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LITTLEWOOD–PALEY CHARACTERIZATION OF DISCRETE MORREY SPACES AND ITS APPLICATION TO THE DISCRETE MARTINGALE TRANSFORM

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The goal of this paper is to develop the Littlewood–Paley theory of discrete Morrey spaces. As an application, we establish the boundedness of martingale transforms. We carefully justify the definition of martingale transforms, since discrete Morrey spaces do not contain discrete Lebesgue spaces as dense subspaces. We also obtain the boundedness of Riesz potentials.

1. Introduction

The goal of this note is to develop the Littlewood–Paley theory of discrete Morrey spaces. As an application, we establish the boundedness of the martingale transforms.

First, we define discrete Morrey spaces. A dyadic interval of integers is the set of integers given by $I(j,k) = \mathbb{Z} \cap [2^j k, 2^j(k+1))$ for some $j \in \mathbb{N}_0 = \{0,1,\ldots\}$ and $k \in \mathbb{Z}$. A dyadic cube in $\mathbb{Z}^n$ is a subset of the form:

$$Q = I(j,k_1) \times I(j,k_2) \times \cdots \times I(j,k_n),$$

where $j \in \mathbb{N}_0$ and $k = (k_1,k_2,\ldots,k_n) \in \mathbb{Z}^n$. The family $\mathcal{D}(\mathbb{Z}^n)$ stands for the set of all dyadic cubes described above, while the subfamily $\mathcal{D}_j(\mathbb{Z}^n)$ collects all dyadic cubes of $I(j,k) = \mathbb{Z} \cap [2^j k, 2^j(k+1))$ with $j \in \mathbb{N}_0$.

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Definition 1.1. Let $1 \le q \le p < \infty$. The space $\mathcal{M}^p_q(\mathbb{Z}^n)$ is the set of all $a = \{a_{\vec{j}}\}_{\vec{j} \in \mathbb{Z}^n}$ for which

$$\|a\|_{\mathcal{M}^p_q(\mathbb{Z}^n)} = \sup_{Q \in \mathcal{D}(\mathbb{Z}^n)} (\sharp Q)^{\frac{1}{p}-\frac{1}{q}} \left(\sum_{\vec{j} \in Q} |a_{\vec{j}}|^q \right)^{\frac{1}{q}}$$

is finite, where $\sharp Q$ stands for the number of elements of the dyadic cube Q.

It is remarkable that the space $\mathcal{M}^p_p(\mathbb{Z}^n)$ boils down to the Lebesgue space $\ell^p(\mathbb{Z}^n)$.

The discrete Morrey space $\mathcal{M}^p_q(\mathbb{Z}^n)$ falls within the scope of the work [1] and has been investigated in [4–7]. Our goal of this paper is to obtain an equivalent norm using the Littlewood–Paley decomposition.

We describe the Littlewood–Paley decomposition. To this end, we start with defining the 1-dimensional Littlewood–Paley operator. For a 1-dimensional sequence $a = \{a_j\}_{j \in \mathbb{Z}}$, we let

$$E_k(a)_j = \frac{1}{2^k} \sum_{l \in Q} a_l,$$

where Q is a unique cube in $\mathcal{D}_k(\mathbb{Z})$ which contains j. Each E_k is called the average operator of generation k. We define $D_k = E_k - E_{k+1}$. The Littlewood–Paley operator $g(a) = \{g(a)_j\}_{j \in \mathbb{Z}}$ is defined by

$$g(a)_j = \left(\sum_{k=0}^{\infty} |D_k(a)_j|^2 \right)^{\frac{1}{2}} \quad (j \in \mathbb{Z})$$

Having defined 1-dimensional operators, we move on to the definition of operators acting of n-fold indexed (multi-indexed) sequences. We let $l = 1, 2, \ldots, n$. The operator $E^{(l)}_k$ acts on the l-th component as E_k with other components unchanged. The difference operator $D^{(l)}_k$ is defined by $D^{(l)}_k = E^{(l)}_k - E^{(l)}_{k+1}$. We write $\vec{E} = (E, E, \ldots, E)$. Let $\vec{X} = (X_1, X_2, \ldots, X_n) \in \{(D, E)\}^n \setminus \{\vec{E}\}$. Define the operator $\vec{X}_k$ by $\vec{X}_k = X^{(1)}_k \circ X^{(2)}_k \circ \cdots \circ X^{(n)}_k$. The discrete Littlewood–Paley operator g is given by the mapping $a = \{a_{\vec{j}}\}_{\vec{j} \in \mathbb{Z}^n} \mapsto g(a) = \{g(a)_{\vec{j}}\}_{\vec{j} \in \mathbb{Z}^n}$, where

$$g(a)_{\vec{j}} = \left(\sum_{k=0}^{\infty} \sum_{\vec{X} \in \{(D,E)\}^n \setminus \{\vec{E}\}} |\vec{X}_k(a)_{\vec{j}}|^2 \right)^{\frac{1}{2}} \quad (\vec{j} \in \mathbb{Z}^n)$$

The next proposition is well known as the Littlewood–Paley characterization of the discrete $\ell^p(\mathbb{Z}^n)$-norm.

Proposition 1.2. *Let $1 < p < \infty$. Then there exists $c_p > 0$ such that*

$$c_p{}^{-1}\|a\|_{\ell^p(\mathbb{Z}^n)} \le \|g(a)\|_{\ell^p(\mathbb{Z}^n)} \le c_p\|a\|_{\ell^p(\mathbb{Z}^n)}$$

for all $a \in \ell^p(\mathbb{Z}^n)$.

In this paper, we will establish the following norm equivalence and then apply it to the boundedness of various operators:

Theorem 1.3. *Let $1 < q \le p < \infty$. Then there exists $c_{p,q} > 0$ such that*

$$c_{p,q}{}^{-1}\|a\|_{\mathcal{M}^p_q(\mathbb{Z}^n)} \le \|g(a)\|_{\mathcal{M}^p_q(\mathbb{Z}^n)} \le c_{p,q}\|a\|_{\mathcal{M}^p_q(\mathbb{Z}^n)}$$

for all $a \in \mathcal{M}^p_q(\mathbb{Z}^n)$.

Theorem 1.3 is a discrete version of [8, Corollary 4.1].

We apply Theorem 1.3 to the boundedness of martingale transforms. Let $\{m^k\}_{k=1}^\infty$ be a sequence of sequences in $\ell^\infty(\mathbb{Z}^n)$. Then define

$$M_m(a) = \{M_m(a)_{\vec{j}}\}_{\vec{j}\in\mathbb{Z}} = \sum_{k=0}^{\infty} E_{k+1}(m^k)D_k(a) = \lim_{N\to\infty}\sum_{k=0}^{N} E_{k+1}(m^k)D_k(a), \quad (1)$$

where $E_{k+1}(m^k)D_k(a) = \{E_{k+1}(m^k)_{\vec{j}}D_k(a)_{\vec{j}}\}_{\vec{j}\in\mathbb{Z}}$. We can not use the density argument. Recall that the support of a multi-indexed sequence $a = \{a_{\vec{j}}\}_{\vec{j}\in\mathbb{Z}^n}$ is the set of all indices $\vec{j}$ for which $a_{\vec{j}} \ne 0$. Since $\mathcal{M}^p_q(\mathbb{Z}^n)$ does not contain the space of finitely supported multi-indexed sequences as a dense subspace (see Remark 4.1), we have to justify the definition of the martingale transform M_m: The existence of the limit defining $M_m(a)$ is not clear. Furthermore, since Theorem 1.3 is applicable for multi-indexed sequences in $\mathcal{M}^p_q(\mathbb{Z}^n)$, we also have to show that $M_m(a) \in \mathcal{M}^p_q(\mathbb{Z}^n)$ for any $a \in \mathcal{M}^p_q(\mathbb{Z}^n)$ before we use Theorem 1.3 to obtain the norm estimate.

We perform this using the predual space $\mathcal{H}^{p'}_{q'}(\mathbb{Z}^n)$ considered in [9].

Definition 1.4. Let $1 < q \le p < \infty$.

1. A multi-indexed sequence $a = \{a_{\vec{j}}\}_{\vec{j}\in\mathbb{Z}^n}$ is said to be a (p',q')-block centered at Q if it is supported on Q and $\|a\|_{\ell^{q'}(\mathbb{Z}^n)} \le (\sharp Q)^{\frac{1}{p}-\frac{1}{q}}$.
2. The block space $\mathcal{H}^{p'}_{q'}(\mathbb{Z}^n)$ is the set of all multi-indexed sequences a of the form: $a = \sum\limits_{j=1}^{\infty} \lambda^{(j)}a^{(j)}$ where the convergence takes place in the topology of $\ell^{p'}(\mathbb{Z}^n)$, $\lambda = \{\lambda^{(j)}\}_{j=1}^\infty \in \ell^1(\mathbb{N})$ and each $a^{(j)}$ is a (p',q')-block centered at $Q_j \in \mathcal{D}(\mathbb{Z}^n)$. The norm is given by $\|a\|_{\mathcal{H}^{p'}_{q'}(\mathbb{Z}^n)} = \inf\|\lambda\|_{\ell^1(\mathbb{Z}^n)}$, where λ and $\{a^{(j)}\}_{j=1}^\infty$ move over all possible representations.

