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STRUCTURE OF UNITAL Q-FRÉCHET ALGEBRAS A SATISFYING: $Ax^2 = Ax$, FOR EVERY $x \in A$

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We show that a unital Q-Fréchet algebra A satisfying $Ax^2 = Ax$, for every $x \in A$, is isomorphic to $\mathbb{C}^n$, $n \in \mathbb{N}^*$.

1. Introduction

We consider algebras A satisfying the condition:

$$Ax^2 = Ax, \text{ for every } x \in A. \qquad (P_1)$$

J. Duncan and A. W. Tullo showed, in ([6], Theorem 1, p. 45), that if A is a unital Banach algebra satisfying (P_1), then A is semi-simple commutative and is of finite-dimension. This type of algebras was studied later by O. H. Cheikh, A. EL Kinani and M. Oudadess in [2]. In particular, they showed that if A is an algebra satisfying (P_1), then A is semi-simple ([2], Proposition 3.1, ii), p. 386). If moreover, A is a m-convex algebra with left (or right) approximate identity satisfying (P_1), then A is commutative ([2], Proposition 3.6, 2), p. 388). In [4], R. Choukri; A. El Kinani considered the algebras A satisfying the condition:

$$xAx = Ax, \text{ for every } x \in A. \qquad (P_2)$$

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Note that if A is commutative, then conditions (P_1) and (P_2) are equivalent. R. Choukri and A. El Kinani showed, in ([4], Théorème 2.1, p. 58), that if A is a unital Q-Fréchet algebra satisfying (P_2), then A is algebraically and topologically isomorphic to a finite product of F-algebras which are fields. The aim of this note is to prove, with a different approach from that given in [4], that if A is a unital Q-Fréchet algebra satisfying (P_1), then A is isomorphic to $\mathbb{C}^n$, $n \in \mathbb{N}^*$. It is worth pointing out that our proof is similar in spirit to the one given in ([6], Theorem 1, p. 45) by J. Duncan and A. W. Tullo, in the case of unital Banach algebras.

2. Definitions and preliminaries

Let (A, τ) be a complex algebra endowed with a locally convex topology given by a family $(|.|_\lambda)_{\lambda \in \Lambda}$ of semi-norms. It is said to be locally m-convex algebra ($l.m.c.a.$ in short) if:

$$|xy|_\lambda \leq |x|_\lambda \, |y|_\lambda \,, \ \forall x, y \in A, \ \forall \lambda \in \Lambda.$$

An algebra A is said to be F-algebra if A is an F-space. An F-algebra is called Fréchet algebra if it is m-convex. A $l.m.c.a.$ with unit is said to be Q-algebra if the group $G(A)$ of its invertible elements is open. Let A be an algebra with unit e. The spectrum of an element x of A, denoted by $Sp_A(x)$, is defined by:

$$Sp_A(x) = \{\lambda \in \mathbb{C} : x - \lambda e \notin G(A)\}.$$

The spectral radius $\rho_A(x)$ of x is given by:

$$\rho_A(x) = \sup\{|\lambda| : \lambda \in Sp_A(x)\}.$$

The Jacobson radical of an algebra A with unit, denoted by $Rad(A)$, is the intersection of all left maximal ideals of A. If $Rad(A) = \{0\}$, we say that A is semi-simple. An element x of an algebra A is said to be idempotent if $x^2 = x$. Two elements, x and y of an algebra is said to be orthogonal if $xy = yx = 0$. An unital algebra A is said to be division algebra if every non-zero element is invertible in A.

The main result of this note is the following:

Theorem 2.1. *Let A be a Q-Fréchet algebra with unit e such that:*

$$Ax^2 = Ax, \textit{ for every } x \in A,$$

then A is isomorphic to $\mathbb{C}^n$, $n \in \mathbb{N}^*$.

