\ 3_9\ 4_7\ 5_7\ 6_7\ 7_7\ 8_7 \\ 2) & l_2 = \infty\ 1_4\ 2_7\ 3_{10}\ 4_7\ 5_7\ 6_5\ 7_8\ 8_8 \\ 3) & l_2 = \infty\ 1_4\ 2_7\ 3_{10}\ 4_7\ 5_7\ 6_6\ 7_6\ 8_9 \\ 4) & l_2 = \infty\ 1_5\ 2_5\ 3_{11}\ 4_7\ 5_7\ 6_7\ 7_7\ 8_7 \end{array}$$

If we denote by l_5 and l_6 the two other $\mu-$invariant blocks, we can write:

$$l_5 = 1_{b_1}\ 2_{b_1}\ 3_{b_1}\ 4_{b_2}\ 5_{b_3}\ 6_{b_4}\ 7_{b_4}\ 8_{b_4}$$

$$l_6 = 1_{c_1}\ 2_{c_1}\ 3_{c_1}\ 4_{c_2}\ 5_{c_3}\ 6_{c_4}\ 7_{c_4}\ 8_{c_4}$$

Appearance multiplicities $b_1,...,b_4,c_1,...,c_4$ satisfy the following conditions which follow from Hamming lengths and Game products

$$H(l_5) = H(l_6) = 378, \quad Sp(l_5,l_1) = Sp(l_6,l_1) = 399$$

and

$$Sp(l_i,l_j) = 399 \quad (i \neq j, i \in \{5,6\}, j \in \{2,3,4,5,6\}).$$

Once the conditions for the blocks l_5 and l_6 are obtained, we construct the last $\mu-$orbit of blocks

$$\begin{aligned} l_7 \quad &= 1_{d_1}\ 2_{d_2}\ 3_{d_3}\ 4_{d_4}\ 5_{d_5}\ 6_{d_6}\ 7_{d_7}\ 8_{d_8} \\ l_7^{\mu} = l_8 \quad &= 1_{d_3}\ 2_{d_1}\ 3_{d_2}\ 4_{d_4}\ 5_{d_5}\ 6_{d_8}\ 7_{d_6}\ 8_{d_7} \\ l_7^{\mu^2} = l_9 \quad &= 1_{d_2}\ 2_{d_3}\ 3_{d_1}\ 4_{d_4}\ 5_{d_5}\ 6_{d_7}\ 7_{d_8}\ 8_{d_6} \end{aligned}$$

Operating as above and keeping in mind the block length $k = 57$, the Hamming length $H(l_7) = 378$, the Game products $Sp(l_7,l_8) = Sp(l_7,l_9) = 399$ and the Game products of the last $\mu-$orbit with the blocks of the other $\mu-$orbits we have found all orbital structures of $\mathscr{D}$ using the collineation group

$$G = \langle \rho, \mu / \rho^{19} = \mu^3 = 1 \rangle.$$

Studying isomorphism and duality between the basic structures, we have found 16 such structures and we list them in the following table:

1)

19	19	19					
3	9	9	7	7	7	7	7
9	3	9	7	7	7	7	7
9	9	3	7	7	7	7	7
7	7	7	3	6	9	9	9
7	7	7	6	12	6	6	6
7	7	7	9	6	3	9	9
7	7	7	9	6	9	3	9
7	7	7	9	6	9	9	3

2)

19	19	19					
3	9	9	7	7	7	7	7
9	3	9	7	7	7	7	7
9	9	3	7	7	7	7	7
7	7	7	3	6	9	9	9
7	7	7	6	12	6	6	6
7	7	7	9	6	5	5	11
7	7	7	9	6	11	5	5
7	7	7	9	6	5	11	5

3)

19	19	19					
3	9	9	7	7	7	7	7
9	3	9	7	7	7	7	7
9	9	3	7	7	7	7	7
7	7	7	6	12	6	6	6
7	7	7	12	6	6	6	6
7	7	7	6	6	4	10	10
7	7	7	6	6	10	4	10
7	7	7	6	6	10	10	4

4)

19	19	19					
3	9	9	7	7	7	7	7
9	3	9	7	7	7	7	7
9	9	3	7	7	7	7	7
7	7	7	6	12	6	6	6
7	7	7	12	6	6	6	6
7	7	7	6	6	6	6	12
7	7	7	6	6	12	6	6
7	7	7	6	6	6	12	6

5)

19	19	19					
4	7	10	7	7	5	8	8
10	4	7	7	7	8	5	8
7	10	4	7	7	8	8	5
7	7	7	3	6	9	9	9
7	7	7	6	12	6	6	6
5	8	8	9	6	10	4	7
8	5	8	9	6	7	10	4
8	8	5	9	6	4	7	10

6)