According to the general theory [9], $\mathcal{M}^p_q(\mathbb{Z}^n)$ admits a predual. One predual of $\mathcal{M}^p_q(\mathbb{Z}^n)$ is the space $\mathcal{H}^{p'}_{q'}(\mathbb{Z}^n)$.

Proposition 1.5. *Let $1 < q' \le p' < \infty$. Then $\mathcal{H}^{p'}_{q'}(\mathbb{Z}^n)$ is a Banach space. Furthermore, the dual of $\mathcal{H}^{p'}_{q'}(\mathbb{Z}^n)$ is isomorphic to $\mathcal{M}^p_q(\mathbb{Z}^n)$. More precisely, we have the following*:

1. *For all $a = \{a_{\vec{j}}\}_{\vec{j}\in\mathbb{Z}^n} \in \mathcal{M}^p_q(\mathbb{Z}^n)$ and $b = \{b_{\vec{j}}\}_{\vec{j}\in\mathbb{Z}^n} \in \mathcal{H}^{p'}_{q'}(\mathbb{Z}^n)$,*
$$\sum_{\vec{j}\in\mathbb{Z}^n} |a_{\vec{j}} b_{\vec{j}}| \le \|a\|_{\mathcal{M}^p_q(\mathbb{Z}^n)} \|b\|_{\mathcal{H}^{p'}_{q'}(\mathbb{Z}^n)}.$$
In particular,
$$a \mapsto L_a(b) = \sum_{\vec{j}\in\mathbb{Z}^n} a_{\vec{j}} b_{\vec{j}}$$
is a bounded linear functional.

2. *Conversely any bounded linear functional over $\mathcal{H}^{p'}_{q'}(\mathbb{Z}^n)$ can be realized as above for some $a \in \mathcal{M}^p_q(\mathbb{Z}^n)$.*

By using Proposition 1.5 we will justify that the limit defining $M_m(a)$ for $a \in \mathcal{M}^p_q(\mathbb{Z}^n)$ exists in the weak-* topology.

Theorem 1.6. *Let $1 < q \le p < \infty$. Assume that*
$$K = \sup_{k\in\mathbb{N}} \|E_{k+1}(m^k)\|_{\ell^\infty(\mathbb{Z}^n)} < \infty.$$
Then the limit defining $M_m(a)$ for $a \in \mathcal{M}^p_q(\mathbb{Z}^n)$ exists in the weak- topology of $\mathcal{M}^p_q(\mathbb{Z}^n)$. The martingale transform $a \in \mathcal{M}^p_q(\mathbb{Z}^n) \mapsto M_m(a) \in \mathcal{M}^p_q(\mathbb{Z}^n)$ is bounded.*

Here we list other conventions of this paper.

- A cube in $\mathbb{Z}^n$ is a subset that can be expressed as
$$Q = Q(a,r) = \{m = (m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n : \max_{j=1,2,\ldots,n} |m_j - a_j| \le r\}$$
for some $a = (a_1, a_2, \ldots, a_n)$ and $r > 0$.
- For multi-indexed sequences $a = \{a_{\vec{j}}\}_{\vec{j}\in\mathbb{Z}^n}$ and $b = \{b_{\vec{j}}\}_{\vec{j}\in\mathbb{Z}^n}$, we write
$$\langle a, b\rangle = \sum_{\vec{j}\in\mathbb{Z}^n} a_{\vec{j}} b_{\vec{j}}$$
as long as the right-hand side converges absolutely.

- Let $A, B \geq 0$. Then $A \lesssim B$ and $B \gtrsim A$ mean that there exists a constant $C > 0$ such that $A \leq CB$, where C depends only on the parameters of importance. The symbol $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$ happen simultaneously, while $A \simeq B$ means that there exists a constant $C > 0$ such that $A = CB$. When we need to emphasize or keep in mind that the constant C depends on the parameters α, β, γ etc, we write $A \lesssim_{\alpha,\beta,\gamma,\ldots} B$ instead of $A \lesssim B$.

Before we conclude this section, we collect some elementary facts that can be derived directly from the above definitions. Observe that any cube $Q \in \mathcal{Q}(\mathbb{Z}^n)$ can be included in the union of dyadic cubes $Q_1, Q_2, \ldots, Q_{3^n}$ satisfying $\ell(Q_j) \leq \ell(Q) < 2\ell(Q_j)$ for each $j = 1, 2, \ldots, 3^n$. A direct consequence of this observation is the norm equivalence: for

$$\|a\|_{\mathcal{M}^p_q(\mathbb{Z}^n)} \sim \sup_{Q \in \mathcal{Q}(\mathbb{Z}^n)} (\sharp Q)^{\frac{1}{p}-\frac{1}{q}} \left(\sum_{\vec{j} \in Q} |a_{\vec{j}}|^q \right)^{\frac{1}{q}}.$$

We organize this paper as follows: In Section 2, we collect some preliminary facts. Theorem 1.3 is proved in Section 3. As an application, we prove Theorem 1.6 in Section 4. Section 5 is an appendix where we prove the boundedness of the fractional integral operator.

2. Preliminaries

2.1. Embedding

We invoke a fundamental embedding result [5, 7]: If $1 \leq r \leq q \leq p < \infty$, then

$$\|a\|_{\mathcal{M}^p_r(\mathbb{Z}^n)} \leq \|a\|_{\mathcal{M}^p_q(\mathbb{Z}^n)} \tag{2}$$

for any multi-indexed sequence $a = \{a_{\vec{j}}\}_{\vec{j} \in \mathbb{Z}^n}$ by Hölder's inequality.

2.2. Maximal operator

For a multi-indexed sequence $a = \{a_{\vec{j}}\}_{\vec{j} \in \mathbb{Z}^n}$, write

$$(M_{\text{dyadic}} a)_{\vec{j}} = \sup_{\vec{j} \in Q \in \mathcal{D}(\mathbb{Z}^n)} \frac{1}{\sharp Q} \sum_{\vec{j}^* \in Q} |a_{\vec{j}^*}|.$$

We define $M_{\text{dyadic}} a = \{(M_{\text{dyadic}} a)_{\vec{j}}\}_{\vec{j} \in \mathbb{Z}^n}$. The correspondence $a \mapsto M_{\text{dyadic}} a$ is called the dyadic maximal operator. Gunawan and Schwanke established that the dyadic maximal operator is bounded on $\mathcal{M}^p_q(\mathbb{Z}^n)$ [4, Theorem 3.2].

Proposition 2.1. *Let $1 < q \le p < \infty$. Then there exists $c_q > 0$ such that*

$$\|M_{\rm dyadic} a\|_{\mathcal{M}^p_q(\mathbb{Z}^n)} \le c_q \|a\|_{\mathcal{M}^p_q(\mathbb{Z}^n)}$$

for all $a \in \mathcal{M}^p_q(\mathbb{Z}^n)$.

2.3. Predual spaces

We invoke the following elementary facts: Since the proof is similar to the classical case as in [3], we content ourselves with the statement.

Lemma 2.2. [3, (9.2)] *For any (p',q')-block $a = \{a_{\vec{j}}\}_{\vec{j} \in \mathbb{Z}^n}$, we have $\|a\|_{\ell^{p'}(\mathbb{Z}^n)} \le 1$.*

A direct consequence of Lemma 2.2 is the following embedding result:

Corollary 2.3. *Let $1 < q \le p < \infty$. Then $\mathcal{H}^{p'}_{q'}(\mathbb{Z}^n)$ is a subset of $\ell^{p'}(\mathbb{Z}^n)$. More quantitatively, $\|a\|_{\ell^{p'}(\mathbb{Z}^n)} \le \|a\|_{\mathcal{H}^{p'}_{q'}(\mathbb{Z}^n)}$ for all $a = \{a_{\vec{j}}\}_{\vec{j} \in \mathbb{Z}^n}$.*

Finally, we invoke [3, Lemma 341].

Proposition 2.4. *Let $1 < q \le p < \infty$ and $Q \in \mathcal{D}(\mathbb{Z}^n)$. Define*

$$R_Q(a)_{\vec{j}} = \begin{cases} a_{\vec{j}} & \vec{j} \in Q, \\ 0 & \vec{j} \notin Q \end{cases}$$

for $a \in \ell^{q'}(\mathbb{Z}^n)$. Then we have

$$\|R_Q(a)\|_{\mathcal{H}^{p'}_{q'}(\mathbb{Z}^n)} \le (\sharp Q)^{\frac{1}{q}-\frac{1}{p}} \|a\|_{\ell^{q'}(\mathbb{Z}^n)}$$

for all $a \in \ell^{q'}(\mathbb{Z}^n)$.