For the proof, we need the following lemma:

Lemma 2.2. *Let $(A,(p_k)_k)$ be a unital Fréchet algebra. If A has an infinite sequence of non zero mutually orthogonal idempotents, then there exists an infinite sequence of complex numbers (λ_n), $\lambda_n > 0$, such that the series $\sum_{n\geq 1} \lambda_n h_n$ converges in $(A,(p_k)_k)$.*

Proof. Let $(h_n)_n$ be an infinite sequence of non zero mutually orthogonal idempotents of A. We are going to construct an infinite sequence of complex numbers (λ_n), $\lambda_n > 0$, such that the series $\sum_{n\geq 1} \lambda_n h_n$ converges in $(A,(p_k)_k)$. For $n = 1$, there exists k_1 such that $p_{k_1}(h_1) \neq 0$. We take $\lambda_1 = \frac{1}{p_{k_1}(h_1)}$. For $n = 2$, there exists $k_2 > k_1$ such that $p_{k_2}(h_2) \neq 0$ and there exists $m_2 > 1$ such that $\frac{1}{2^{m_2} p_{k_2}(h_2)} < \lambda_1$. We take $\lambda_2 = \frac{1}{2^{m_2} p_{k_2}(h_2)}$. So, by induction, we build two strictly increasing sequences $(k_n)_{n\geq 1}$ and $(m_n)_{n\geq 1}$ such that:
(i) $p_{k_n}(h_n) \neq 0$, for every n,
(ii) the sequence $(\lambda_n)_n$, where:

$$\lambda_n = \frac{1}{n^{m_n} p_{k_n}(h_n)}, \text{ for every } n \geq 1, \text{ is strictly decreasing.}$$

Let us now show that the series $\sum_n \lambda_n h_n$ is absolutely convergent in A. Take $N \geq 1$. Then:

$$\sum_{n\geq 1} p_N(\lambda_n h_n) = \sum_{1\leq n\leq N-1} |\lambda_n|\, p_N(h_n) + \sum_{n\geq N} |\lambda_n|\, p_N(h_n).$$

As the sequence $(p_n)_n$ is increasing, one has:

$$\sum_{n\geq N} |\lambda_n|\, p_N(h_n) \leq \sum_{n\geq N} |\lambda_n|\, p_n(h_n).$$

Using again the fact that the sequence $(p_n)_n$ is increasing and that $k_n \geq n$, we obtain:

$$\sum_{n\geq N} |\lambda_n|\, p_n(h_n) \leq \sum_{n\geq N} |\lambda_n|\, p_{k_n}(h_n).$$

Using (ii), one has:

$$\sum_{n\geq N} |\lambda_n|\, p_{k_n}(h_n) \leq \sum_{n\geq N} \frac{1}{n^{m_n}} < \infty.$$

It follows that the series $\sum_n \lambda_n h_n$ is convergent in $(A,(p_k)_k)$. Consequently, $a = \sum_{n=0}^{+\infty} \lambda_n h_n$ belongs to A. □