19	19	19					
4	7	10	7	7	5	8	8
10	4	7	7	7	8	5	8
7	10	4	7	7	8	8	5
7	7	7	3	6	9	9	9
7	7	7	6	12	6	6	6
6	6	9	9	6	10	7	4
9	6	6	9	6	4	10	7
6	9	6	9	6	7	4	10

7)

19	19	19					
4	7	10	7	7	5	8	8
10	4	7	7	7	8	5	8
7	10	4	7	7	8	8	5
7	7	7	6	12	6	6	6
7	7	7	12	6	6	6	6
5	8	8	6	6	11	5	8
8	5	8	6	6	8	11	5
8	8	5	6	6	5	8	11

8)

19	19	19					
4	7	10	7	7	5	8	8
10	4	7	7	7	8	5	8
7	10	4	7	7	8	8	5
7	7	7	6	12	6	6	6
7	7	7	12	6	6	6	6
6	6	9	6	6	11	8	5
9	6	6	6	6	5	11	8
6	9	6	6	6	8	5	11

9)

19	19	19					
4	7	10	7	7	6	6	9
10	4	7	7	7	9	6	6
7	10	4	7	7	6	9	6
7	7	7	3	6	9	9	9
7	7	7	6	12	6	6	6
5	8	8	9	6	10	7	4
8	5	8	9	6	4	10	7
8	8	5	9	6	7	4	10

10)

19	19	19					
4	7	10	7	7	6	6	9
10	4	7	7	7	9	6	6
7	10	4	7	7	6	9	6
7	7	7	3	6	9	9	9
7	7	7	6	12	6	6	6
6	6	9	9	6	7	10	4
9	6	6	9	6	4	7	10
6	9	6	9	6	10	4	7

11)

19	19	19					
4	7	10	7	7	6	6	9
10	4	7	7	7	9	6	6
7	10	4	7	7	6	9	6
7	7	7	6	12	6	6	6
7	7	7	12	6	6	6	6
5	8	8	6	6	11	8	5
8	5	8	6	6	5	11	8
8	8	5	6	6	8	5	11

12)

19	19	19					
4	7	10	7	7	6	6	9
10	4	7	7	7	9	6	6
7	10	4	7	7	6	9	6
7	7	7	6	12	6	6	6
7	7	7	12	6	6	6	6
6	6	9	6	6	8	11	5
9	6	6	6	6	5	8	11
6	9	6	6	6	11	5	8

13)

19	19	19					
5	5	11	7	7	7	7	7
11	5	5	7	7	7	7	7
5	11	5	7	7	7	7	7
7	7	7	3	6	9	9	9
7	7	7	6	12	6	6	6
7	7	7	9	6	3	9	9
7	7	7	9	6	9	3	9
7	7	7	9	6	9	9	3

14)

19	19	19					
5	5	11	7	7	7	7	7
11	5	5	7	7	7	7	7
5	11	5	7	7	7	7	7
7	7	7	3	6	9	9	9
7	7	7	6	12	6	6	6
7	7	7	9	6	5	5	11
7	7	7	9	6	11	5	5
7	7	7	9	6	5	11	5

15)

19	19	19					
5	5	11	7	7	7	7	7
11	5	5	7	7	7	7	7
5	11	5	7	7	7	7	7
7	7	7	6	12	6	6	6
7	7	7	12	6	6	6	6
7	7	7	6	6	4	10	10
7	7	7	6	6	10	4	10
7	7	7	6	6	10	10	4

16)

19	19	19					
5	5	11	7	7	7	7	7
11	5	5	7	7	7	7	7
5	11	5	7	7	7	7	7
7	7	7	12	6	6	6	6
7	7	7	6	12	6	6	6
7	7	7	6	6	12	6	6
7	7	7	6	6	6	12	6
7	7	7	6	6	6	6	12

In this way we have proved the following:

Theorem 2.2. *Up to isomorphism and duality there are exactly sixteen orbital structures for a symmetric block design with parameters* $(153,57,21)$ *admitting the group* $G = \langle \rho, \mu\rho^{19} = \mu^3 = 1\rangle$ *of order 57 as a collineation group. The collineation* ρ *of order 19 fixes one point, whereas the collineation* μ *has the form* $\mu = (\infty)(1,2,3)(4)(5)(6,7,8)$.

Note 2.3. The actual indexing of these 16 orbital structures in order to produce an example is still an open problem.

Note 2.4. It is clear that in the orbital structure 4) acts the collineation $\tau = (\infty)(1)(2,3)(4)(5,6)(7,8)$, which fixes 39 blocks (points) of the design $\mathscr{D}$.

Moreover, the multiplicities of appearance in this orbital structure are $\equiv 0, 1 (\bmod 3)$. Because of this fact in orbital structure 4) the collineation μ fixes all $\rho-$orbital numbers.