3. Littlewood–Paley decomposition–Proof of Theorem 1.3

Recall that $g(a)$ contains the operators D_k in its definition, which annihilate the constant multi-indexed sequence $\{1\}_{\vec{j} \in \mathbb{Z}^n}$. Therefore, seemingly the quantity $\|g(a)\|_{\mathcal{M}^p_q(\mathbb{Z}^n)}$ loses something that $\|a\|_{\mathcal{M}^p_q(\mathbb{Z}^n)}$ has. This is the case if we consider a multi-indexed sequence a that does not necessarily belong to $\mathcal{M}^p_q(\mathbb{Z}^n)$. To establish that this does not apply for any multi-indexed sequence in $\mathcal{M}^p_q(\mathbb{Z}^n)$, we use the following lemma:

Lemma 3.1. *Let $R \in \mathcal{D}(\mathbb{Z}^n)$ and $1 < q \le p < \infty$. Then for each $a \in \mathcal{M}^p_q(\mathbb{Z}^n)$ and for each multi-indexed sequence b which is supported on R, we have*

$$\lim_{N\to\infty} \langle E_N(a), E_N(b)\rangle = 0.$$

Proof. Normalization allows us to assume $\sum_{\vec{j}\in R} |b_{\vec{j}}|^{q'} = 1$. Let $\sharp R = 2^{nM}$. Consider an increasing sequence $\{Q_m\}_{m=1}^\infty \subset \mathcal{D}(\mathbb{Z}^n)$ satisfying $Q_0 = R$, $\sharp Q_{m+1} = 2^n \sharp Q_m$. A geometric observation shows that $\bigcup_{m=0}^\infty Q_m$ is nothing but a quadrant S of $\mathbb{Z}^n$. That is, S is the Cartesian n-fold product of the sets $[0,\infty)\cap\mathbb{Z}$ or $(-\infty,0)\cap\mathbb{Z}$. We decompose

$$\bigcup_{m=0}^\infty Q_m = Q_0 \cup \bigcup_{m=0}^\infty (Q_{m+1}\setminus Q_m).$$

Then we have

$$\begin{aligned}
|\langle E_N(a), E_N(b)\rangle| &= \left|\sum_{\vec{j}\in\mathbb{Z}^n} E_N(a)_{\vec{j}} E_N(b)_{\vec{j}}\right| \\
&\le \sum_{\vec{j}\in\mathbb{Z}^n} |E_N(a)_{\vec{j}}||E_N(b)_{\vec{j}}| \\
&= \sum_{\vec{j}\in Q_0} |E_N(a)_{\vec{j}}||E_N(b)_{\vec{j}}| + \sum_{m=0}^\infty \sum_{\vec{j}\in Q_{m+1}\setminus Q_m} |E_N(a)_{\vec{j}}||E_N(b)_{\vec{j}}|.
\end{aligned}$$

For the first term, we employ Hölder's inequality and Proposition 2.1 to have

$$\begin{aligned}
\sum_{\vec{j}\in Q_0} |E_N(a)_{\vec{j}}||E_N(b)_{\vec{j}}| &\le \|E_N(a)\|_{\ell^q(Q_0)} \|E_N(b)\|_{\ell^{q'}(Q_0)} \\
&\le (\sharp Q_0)^{\frac{1}{q}-\frac{1}{p}} \|E_N(a)\|_{\mathcal{M}^p_q(\mathbb{Z}^n)} \left(\sum_{\vec{j}\in Q_0} \left|\frac{1}{2^{nN}} \sum_{\vec{j}*\in Q_0} b_{\vec{j}*}\right|^{q'}\right)^{\frac{1}{q'}} \\
&\le \frac{1}{2^{nN}} (\sharp Q_0)^{\frac{1}{q}-\frac{1}{p}+\frac{1}{q'}} \|E_N(a)\|_{\mathcal{M}^p_q(\mathbb{Z}^n)} \left|\sum_{\vec{j}*\in Q_0} b_{\vec{j}*}\right| \\
&\le \frac{1}{2^{nN}} (\sharp Q_0)^{\frac{1}{q}-\frac{1}{p}+\frac{1}{q'}} \|M_{\text{dyadic}} a\|_{\mathcal{M}^p_q(\mathbb{Z}^n)} \left|\sum_{\vec{j}*\in Q_0} b_{\vec{j}*}\right| \\
&\lesssim \frac{1}{2^{nN}} (\sharp Q_0)^{1-\frac{1}{p}} \|a\|_{\mathcal{M}^p_q(\mathbb{Z}^n)} \left|\sum_{\vec{j}*\in Q_0} b_{\vec{j}*}\right|.
\end{aligned}$$

This term tends to 0 as $N \to \infty$.

For the second term, we first choose a dyadic cube $S \in \mathcal{D}_N(\mathbb{Z}^n)$ which contains Q_0. Then we obtain an increasing sequence $Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_l = S$ with the property that there is no intermediate dyadic cube between Q_{j-1} and Q_j for all $j = 1, 2, \ldots, l$, where $l = N - M$. Suppose $\vec{j} \in Q_{m+1} \setminus Q_m$ with $m = 0, 1, \ldots$. Then

$$E_N(b)_{\vec{j}} = \begin{cases} \dfrac{1}{2^{nN}} \displaystyle\sum_{\vec{j}* \in Q_0} b_{\vec{j}*} & \text{if } m+1 \leq l, \\ 0 & \text{if } m+1 > l. \end{cases}$$

If we insert this expression into the second term, then we have

$$\begin{aligned} \sum_{m=0}^{\infty} \sum_{\vec{j} \in Q_{m+1} \setminus Q_m} |E_N(a)_{\vec{j}}||E_N(b)_{\vec{j}}| &= \sum_{m=0}^{l-1} \sum_{\vec{j} \in Q_{m+1} \setminus Q_m} |E_N(a)_{\vec{j}}| \left| \frac{1}{2^{nN}} \sum_{\vec{j}* \in Q_0} b_{\vec{j}*} \right| \\ &= \frac{1}{2^{nN}} \left| \sum_{\vec{j}* \in Q_0} b_{\vec{j}*} \right| \sum_{m=0}^{l-1} \sum_{\vec{j} \in Q_{m+1} \setminus Q_m} |E_N(a)_{\vec{j}}|. \end{aligned}$$

By the triangle inequality, the definition of the Morrey norm $\|\cdot\|_{\mathcal{M}^p_q(\mathbb{Z}^n)}$ and Proposition 2.1,

$$\begin{aligned} &\sum_{m=0}^{\infty} \sum_{\vec{j} \in Q_{m+1} \setminus Q_m} |E_N(a)_{\vec{j}}||E_N(b)_{\vec{j}}| \\ &\leq \frac{1}{2^{nN}} \left| \sum_{\vec{j}* \in Q_0} b_{\vec{j}*} \right| \sum_{m=0}^{l-1} \sum_{\vec{j} \in Q_{m+1}} |E_N(a)_{\vec{j}}| \\ &\leq \frac{1}{2^{nN}} \left| \sum_{\vec{j}* \in Q_0} b_{\vec{j}*} \right| \sum_{m=0}^{l-1} (\sharp Q_{m+1})^{1-\frac{1}{p}} \|E_N(a)\|_{\mathcal{M}^p_q(\mathbb{Z}^n)} \\ &\leq \frac{1}{2^{nN}} \left| \sum_{\vec{j}* \in Q_0} b_{\vec{j}*} \right| \|M_{\mathrm{dyadic}} a\|_{\mathcal{M}^p_q(\mathbb{Z}^n)} \sum_{m=0}^{l-1} (\sharp Q_{m+1})^{1-\frac{1}{p}} \\ &\lesssim \frac{1}{2^{nN}} \left| \sum_{\vec{j}* \in Q_0} b_{\vec{j}*} \right| \|a\|_{\mathcal{M}^p_q(\mathbb{Z}^n)} \sum_{m=0}^{l-1} (\sharp Q_{m+1})^{1-\frac{1}{p}}. \end{aligned}$$

Since $\sharp Q_{m+1} = 2^{n(M+m+1)}$, $p < \infty$ and $l = N - M$,

$$\sum_{m=0}^{l-1} (\sharp Q_{m+1})^{1-\frac{1}{p}} \lesssim 2^{\frac{nN}{p'}}.$$

As a result,

$$\begin{aligned}\sum_{m=0}^{\infty}\sum_{\vec{j}\in Q_{m+1}\setminus Q_m}|E_N(a)_{\vec{j}}||E_N(b)_{\vec{j}}| &\lesssim \frac{1}{2^{nN}}\times 2^{\frac{nN}{p'}}\times\left|\sum_{\vec{j}*\in Q_0}b_{\vec{j}*}\right|\|a\|_{\mathcal{M}^p_q(\mathbb{Z}^n)}\\ &=2^{-\frac{nN}{p}}\times\left|\sum_{\vec{j}*\in Q_0}b_{\vec{j}*}\right|\|a\|_{\mathcal{M}^p_q(\mathbb{Z}^n)}\\ &\to 0\quad(N\to\infty).\end{aligned}$$

This completes the estimate for the second term. □

3.1. Proof of the right inequality

It suffices to show that

$$(\sharp Q)^{\frac{1}{p}-\frac{1}{q}}\left(\sum_{\vec{j}\in Q}g(a)_{\vec{j}}^q\right)^{\frac{1}{q}}\le c_{p,q}\|a\|_{\mathcal{M}^p_q(\mathbb{Z}^n)}$$

for each $Q\in\mathcal{D}(\mathbb{Z}^n)$. To specify we let $Q\in\mathcal{D}_N(\mathbb{Z}^n)$.