Proof of theorem 2.1. The following proof go along the lines of [6] using previous lemma. Note first that A is semi-simple ([2], Proposition 3.1, ii), p. 386), and A is commutative ([2], Proposition 3.6, 2), p. 388). We will now show that A is finite-dimensional. Suppose that A has no proper idempotents. Given $x \in A$, $x \neq 0$, there exists $y \in A$ such that $yx^2 = x$. One has $(yx)^2 = yyx^2 = yx$, then yx is idempotent. but $yx \neq 0$, so $yx = e$. Thus x is invertible in A. Therefore A is a division algebra. It follows from ([8], Proposition 2.9, b), p. 13), that A is isomorphic to $\mathbb{C}$. The algebra A cannot contain an infinite sequence of pairwise orthogonal idempotents. Indeed, suppose $(h_n)_n$ is such a sequence. Using lemma 2.2, there exists an infinite sequence of complex numbers (λ_n), $\lambda_n > 0$, such that the series $\sum\limits_{n\geq 1} \lambda_n h_n$ converges in $(A,(p_k)_k)$. Put $x = \sum\limits_{n=1}^{+\infty} \lambda_n h_n$, $x \in A$. The algebra A is a Q-algebra, then by (**[7]**, Theorem 6.18, p. 85), for all $n \in \mathbb{N}^*$, $0 < \lambda_n \leq \lambda_n \rho_A(h_n) \leq \lambda_n p_N(h_n)$, $N \in \mathbb{N}^*$. Then the sequence (λ_n) converges to 0. Let $y \in A$ such that $yx^2 = x$. Let $n \in \mathbb{N}^*$, one has $xh_n = (\sum\limits_{k=1}^{+\infty} \lambda_k h_k)h_n = \lambda_n h_n$, which give $\lambda_n h_n = xh_n = yx^2 h_n = \lambda_n^2 y h_n$. Or $\lambda_n \neq 0$, then $h_n = \lambda_n y h_n$. The algebra A is a complete commutative m-convex algebra, then by ([8], Corollary 5.7, p. 23), $\rho_A(yh_n) \leq \rho_A(y)\rho_A(h_n)$. which implies, $1 = \rho_A(h_n) = \lambda_n \rho_A(yh_n) \leq \lambda_n \rho_A(y)\rho_A(h_n) \leq \lambda_n \rho_A(y)$. Finally, we get that for all $n \in \mathbb{N}^*$, $1 \leq \lambda_n \rho_A(y)$, which is impossible. Let $\{h_1,...,h_m\}$ be a family of pairwise orthogonal non-zero idempotents. For each $j \in \{1,...,m\}$, either Ah_j has no proper idempotents or there exists non-zero idempotents p_j in Ah_j. Put $q_j = h_j - p_j$. One has $p_j q_j = p_j(h_j - p_j) = p_j - p_j^2 = 0$, moreover $q_j^2 = (h_j - p_j)^2 = h_j^2 - p_j - p_j + p_j^2 = h_j - p_j = q_j$. Then p_j and q_j are two nonzero idempotents in Ah_j, such that $h_j = p_j + q_j$ and $p_j q_j = 0$. Since A cannot contain an infinite sequence of pairwise orthogonal idempotents, we may suppose that $\{h_1,...,h_m\}$ is chosen so that for each $j \in \{1,...,m\}$, Ah_j has no proper idempotents, and $e = h_1 + ... + h_m$. Thus $A = Ah_1 + ... + Ah_m$. For each $j \in \{1,...,m\}$, for each $x \in Ah_j$, $Ah_j x^2 = Ax^2 h_j = Axh_j = Ah_j x$, then the algebra Ah_j satisfies the given condition (P_1). It follows that Ah_j is isomorphic to $\mathbb{C}$. Thus $\dim(A) \leq m$. The algebra A is finite-dimensional, then A admits a finite number of non-zero characters. Let us note $\chi_1,...,\chi_n$ the non-zero characters of A. For every $i \neq j \in [\![1,n]\!]$, $A = ker(\chi_i) + ker(\chi_j)$. Using the chinese theorem ([1], Proposition 5, p. 72), we obtain:

$$A/\bigcap_{i=1}^{n} Ker(\chi_i) \simeq \prod_{i=1}^{n} A/Ker(\chi_i).$$

Since $A/Ker(\chi_i) \simeq Im(\chi_i) = \mathbb{C}$, we obtain

$$A/\bigcap_{i=1}^{n} Ker(\chi_i) \simeq \mathbb{C}^n.$$

The algebra A is a Q-Fréchet commutative semi-simple algebra, then

$$\{0\} = Rad(A) = \bigcap_{i=1}^{n} Ker(\chi_i).$$

Finally:

$$A \simeq \mathbb{C}^n.$$

□

Remark 2.3. The result of the previous theorem is not valid without the Q-property. Indeed the algebra of complex sequences $\mathbb{C}^{\mathbb{N}}$ endowed with the product topology is a unital semi-simple commutative Fréchet algebra of infinite dimension satisfying the condition (P_1)

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