These collineations generate the group $G \times \langle\tau\rangle = \langle\rho, \mu\rangle \times \langle\tau\rangle$, which operates as the full collineation group of the design $\mathscr{D}$. In the orbital structure 16) operates the collineation $\pi = (1)(2,3)(4)(5,6)(7,8)$, so it can be indexed by the collineation group $G \times \langle\pi\rangle = \langle\rho, \mu\rangle \times \langle\pi\rangle$, where the collineations ρ, μ operate as described above and the collineation π acts as $\pi = (1)(2,3)(4)(5,6)(7,8)$ on $\rho-$orbital numbers, while it acts fixed-point-free on indices.

Annex 1

```
#include <stdio.h> void main(){ int
a1,a2,a3,a4,a5,a6,a7,a8,a9,hemi; double wz; FILE *fo;
       printf("** (153,57,21) Orbital types for the
    line L1 **\n\n");
        if((fo=fopen("d153L1.TXT","wt"))==NULL) {
  printf("\a\a Can not open the file %s\n","d153L1.TXT");
    exit(0);
  }
wz=0;
for(a1=0; a1<=24; a1++) {
 for(a2=a1; a2<=24; a2++) {
  for(a3=a2; a3<=24; a3++) {
   for(a4=a3; a4<=24; a4++) {
    for(a5=a4; a5<=24; a5++) {
     for(a6=a5; a6<=24; a6++) {
      for(a7=a6; a7<=24; a7++) {
       for(a8=a7; a8<=24; a8++) {
   a9=57-(a1+a2+a3+a4+a5+a6+a7+a8);
   if ((a9<a8) | (a9>24))  continue;
   hemi=a1*(a1-1)+a2*(a2-1)+a3*(a3-1)+a4*(a4-1)+
        a5*(a5-1)+a6*(a6-1)+a7*(a7-1)+a8*(a8-1)+a9*(a9-1);
   if (hemi != 336) continue;
   wz++;
   printf("%8.0f   %2d %2d %2d %2d %2d %2d %2d %2d %2d \n",
   wz,a1,a2,a3,a4,a5,a6,a7,a8,a9);
   /*fprintf(fo,"%8.0f   %2d %2d %2d %2d %2d %2d %2d %2d
                                                  \n", */
```

```
   fprintf(fo,"%8.0f  {%2d ,%2d ,%2d ,%2d ,%2d ,%2d ,%2d ,
                                                          , %2d }\n",
    wz,a1,a2,a3,a4,a5,a6,a7,a8,a9);
           } /* for a8 */
         } /* for a7 */
       } /* for a6 */
     } /* for a5 */
    } /* for a4 */
} /* for a3 */
  } /* for a2 */
 } /* for a1 */
   fclose(fo);
      printf("End !\n");
  }
```

Annex 2

```
#include <stdio.h>
void main()
{
int o1,o2,o3,o4,o5,o6,o7,o8,o9,hemi,sp21,sp32,sp42, sp52,
                                                    sp62;
double wz;
FILE *fo;
     printf("** (153,57,21) - orbital structures  **\n\n");
  if((fo=fopen("D153L2.TXT","wt"))==NULL) {
  printf("\a\a Can not open file %s\n","D153L2.TXT");
    exit(0);
  }
wz=0;
    for(o1=18; o1>=0; o1--) {
     for(o2=18; o2>=0; o2--) {
      for(o3=9; o3>=0; o3--) {
       for(o4=13; o4>=0; o4--) {
              o5=57-(o1+o2+4*o3+2*o4);
                   if (o5<0 | o5>18)
                              continue;
                   hemi=0;
                   hemi=o1*o1+o2*o2+4*o3*o3+2*o4*o4+o5*o5;
```

```
            if (hemi != 393)
           continue;
          sp21=0;
          sp21=o1+7*(o2+4*o3+2*o4+o5);
        if (sp21 != 357)
      continue;
     sp32=0;
     sp32=o1*o1+2*o2*o3+4*o3*o4+2*o3*o5;
   if (sp32 != 357)
 continue;
sp42=0;
sp42=o1*o1+2*o2*o4+4*o3*o3+2*o4*o5;
        if (sp42 != 357)
           continue;
              sp52=0;
              sp52=o1*o1+2*o2*o3+4*o3*o4+2*o3*o5;
             if (sp52 != 357)
                continue;
                   sp62=0;
                   sp62=o1*o1+2*o2*o5+4*o3*o3+2*o4*o4;
                  if (sp62 != 357)
                     continue;
   wz++;
    /* solution */
   printf("%8.0f   %2d %2d %2d %2d %2d \n",
                                    wz,o1,o2,o3,o4,o5);
   fprintf(fo,"%8.0f   %2d %2d %2d %2d %2d \n",
                  wz,o1,o2,o3,o4,o5);
  } /* for o4 */
  } /* for o3 */
  } /* for o2 */
  } /* for o1 */
  fclose(fo);
     printf("End !\n");
 }
```

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