We write $a=a_Q^+ + a_Q^- = \{(a_Q^+)_{\vec{j}}\}_{\vec{j}\in\mathbb{Z}^n}+\{(a_Q^-)_{\vec{j}}\}_{\vec{j}\in\mathbb{Z}^n}$, where

$$(a_Q^+)_{\vec{j}}=\chi_Q(\vec{j})a_{\vec{j}},\quad (a_Q^-)_{\vec{j}}=a_{\vec{j}}-(a_Q^+)_{\vec{j}}.$$

Matters are reduced to the proof of

$$(\sharp Q)^{\frac{1}{p}-\frac{1}{q}}\left(\sum_{\vec{j}\in Q}g(a_Q^+)_{\vec{j}}^q\right)^{\frac{1}{q}}+(\sharp Q)^{\frac{1}{p}-\frac{1}{q}}\left(\sum_{\vec{j}\in Q}g(a_Q^-)_{\vec{j}}^q\right)^{\frac{1}{q}}\le c_{p,q}\|a\|_{\mathcal{M}^p_q(\mathbb{Z}^n)}$$

for each $Q\in\mathcal{D}(\mathbb{Z}^n)$.

As for a_Q^+, we employ Proposition 1.2 to have

$$\begin{aligned}(\sharp Q)^{\frac{1}{p}-\frac{1}{q}}\left(\sum_{\vec{j}\in Q}g(a_Q^+)_{\vec{j}}^q\right)^{\frac{1}{q}} &\le (\sharp Q)^{\frac{1}{p}-\frac{1}{q}}\|g(a_Q^+)\|_{\ell^q(\mathbb{Z}^n)}\\ &\le c_q(\sharp Q)^{\frac{1}{p}-\frac{1}{q}}\|a_Q^+\|_{\ell^q(\mathbb{Z}^n)}\\ &= c_q(\sharp Q)^{\frac{1}{p}-\frac{1}{q}}\left(\sum_{\vec{j}\in Q}|a_{\vec{j}}|^q\right)^{\frac{1}{q}}.\end{aligned}$$

Thus, we are left with the task of dealing with a_Q^-.

It follows from the definition of $g(a_Q^-)$ that

$$g(a_Q^-)_{\vec{j}} = \left(\sum_{k=0}^{\infty} |D_k(a_Q^-)_{\vec{j}}|^2\right)^{\frac{1}{2}}.$$

Suppose $\vec{j} \in Q$. Then we have

$$g(a_Q^-)_{\vec{j}} \le \sum_{k=0}^{\infty} |D_k(a_Q^-)_{\vec{j}}| \le \sum_{k=0}^{\infty} (|E_k(a_Q^-)_{\vec{j}}| + |E_{k+1}(a_Q^-)_{\vec{j}}|) \le 2\sum_{k=0}^{\infty} |E_k(a_Q^-)_{\vec{j}}|$$

by the triangle inequality. Denote by Q_k the unique cube in $\mathcal{D}_k(\mathbb{Z}^n)$ that contains Q. A geometric observation shows that

$$E_k(a_Q^-)_{\vec{j}} = \begin{cases} 0 & \text{if } k \le N, \\ \frac{1}{2^{nk}} \sum\limits_{\vec{j}^* \in Q_k} (a_Q^-)_{\vec{j}^*} & \text{if } k < N. \end{cases}$$

If we insert this expression into the definition of $g(a_Q^-)$, then we obtain

$$\begin{aligned} g(a_Q^-)_{\vec{j}} \le 2\sum_{k=N+1}^{\infty} |E_k(a_Q^-)_{\vec{j}}| &= 2\sum_{k=N+1}^{\infty} \frac{1}{2^{nk}} \left|\sum_{\vec{j}^* \in Q_k} (a_Q^-)_{\vec{j}^*}\right| \\ &\le 2\sum_{k=N+1}^{\infty} \frac{1}{2^{nk}} \sum_{\vec{j}^* \in Q_k} |(a_Q^-)_{\vec{j}^*}| \\ &\le 2\sum_{k=N+1}^{\infty} \frac{1}{2^{nk}} \sum_{\vec{j}^* \in Q_k} |a_{\vec{j}^*}|. \end{aligned}$$

Consequently,

$$(\sharp Q)^{\frac{1}{p}-\frac{1}{q}} \left(\sum_{\vec{j} \in Q} g(a_Q^-)_{\vec{j}}^q\right)^{\frac{1}{q}} \le 2(\sharp Q)^{\frac{1}{p}} \sum_{k=N+1}^{\infty} \frac{1}{2^{nk}} \sum_{\vec{j}^* \in Q_k} |a_{\vec{j}^*}|.$$

Recall that $\sharp Q = 2^{nN}$ and that $\sharp Q_k = 2^{nk}$. Therefore,

$$(\sharp Q)^{\frac{1}{p}-\frac{1}{q}} \left(\sum_{\vec{j} \in Q} g(a_Q^-)_{\vec{j}}^q\right)^{\frac{1}{q}} \le 2^{1+\frac{nN}{p}} \sum_{k=N+1}^{\infty} (\sharp Q_k)^{-1} \sum_{\vec{j}^* \in Q_k} |a_{\vec{j}^*}|.$$

By the definition of the Morrey norm $\|a\|_{\mathcal{M}_1^p(\mathbb{Z}^n)}$ and embedding (2),

$$(\sharp Q)^{\frac{1}{p}-\frac{1}{q}}\left(\sum_{\vec{j}\in Q} g(a_Q^-)_{\vec{j}}^q\right)^{\frac{1}{q}} \le 2^{1+\frac{nN}{p}} \sum_{k=N+1}^{\infty} (2^{nk})^{-\frac{1}{p}} \|a\|_{\mathcal{M}_1^p(\mathbb{Z}^n)}$$
$$\lesssim \|a\|_{\mathcal{M}_q^p(\mathbb{Z}^n)}.$$

Thus, the proof is complete.

3.2. Proof of the left inequality

Let $R \in \mathcal{D}(\mathbb{Z}^n)$. It suffices to show that

$$(\sharp R)^{\frac{1}{p}-\frac{1}{q}}\left(\sum_{\vec{j}\in R} |a_{\vec{j}}|^q\right)^{\frac{1}{q}} \le c_{p,q}\|g(a)\|_{\mathcal{M}_q^p(\mathbb{Z}^n)}.$$

We linearize the left-hand side. By Hölder's inequality,

$$\left(\sum_{\vec{j}\in R} |a_{\vec{j}}|^q\right)^{\frac{1}{q}} = \sup\left\{\left|\sum_{\vec{j}\in R} a_{\vec{j}} b_{\vec{j}}\right| : b=\{b_{\vec{j}}\}_{\vec{j}\in R}, \left(\sum_{\vec{j}\in R} |b_{\vec{j}}|^{q'}\right)^{\frac{1}{q'}} \le 1\right\} \tag{3}$$

Extend b to an element in $\ell^{q'}(\mathbb{Z}^n)$ by letting $b_{\vec{j}}=0$ outside R. Then we have

$$\left|\sum_{\vec{j}\in R} a_{\vec{j}} b_{\vec{j}}\right| = |\langle a,b\rangle| = \left|\langle E_N(a),E_N(b)\rangle + \sum_{k=0}^{N-1}\langle D_k(a),D_k(b)\rangle\right|$$
$$\le |\langle E_N(a),E_N(b)\rangle| + \sum_{k=0}^{\infty}|\langle D_k(a),D_k(b)\rangle|$$

for all $N\in\mathbb{N}$. By using the Cauchy–Schwarz inequality twice, we have

$$\sum_{k=0}^{\infty}|\langle D_k(a),D_k(b)\rangle| \le \sum_{\vec{j}\in\mathbb{Z}^n}\sum_{k=0}^{\infty}|D_k(a)_{\vec{j}}||D_k(b)_{\vec{j}}|$$
$$\le \sum_{\vec{j}\in\mathbb{Z}^n}\sqrt{\sum_{k=0}^{\infty}|D_k(a)_{\vec{j}}|^2}\sqrt{\sum_{k=0}^{\infty}|D_k(b)_{\vec{j}}|^2}$$
$$= \sum_{\vec{j}\in\mathbb{Z}^n} g(a)_{\vec{j}}\, g(b)_{\vec{j}}.$$

Inserting this inequality into (3), we have

$$\left(\sum_{\vec{j}\in R}|a_{\vec{j}}|^q\right)^{\frac{1}{q}}$$
$$\leq \sup\left\{|\langle E_N(a),E_N(b)\rangle|+\sum_{\vec{j}\in\mathbb{Z}^n} g(a)_{\vec{j}}g(b)_{\vec{j}} : \operatorname{supp}(b)\subset R, \|b\|_{\ell^{q'}(\mathbb{Z}^n)}\leq 1\right\}$$

for all $N\in\mathbb{N}$. Fix $b\in\ell^{q'}(\mathbb{Z}^n)$ such that

$$\|b\|_{\ell^{q'}(\mathbb{Z}^n)}=1, \quad \operatorname{supp}(b)\subset R. \tag{4}$$

Recall that

$$\lim_{N\to\infty}\langle E_N(a),E_N(b)\rangle=0$$

according to Lemma 3.1. Thus, it remains to show

$$\sum_{\vec{j}\in\mathbb{Z}^n} g(a)_{\vec{j}}g(b)_{\vec{j}}\leq c_{p,q}\|g(a)\|_{\mathcal{M}^p_q(\mathbb{Z}^n)}$$

for all $b\in\ell^{q'}(\mathbb{Z}^n)$ supported in R with $\|b\|_{\ell^{q'}(\mathbb{Z}^n)}=1$. Let $\{Q_m\}_{m=0}^\infty$ be the same exhausting sequence of a quadrant S as in the proof of Lemma 3.1. In particular, we let $Q_0=R$. Then notice that $g(b)_{\vec{j}}=0$ outside S. Thus,

$$\sum_{\vec{j}\in\mathbb{Z}^n} g(a)_{\vec{j}}g(b)_{\vec{j}}=\sum_{\vec{j}\in Q_0} g(a)_{\vec{j}}g(b)_{\vec{j}}+\sum_{m=0}^\infty\sum_{\vec{j}\in Q_{m+1}\setminus Q_m} g(a)_{\vec{j}}g(b)_{\vec{j}}.$$

As for the first term, we employ Hölder's inequality and Proposition 1.2 to have

$$\begin{aligned}\sum_{\vec{j}\in Q_0} g(a)_{\vec{j}}g(b)_{\vec{j}} &\leq \|g(a)\|_{\ell^q(Q_0)}\|g(b)\|_{\ell^{q'}(Q_0)}\\ &\leq c_{q'}\|g(a)\|_{\ell^q(Q_0)}\|b\|_{\ell^{q'}(Q_0)}\\ &\leq c_{q'}\|g(a)\|_{\ell^q(Q_0)}\\ &= c_{q'}\left(\sum_{\vec{j}\in Q_0}|g(a)|^q\right)^{\frac{1}{q}}\\ &\leq c_{q'}(\sharp Q_0)^{\frac{1}{q}-\frac{1}{p}}\|g(a)\|_{\mathcal{M}^p_q(\mathbb{Z}^n)}.\end{aligned}$$

It remains to handle the second term. Fix $\vec{j}\in Q_{m+1}\setminus Q_m$ and consider

$$g(b)_{\vec{j}}=\left(\sum_{k=0}^\infty|D_k(b)_{\vec{j}}|^2\right)^{\frac{1}{2}}.$$

Then, since $\sharp Q_m = 2^{n(N+m)}$ and $\sharp Q_{m+1} = 2^{n(N+m+1)}$, we have

$$E_k(b)_{\vec{j}} = \begin{cases} 0 & \text{if } k \leq N+m, \\ \dfrac{1}{2^{nk}} \displaystyle\sum_{\vec{j}* \in Q_0} b_{\vec{j}*} & \text{if } k > N+m. \end{cases}$$

Inserting this expression into $D_k(b)_{\vec{j}}$, we obtain

$$\begin{aligned} D_k(b)_{\vec{j}} &= E_k(b)_{\vec{j}} - E_{k+1}(b)_{\vec{j}} \\ &= \begin{cases} 0 & \text{if } k < N+m, \\ -\dfrac{1}{2^{n(N+m+1)}} \displaystyle\sum_{\vec{j}* \in Q_0} b_{\vec{j}*} & \text{if } k = N+m, \\ \dfrac{1}{2^{nk}} \displaystyle\sum_{\vec{j}* \in Q_0} b_{\vec{j}*} - \dfrac{1}{2^{n(k+1)}} \displaystyle\sum_{\vec{j}* \in Q_0} b_{\vec{j}*} & \text{if } k > N+m. \end{cases} \end{aligned}$$

As a result,

$$\sum_{k=0}^{\infty} |D_k(b)_{\vec{j}}|^2 \sim \left(\sum_{\vec{j}* \in Q_0} b_{\vec{j}*} \right)^2 \sum_{k=N+m+1}^{\infty} \frac{1}{2^{2nk}} \sim \left(\sum_{\vec{j}* \in Q_0} b_{\vec{j}*} \right)^2 \frac{1}{2^{2n(N+m+1)}}.$$

Hence from (4), we conclude

$$g(b)_{\vec{j}} \lesssim \frac{1}{2^{n(N+m+1)}} \|b\|_{\ell^{q'}(\mathbb{Z}^n)} \lesssim \frac{1}{2^{n(N+m+1)}}.$$

If we insert $\sharp Q_{m+1} = 2^{n(N+m+1)}$ into the above estimate and use embedding (2), then we obtain

$$\begin{aligned} \sum_{m=0}^{\infty} \sum_{\vec{j} \in Q_{m+1} \setminus Q_m} g(a)_{\vec{j}} g(b)_{\vec{j}} &\lesssim \sum_{m=0}^{\infty} \frac{1}{2^{n(N+m+1)}} \sum_{\vec{j} \in Q_{m+1} \setminus Q_m} g(a)_{\vec{j}} \\ &\lesssim \|g(a)\|_{\mathcal{M}_1^p(\mathbb{Z}^n)} \sum_{m=0}^{\infty} \left(\frac{1}{2^{n(N+m+1)}} \right)^{\frac{1}{p}} \\ &\lesssim_{p,q} (\sharp Q_0)^{-\frac{1}{p}} \|g(a)\|_{\mathcal{M}_1^p(\mathbb{Z}^n)} \\ &\lesssim_{p,q} (\sharp Q_0)^{-\frac{1}{p}} \|g(a)\|_{\mathcal{M}_q^p(\mathbb{Z}^n)}. \end{aligned}$$

In total,

$$\left(\sum_{\vec{j} \in R} |a_{\vec{j}}|^q \right)^{\frac{1}{q}} \lesssim (\sharp Q_0)^{\frac{1}{q} - \frac{1}{p}} \|g(a)\|_{\mathcal{M}_q^p(\mathbb{Z}^n)} + (\sharp Q_0)^{-\frac{1}{p}} \|g(a)\|_{\mathcal{M}_q^p(\mathbb{Z}^n)}.$$

Multiply both sides by $(\sharp Q_0)^{\frac{1}{p}-\frac{1}{q}}$ and use the norm $\|g(a)\|_{\mathcal{M}^p_q(\mathbb{Z}^n)}$ to have

$$(\sharp Q_0)^{\frac{1}{p}-\frac{1}{q}} \left(\sum_{\vec{j} \in Q_0} |a_{\vec{j}}|^q \right)^{\frac{1}{q}} \lesssim_{p,q} \|g(a)\|_{\mathcal{M}^p_q(\mathbb{Z}^n)}.$$

The cube $R = Q_0$ being arbitrary, we obtain the desired result.

4. Applications to martingale transforms

We apply Theorem 1.3 to martingale transforms. For $N \in \mathbb{N}$, $b \in \mathcal{M}^p_q(\mathbb{Z}^n)$ and multi-indexed sequences $m^0, m^1, \ldots$ satisfying $|E_{k+1}(m^k)| \le K$ for each $k \in \mathbb{N}_0$, we define the martingale transform $M_m(b)$ of a multi-indexed sequence b by

$$M_m(b) = \sum_{k=0}^{\infty} E_{k+1}(m^k) D_k(b).$$

If $m^k = 0$ for $k \gg 1$, then we call $M_m(b)$ a finite martingale transform. Thus, a finite martingale transform takes the form

$$M_{m,(N)}(b) = \sum_{k=0}^{N} E_{k+1}(m^k) D_k(b).$$

We consider finite martingale transforms in Section 4.1. Based on the observations in Section 4.1, we move on to the general case in Section 4.2.

4.1. Finite martingale transform

For $N \in \mathbb{N}$, $b \in \mathcal{M}^p_q(\mathbb{Z}^n)$ and multi-indexed sequences $m^0, m^1, \ldots$ satisfying

$$|E_{k+1}(m^k)| \le K \tag{5}$$

for each $k \in \mathbb{N}_0$, we deal with the finite martingale transform $M_{m,(N)}(b)$ of a multi-indexed sequence b by

$$M_{m,(N)}(b) = \sum_{k=0}^{N} E_{k+1}(m^k) D_k(b).$$

Note that $M_{m,(N)}(b) \in \mathcal{M}^p_q(\mathbb{Z}^n)$ whenever $b \in \mathcal{M}^p_q(\mathbb{Z}^n)$. In fact,

$$\begin{aligned} &\sup_{Q \in \mathcal{D}(\mathbb{Z}^n)} (\sharp Q)^{\frac{1}{p}-\frac{1}{q}} \left(\sum_{\vec{j} \in Q} |M_{m,(N)}(b)_{\vec{j}}|^q \right)^{\frac{1}{q}} \\ &\le (N+1) \sup_{Q \in \mathcal{D}(\mathbb{Z}^n)} (\sharp Q)^{\frac{1}{p}-\frac{1}{q}} \left(\sum_{\vec{j} \in Q} \max_k |E_{k+1}(m^k)_{\vec{j}} D_k(b)_{\vec{j}}|^q \right)^{\frac{1}{q}}. \end{aligned}$$

From Proposition 2.1 and (5), we have

$$\begin{aligned}
&\sup_{Q\in\mathcal{D}(\mathbb{Z}^n)}(\sharp Q)^{\frac{1}{p}-\frac{1}{q}}\left(\sum_{\vec{j}\in Q}|M_{m,(N)}(b)_{\vec{j}}|^q\right)^{\frac{1}{q}}\\
&\le K(N+1)\sup_{Q\in\mathcal{D}(\mathbb{Z}^n)}(\sharp Q)^{\frac{1}{p}-\frac{1}{q}}\left(\sum_{\vec{j}\in Q}\max_k|D_k(b)_{\vec{j}}|^q\right)^{\frac{1}{q}}\\
&\le K(N+1)\sup_{Q\in\mathcal{D}(\mathbb{Z}^n)}(\sharp Q)^{\frac{1}{p}-\frac{1}{q}}\left(\sum_{\vec{j}\in Q}(M_{\rm dyadic}b)_{\vec{j}}^q\right)^{\frac{1}{q}}\\
&\lesssim K(N+1)\|b\|_{\mathcal{M}^p_q(\mathbb{Z}^n)}\\
&<\infty.
\end{aligned}$$

Hence, the linear functional $L_{M_{m,(N)}(b)}:\mathcal{H}^{p'}_{q'}(\mathbb{Z}^n)\to\mathbb{C}$, given by

$$L_{M_{m,(N)}(b)}(a)=\langle M_{m,(N)}(b),a\rangle \quad (a\in\mathcal{H}^{p'}_{q'}(\mathbb{Z}^n)),$$

is bounded. In Section 4.2, we will show that

$$\lim_{N\to\infty}\langle M_{m,(N)}(b),a\rangle$$

exists for all $a\in\mathcal{H}^{p'}_{q'}(\mathbb{Z}^n)$. Once this is achieved, we can say that there exists an element $M_m(b)\in\mathcal{M}^p_q(\mathbb{Z}^n)$ such that

$$M_{m,(N)}(b)\to M_m(b) \qquad (N\to\infty)$$

in the weak-* topology. By considering the coupling of this equality with $\mathbf{e}_{\vec{j}}$, we learn that

$$M_{m,(N)}(b)_{\vec{j}}\to M_m(b)_{\vec{j}} \qquad (N\to\infty)$$

for each $\vec{j}\in\mathbb{Z}^n$.

We concentrate on the proof of Theorem 1.6 for finite martingale transforms. In this case, there is no need to consider the convergence defining the finite martingale transform.

Theorem 1.6 for finite martingale transforms. Let $b\in\mathcal{M}^p_q(\mathbb{Z}^n)$. As we have remarked above, $M_{m,(N)}(b)\in\mathcal{M}^p_q(\mathbb{Z}^n)$. Thus, from Theorem 1.3, we deduce

$$\|g(M_{m,(N)}(b))\|_{\mathcal{M}^p_q(\mathbb{Z}^n)}\gtrsim\|M_{m,(N)}(b)\|_{\mathcal{M}^p_q(\mathbb{Z}^n)}.$$

Thus, it suffices to show that

$$(\sharp Q)^{\frac{1}{p}-\frac{1}{q}}\left(\sum_{\vec{j}\in Q}|g(M_{m,(N)}(b)_{\vec{j}})|^q\right)^{\frac{1}{q}} \lesssim_{p,q} \|b\|_{\mathcal{M}^p_q(\mathbb{Z}^n)}$$

for all $Q\in\mathcal{D}(\mathbb{Z}^n)$. It follows from the definition of $g(b)$ that

$$g(M_{m,(N)}(b))_{\vec{j}}=\left(\sum_{k=0}^{\infty}|D_k(M_{m,(N)}(b))_{\vec{j}}|^2\right)^{\frac{1}{2}}=\left(\sum_{k=0}^{\infty}|E_{k+1}(m^k)_{\vec{j}}D_k(b)_{\vec{j}}|^2\right)^{\frac{1}{2}}.$$

Thus,

$$\sum_{\vec{j}\in Q}|g(M_{m,(N)}(b)_{\vec{j}})|^q=\sum_{\vec{j}\in Q}\left(\sum_{k=0}^{\infty}|E_{k+1}(m^k)_{\vec{j}}D_k(b)_{\vec{j}}|^2\right)^{\frac{q}{2}}.$$

Recall that we are assuming

$$|E_{k+1}(m^k)_{\vec{j}}|\le\|E_{k+1}(m^k)\|_{\ell^\infty(\mathbb{Z}^n)}\le K<\infty$$

for each $\vec{j}\in\mathbb{Z}^n$ and $k=0,1,2,\ldots$. Thus,

$$\begin{aligned}\sum_{\vec{j}\in Q}|g(M_{m,(N)}(b)_{\vec{j}})|^q&\le\sum_{\vec{j}\in Q}\left(\sum_{k=0}^{\infty}\|E_{k+1}(m^k)\|^2_{\ell^\infty(\mathbb{Z}^n)}|D_k(b)_{\vec{j}}|^2\right)^{\frac{q}{2}}\\&\le\sup_k\|E_{k+1}(m^k)\|_{\ell^\infty(\mathbb{Z}^n)}{}^q\sum_{\vec{j}\in Q}\left(\sum_{k=0}^{\infty}|D_k(b)_{\vec{j}}|^2\right)^{\frac{q}{2}}.\end{aligned}$$

Once again from the definition of $g(b)$, we have

$$\left(\sum_{k=0}^{\infty}|D_k(b)_{\vec{j}}|^2\right)^{\frac{q}{2}}=g(b)_{\vec{j}}{}^q.$$

If we insert this expression into the above inequality, then we obtain

$$\sum_{\vec{j}\in Q}|g(M_{m,(N)}(b)_{\vec{j}})|^q\lesssim_q K^q\sum_{\vec{j}\in Q}|g(b)_{\vec{j}}|^q.$$

Hence,

$$\begin{aligned}(\sharp Q)^{\frac{1}{p}-\frac{1}{q}}\left(\sum_{\vec{j}\in Q}|g(M_{m,(N)}(b)_{\vec{j}})|^q\right)^{\frac{1}{q}}&\lesssim_q K(\sharp Q)^{\frac{1}{p}-\frac{1}{q}}\left(\sum_{\vec{j}\in Q}|g(b)_{\vec{j}}|^q\right)^{\frac{1}{q}}\\&\lesssim_q K\|g(b)\|_{\mathcal{M}^p_q(\mathbb{Z}^n)}.\end{aligned}$$

Once again from Theorem 1.3, we have

$$(\sharp Q)^{\frac{1}{p}-\frac{1}{q}}\left(\sum_{\vec{j}\in Q}|g(M_{m,(N)}(b)_{\vec{j}})|^q\right)^{\frac{1}{q}} \lesssim_{p,q} K\|g(b)\|_{\mathcal{M}^p_q(\mathbb{Z}^n)}$$
$$\lesssim_{p,q} K\|b\|_{\mathcal{M}^p_q(\mathbb{Z}^n)}.$$

This proves Theorem 1.6 for finite martingale transforms. □

4.2. Proof of Theorem 1.6–General case

We will establish that the limit $\langle M_{m,(N)}(b),a\rangle$ as $N\to\infty$ exists for all $b\in\mathcal{M}^p_q(\mathbb{Z}^n)$ and $a\in\mathcal{H}^{p'}_{q'}(\mathbb{Z}^n)$. This amounts to showing that $\{\langle M_{m,(N)}(b),a\rangle\}_{N=1}^{\infty}$ is a Cauchy sequence.

Let us start with the case where a is a (p',q')-block centered at Q. Let $n_1,n_2\in\mathbb{N}$ satisfy $n_1>n_2$. Suppose $\sharp Q=2^{nN}$. By linearity, we have

$$\langle M_{m,(n_1)}(b),a\rangle-\langle M_{m,(n_2)}(b),a\rangle=\langle M_{m,(n_1)}(b)-M_{m,(n_2)}(b),a\rangle.$$

By the Cauchy–Schwarz inequality, we have

$$\begin{aligned}&|\langle M_{m,(n_1)}(b)-M_{m,(n_2)}(b),a\rangle|\\
&\le\sum_{k=0}^{\infty}|\langle D_k(M_{m,(n_1)}(b)-M_{m,(n_2)}(b)),D_k(a)\rangle|\\
&\le\sum_{\vec{j}\in\mathbb{Z}^n}\sum_{k=n_2}^{n_1}|D_k(M_{m,(n_1)}(b)-M_{m,(n_2)}(b))_{\vec{j}}||D_k(a)_{\vec{j}}|.\end{aligned}$$

By the Cauchy–Schwarz inequality and Proposition 1.5, we have

$$\begin{aligned}&|\langle M_{m,(n_1)}(b)-M_{m,(n_2)}(b),a\rangle|\\
&\le\sum_{\vec{j}\in\mathbb{Z}^n}\sqrt{\sum_{k=n_2}^{n_1}|D_k(M_{m,(n_1)}(b)-M_{m,(n_2)}(b))_{\vec{j}}|^2}\sqrt{\sum_{k=n_2}^{n_1}|D_k(a)_{\vec{j}}|^2}\\
&\le\left\|\sqrt{\sum_{k=n_2}^{n_1}|D_k(M_{m,(n_1)}(b)-M_{m,(n_2)}(b))|^2}\right\|_{\mathcal{M}^p_q(\mathbb{Z}^n)}\left\|\sqrt{\sum_{k=n_2}^{n_1}|D_k(a)|^2}\right\|_{\mathcal{H}^{p'}_{q'}(\mathbb{Z}^n)}.\end{aligned}$$

Since $|E_{k+1}(m^k)|\le K$, thanks to what we did for finite martingale transforms,

$$\left\|\sqrt{\sum_{k=n_2}^{n_1}|D_k(M_{m,(n_1)}(b)-M_{m,(n_2)}(b))|^2}\right\|_{\mathcal{M}^p_q(\mathbb{Z}^n)}\lesssim K\|b\|_{\mathcal{M}^p_q(\mathbb{Z}^n)}.$$

Let $\vec{j} \in \mathbb{Z}^n$. We decompose

$$\sqrt{\sum_{k=n_2}^{n_1} |D_k(a)_{\vec{j}}|^2} = \chi_Q(\vec{j})\sqrt{\sum_{k=n_2}^{n_1} |D_k(a)_{\vec{j}}|^2} + \chi_{\mathbb{Z}^n \setminus Q}(\vec{j})\sqrt{\sum_{k=n_2}^{n_1} |D_k(a)_{\vec{j}}|^2}.$$

As for the first term, we have

$$\left\| \chi_Q \sqrt{\sum_{k=n_2}^{n_1} |D_k(a)|^2} \right\|_{\mathcal{H}^{p'}_{q'}(\mathbb{Z}^n)} \leq (\sharp Q)^{\frac{1}{q}-\frac{1}{p}} \left\| \chi_Q \sqrt{\sum_{k=n_2}^{n_1} |D_k(a)|^2} \right\|_{\ell^{q'}(\mathbb{Z}^n)}$$

thanks to Proposition 2.4. Due to Proposition 1.2, we have

$$\left\| \chi_Q \sqrt{\sum_{k=n_2}^{n_1} |D_k(a)|^2} \right\|_{\ell^{q'}(\mathbb{Z}^n)} \leq \|g(a)\|_{\ell^{q'}(\mathbb{Z}^n)} \lesssim \|a\|_{\ell^{q'}(\mathbb{Z}^n)}$$

whenever integers n_1 and n_2 satisfy $n_1 > n_2 \geq 1$. By the dominated convergence theorem, we have

$$\lim_{n_1,n_2\to\infty} \left\| \chi_Q \sqrt{\sum_{k=n_2}^{n_1} |D_k(a)|^2} \right\|_{\ell^{q'}(\mathbb{Z}^n)} = \left\| \lim_{n_1,n_2\to\infty} \chi_Q \sqrt{\sum_{k=n_2}^{n_1} |D_k(a)|^2} \right\|_{\ell^{q'}(\mathbb{Z}^n)} = 0.$$

We move on to the second term. Let $\vec{j} \notin Q$. Then for each $k \in \mathbb{N}_0$,

$$E_k(a)_{\vec{j}} = \begin{cases} \dfrac{\chi_{Q_k}(\vec{j})}{2^{nk}} \displaystyle\sum_{\vec{j}*\in Q} a_{\vec{j}*} & \text{if } k > N \text{ and } Q \subset Q_k, \\ 0 & \text{otherwise,} \end{cases}$$

where $\sharp Q_k = 2^{nk}$. Furthermore, since

$$\sqrt{\sum_{k=n_2}^{n_1} |D_k(a)_{\vec{j}}|^2} \leq \sum_{k=n_2}^{n_1} |D_k(a)_{\vec{j}}| = \sum_{k=n_2}^{n_1} |E_k(a)_{\vec{j}} - E_{k+1}(a)_{\vec{j}}| \leq 2\sum_{k=n_2}^{n_1+1} |E_k(a)_{\vec{j}}|,$$

if $n_2 > N$, then we have

$$\begin{aligned} \left\| \left\{ \sqrt{\sum_{k=n_2}^{n_1} |D_k(a)_{\vec{j}}|^2} \right\}_{\vec{j}\in\mathbb{Z}^n} \right\|_{\mathcal{H}^{p'}_{q'}(\mathbb{Z}^n)} &\leq \left\| \sum_{k=n_2}^{n_1} 2 \left| \chi_{Q_k} \frac{1}{2^{nk}} \sum_{\vec{j}*\in Q} a_{\vec{j}*} \right| \right\|_{\mathcal{H}^{p'}_{q'}(\mathbb{Z}^n)} \\ &\lesssim \sum_{k=n_2}^{n_1} \left\| \chi_{Q_k} \frac{1}{2^{nk}} \sum_{\vec{j}*\in Q} a_{\vec{j}*} \right\|_{\mathcal{H}^{p'}_{q'}(\mathbb{Z}^n)} \\ &\lesssim \sum_{k=n_2}^{n_1} \frac{(\sharp Q_k)^{\frac{1}{q}-\frac{1}{p}}}{2^{nk}} \left\| \left\{ \sum_{\vec{j}*\in Q} a_{\vec{j}*} \right\}_{\vec{j}\in\mathbb{Z}^n} \right\|_{\ell^{q'}(Q_k)}. \end{aligned}$$

Since $\sharp Q_k = 2^{nk}$,

$$\left\| \left\{ \frac{1}{2^{nk}} \sum_{\vec{j}*\in Q} a_{\vec{j}*} \right\}_{\vec{j}\in\mathbb{Z}^n} \right\|_{\ell^{q'}(Q_k)} = \left(\sum_{\vec{j}\in Q_k} \left| \frac{1}{2^{nk}} \sum_{\vec{j}*\in Q} a_{\vec{j}*} \right|^{q'} \right)^{\frac{1}{q'}}$$

$$= (\sharp Q_k)^{\frac{1}{q'}-1} \left| \sum_{\vec{j}*\in Q} a_{\vec{j}*} \right|.$$

If we insert this equality into the above estimate, then we have

$$\left\| \left\{ \sqrt{\sum_{k=n_2}^{n_1} |D_k(a)_{\vec{j}}|^2} \right\}_{\vec{j}\in\mathbb{Z}^n} \right\|_{\mathcal{H}^{p'}_{q'}(\mathbb{Z}^n)} \lesssim \left| \sum_{\vec{j}*\in Q} a_{\vec{j}*} \right| \sum_{k=n_2}^{n_1} (\sharp Q_k)^{-\frac{1}{p}}$$

$$\lesssim \left| \sum_{\vec{j}*\in Q} a_{\vec{j}*} \right| \sum_{k=n_2}^{\infty} (\sharp Q_k)^{-\frac{1}{p}}$$

$$= \left| \sum_{\vec{j}*\in Q} a_{\vec{j}*} \right| \frac{2^{-\frac{n_2 k}{p}}}{1-2^{-\frac{n}{p}}}.$$

Since $p, n < \infty$, the last term vanishes as $n_2 \to \infty$. This implies that the limit defining $\langle M_{m,(N)}(b), a\rangle$ exists as long as a is a (p', q')-block.

Next, we remove the assumption that a is a (p', q')-block. Let $a \in \mathcal{H}^{p'}_{q'}(\mathbb{Z}^n)$. Then there exist $\lambda = \{\lambda^{(j)}\}_{j=1}^{\infty} \in \ell^1(\mathbb{N})$ and a collection $\{a^{(j)}\}_{j=1}^{\infty}$ of (p', q')-blocks such that $a = \sum\limits_{j=1}^{\infty} \lambda^{(j)} a^{(j)}$. From this expression of a, we deduce

$$\langle M_{m,(N)}(b), a\rangle = \left\langle M_{m,(N)}(b), \sum_{j=1}^{\infty} \lambda^{(j)} a^{(j)} \right\rangle = \sum_{j=1}^{\infty} \lambda^{(j)} \langle M_{m,(N)}(b), a^{(j)}\rangle.$$

As we have established, the limit of $\langle M_{m,(N)}(b), a^{(j)}\rangle$ as $N \to \infty$ exists for each j. Meanwhile, since $M_{m,(N)}(b) \in \mathcal{M}^p_q(\mathbb{Z}^n)$,

$$|\langle M_{m,(N)}(b), a^{(j)}\rangle| \le \|M_{m,(N)}(b)\|_{\mathcal{M}^p_q(\mathbb{Z}^n)} \|a^{(j)}\|_{\mathcal{H}^{p'}_{q'}(\mathbb{Z}^n)} \lesssim K\|b\|_{\mathcal{M}^p_q(\mathbb{Z}^n)}$$

by what we proved in Section 4.1. By the dominated convergence theorem, we conclude

$$\lim_{N\to\infty} \sum_{j=1}^{\infty} \lambda^{(j)} \langle M_{m,(N)}(b), a^{(j)}\rangle = \sum_{j=1}^{\infty} \lim_{N\to\infty} \lambda^{(j)} \langle M_{m,(N)}(b), a^{(j)}\rangle.$$

In particular, the limit $\lim_{N\to\infty}\langle M_{m,(N)}(b), a^{(j)}\rangle$ exists.

We end this section with the remark that finitely supported multi-indexed sequences do not form a dense subspace in $\mathcal{M}^p_q(\mathbb{Z}^n)$; if we let X be the set of all finitely supported sequences, then $\overline{X} \subsetneq \mathcal{M}^p_q(\mathbb{Z}^n)$. This means that we are not allowed to use the "so-called" density argument.

Remark 4.1. Let $n=1$. Define $a=\{\chi_{\mathbb{Z}}(\log_2|j|)\}_{j\in\mathbb{Z}}$, where it is understood that $\log_2 0=-\infty$ and hence $\chi_{\mathbb{Z}}(\log_2|0|)=0$. Notice that any cube $Q\in\mathcal{D}_k$ can contain at most k points in the support of a: $\sharp(Q\cap\operatorname{supp}(a))\le k$. If we take $Q=\mathbb{Z}\cap[0,2^k)$, then $\sharp(Q\cap\operatorname{supp}(a))=k$. Observe also that

$$\|a\|_{\mathcal{M}^p_q(\mathbb{Z}^n)}=\sup_{Q\in\mathcal{D}(\mathbb{Z}^n)}(\sharp Q)^{\frac{1}{p}-\frac{1}{q}}(\sharp(Q\cap\operatorname{supp}(a)))^{\frac{1}{q}}=\sup_{k\in\mathbb{N}_0}2^{\frac{k}{p}-\frac{k}{q}}k^{\frac{1}{q}}<\infty.$$

Therefore, $a\in\mathcal{M}^p_q(\mathbb{Z}^n)$ whenever $1\le q<p<\infty$. However, since

$$\|a-b\|_{\mathcal{M}^p_q(\mathbb{Z}^n)}\ge 1$$

for any $b\in X$, a is not in the closure of the space of finitely supported multi-indexed sequences.

5. Appendix–dyadic Riesz potential

For $a\in\mathcal{M}^p_q(\mathbb{Z}^n)$, we set

$$R_\alpha a=\sum_{N=0}^{\infty}2^{N\alpha}D_N a.$$

For the time being, let $0<\alpha<n$. The next lemma contains a flavor of the original observation by Morrey. This observation allows us to conclude that the function f is Hölder continuous if f has a derivative in some classical Morrey spaces.

Lemma 5.1. *For all $a\in\mathcal{M}^p_q(\mathbb{Z}^n)$ $\|D_N a\|_{\ell^\infty(\mathbb{Z}^n)}\lesssim 2^{-\frac{nN}{p}}\|a\|_{\mathcal{M}^p_q(\mathbb{Z}^n)}$.*

Proof. Observe that $D_N a$ is constant on each $Q\in\mathcal{D}_N(\mathbb{Z}^n)$. Hence

$$\|D_N a\|_{\ell^\infty(\mathbb{Z}^n)}=\sup_{Q\in\mathcal{D}_N}(\sharp Q)^{-\frac{1}{q}}\|D_N a\|_{\ell^q(Q)}.$$

Let $Q \in \mathcal{D}_N(\mathbb{Z}^n)$, or equivalently $\sharp Q = 2^{nN}$. It follows from the definition of $M_{\text{dyadic}}a$ and Proposition 2.1 that

$$\begin{aligned}\sup_{Q\in\mathcal{D}_N} (\sharp Q)^{-\frac{1}{q}} \|D_N a\|_{\ell^q(Q)} &\le \sup_{Q\in\mathcal{D}_N} (\sharp Q)^{-\frac{1}{q}} 2\|M_{\text{dyadic}}a\|_{\ell^q(Q)} \\ &\lesssim \sup_{Q\in\mathcal{D}_N} (\sharp Q)^{-\frac{1}{p}} (\sharp Q)^{\frac{1}{p}-\frac{1}{q}} \|M_{\text{dyadic}}a\|_{\ell^q(Q)} \\ &\le \sup_{Q\in\mathcal{D}_N} (\sharp Q)^{-\frac{1}{p}} \|M_{\text{dyadic}}a\|_{\mathcal{M}^p_q(\mathbb{Z}^n)} \\ &\le c_q 2^{-\frac{nN}{p}} \|a\|_{\mathcal{M}^p_q(\mathbb{Z}^n)}.\end{aligned}$$

Putting together these observations, we obtain the desired result. □

A direct consequence of Lemma 5.1 is that

$$|(R_\alpha a)_{\vec{j}}| \le \sum_{N=0}^{\infty} 2^{N\alpha} |(D_N a)_{\vec{j}}| \lesssim \sum_{N=0}^{\infty} \min\left(2^{N\alpha} \sup_{k\in\mathbb{N}} |D_k(a)_{\vec{j}}|, 2^{N\alpha-\frac{nN}{p}} \|a\|_{\mathcal{M}^p_q(\mathbb{Z}^n)}\right).$$

If

$$0 < \alpha < \frac{n}{p},$$

then

$$|(R_\alpha a)_{\vec{j}}| \le \sum_{N=0}^{\infty} 2^{N\alpha} |D_N a| \le K \|a\|_{\mathcal{M}^p_q(\mathbb{Z}^n)}^{\frac{p\alpha}{n}} \sup_{k\in\mathbb{N}} |D_k(a)_{\vec{j}}|^{1-\frac{p\alpha}{n}}$$

for some positive constant $K > 0$. As a result, by taking the $\mathcal{M}^s_t(\mathbb{Z}^n)$-norm, we obtain the following theorem, which corresponds to the discrete version of a result in [2, 11**?**]:

Theorem 5.2. *Let* $1 < q \le p < \infty$ *and* $1 < t \le s < \infty$ *satisfy* $\frac{1}{p} - \frac{\alpha}{n} = \frac{1}{s}$ *and* $\frac{t}{s} = \frac{q}{p}$. *Then*

$$\left\| \sum_{N=0}^{\infty} 2^{N\alpha} |D_N a| \right\|_{\mathcal{M}^s_t(\mathbb{Z}^n)} \lesssim \|a\|_{\mathcal{M}^p_q(\mathbb{Z}^n)}$$

for all $a \in \mathcal{M}^p_q(\mathbb{Z}^n)$. *In particular,*

$$\|R_\alpha a\|_{\mathcal{M}^s_t(\mathbb{Z}^n)} \lesssim \|a\|_{\mathcal{M}^p_q(\mathbb{Z}^n)}$$

for all $a \in \mathcal{M}^p_q(\mathbb{Z}^n)$.

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